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Some New Approaches for Computation of Domination Polynomial of Specific Graphs

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Abstract. Let $G = (V, E)$ be a simple graph. The domination polynomial of G is the polynomial $D(G, x) = \sum_{i=0}^{n} d(G, i)x^i$, where $d(G, i)$ is the number of dominating sets of G of size i. In this paper, we present some new approaches for computation of domination polynomial of specific graphs.

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1. Introduction

Let $G = (V, E)$ be a simple graph. Graph polynomials are a well-developed area useful for analyzing properties of graphs. Domination polynomial is a new graph polynomial which introduced by the first author for the first time as a PhD thesis (see [3]) in the literature.

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Let $G = (V, E)$ be a graph of order $|V| = n$. For any vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V | uv \in E\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood of S is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a dominating set if $N[S] = V$, or equivalently, every vertex in $V \backslash S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. For a detailed treatment of this parameter, the reader is referred to [10]. Let $\mathcal{D}(G, i)$ be the family of dominating sets of a graph G with cardinality i and let $d(G, i) = |\mathcal{D}(G, i)|$. The *domination polynomial* $D(G, x)$ of G is defined as $D(G, x) = \sum_{i=0}^{|V(G)|} d(G, i)x^i$. The path P_4 on 4 vertices, for example, has one dominating set of cardinality 4, four dominating sets of cardinality 3, and four dominating sets of cardinality 2; its domination polynomial is then $D(P_4, x) = x^4 + 4x^3 + 4x^2$. For more information of this polynomial see [1, 2, 3, 6, 7, 8].

The *corona* of two graphs G_1 and G_2, as defined by Frucht and Harary in [9], is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2. The corona $G \circ K_1$, in particular, is the graph constructed from a copy of G, where for each vertex $v \in V(G)$, a new vertex v' and a pendant edge vv' are added.

In the next section, we present some new approaches for computation of domination polynomial of specific graphs. In Section 3, we consider the domination polynomial of a graph with a handle. In the last section we give a new approach for computing the domination polynomial of graphs of the form $H \circ K_1$.

2. New Approaches for Computation of Domination Polynomial of Specific Graphs

In [8] the structures of dominating sets of paths has studied. Using that structures there is the following recurrence relation for domination polynomial of paths.

Theorem 2.1. ([8]) $D(P_n, x) = x\big[D(P_{n-1}, x)+D(P_{n-2}, x)+D(P_{n-3}, x)\big]$, *with the initial values* $D(P_0, x) = 1, D(P_1, x) = x, D(P_2, x) = 2x + x^2$.

We shall give a simple proof for this recurrence relation. First we state a formula for the domination polynomial of a graph in terms of the domination polynomials of several other graphs which have fewer vertices or edges ([4, 5, 11]).

The vertex contraction G/v of a graph G by a vertex v is the operation under which all vertices in $N(v)$ are joined to each other and then v is deleted (see [12]).

Theorem 2.2. ([4, 5, 11]) *For any vertex v in a graph G we have*

$$D(G, x) = xD(G/v, x) + D(G-v, x) + xD(G-N[v], x) - (x+1)p_v(G, x)$$

where $p_v(G, x)$ is the polynomial counting those dominating sets for $G - N[v]$ which additionally dominate the vertices of $N(v)$ in G.

The following theorems are special cases which have a simpler recurrence relations. These results also appeared in [11] but as mentioned in page 4 of [11], were proved independently.

Theorem 2.3. ([4, 11]) *If $u, v \in V(G)$, $uv \in E(G)$ and $N[v] \subseteq N[u]$, then*

$$D(G, x) = xD(G/u, x) + D(G - u, x) + xD(G - N[u], x).$$

Proof. Since $N[v] \subseteq N[u]$, there are no edges from v to $G - N[u]$ and hence no dominating set for $G-N[u]$ can dominate $v \in N(u)$. Therefore $P_u(G, x) = 0$ and the result follows from Theorem 2.2. $\square$

Theorem 2.4. ([4, 11]) *If $u, w \in V(G)$ and $N(w) = N(u)$ then*

$$D(G, x) = xD(G/u, x) + D(G - u, x) - xD(G - N[u] - w, x).$$

Proof. In this case w is of degree 0 in $G - N[u]$ and hence w must exist in any dominating set for $G - N[u]$. Since $N(w) = N(u)$ we know that $N(u)$ is dominated by every dominating set for $G - N[u]$. Thus we have $p_u(G, x) = D(G - N[u], x)$ and the result follows from Theorem 2.3. $\square$

Corollary 2.5. *Let $G = (V, E)$ be a graph, v be a vertex of degree 1 in G and let u be its neighbor. Then*

$$D(G, x) = x\big(D(G/u, x) + D(G - u - v, x) + D(G - N[u], x)\big).$$

Proof. We have $N[v] = \{u, v\} \subseteq N[u]$ and so, by Theorem 2.3:

$$\begin{aligned} D(G, x) &= xD(G/u, x) + D(G - u, x) + xD(G - N[u], x) \\ &= xD(G/u, x) + xD(G - u - v, x) + xD(G - N[u], x) \\ &= x(D(G/u, x) + D(G - u - v, x) + D(G - N[u], x)). \quad \square \end{aligned}$$

Theorem 2.6.([8]) $D(P_n, x) = x\big[D(P_{n-1}, x) + D(P_{n-2}, x) + D(P_{n-3}, x)\big]$.

Proof. Suppose that the vertex v of the path graph P_n has degree 1 and u be its neighbor. Since $P_n/v = P_{n-1}$, $P_n - u - v = P_{n-2}$ and $P_n - N[u] = P_{n-3}$, we have the result by Corollary 2.5. □

As another corollary of Theorems 2.2 and 2.3 we consider graphs which contain vertices of degree $|V(G)| - 2$ (see [11]).

Corollary 2.7. *Let $G = (V, E)$ be a graph. Let u be a vertex of degree $|V(G)| - 2$ (i.e. the vertex u is adjacent to all vertices except w). If there exists a vertex $v \in N(u)$ such that $\{v, w\} \notin E(G)$ then*

$$D(G, x) = D(G/u, x) + D(G - u, x) + x^2.$$

Otherwise,

$$D(G, x) = x(1 + x)((1 + x)^{|V(G)|-2} - 1) + D(G - u, x).$$

Proof. If there exists a vertex $v \in N(u)$ such that $\{v, w\} \in E(G)$ then $N[v] \subseteq N[u]$ and by Theorem 2.3 we have:

$$\begin{aligned} D(G, x) &= xD(G/u, x) + D(G - u, x) + xD(G - N[u], x) \\ &= xD(G/u, x) + D(G - u, x) + x^2. \end{aligned}$$

Another case is $N(w) = N(u)$ and hence by Theorem 2.3

$$\begin{aligned} D(G,x) &= xD(G/u,x) + D(G-u,x) - D(G-N[u],x) \\ &= xD(G/u,x) + D(G-u,x) - x \\ &= x((1+x)^{|V(G)|-1} - 1) + D(G-u,x) - x \\ &= x(1+x)((1+x)^{|V(G)|-2} - 1) + D(G-u,x). \quad \square \end{aligned}$$

3. Domination Polynomial of a Graph with a Handle

Let P_{m+1} be a path with vertices labeled by $y_0, y_1, \ldots, y_m$, for $m \geqslant 0$ and let v_0 be a specific vertex of a graph G. Denote by $G_{v_0}(m)$ a graph obtained from G by identifying the vertex v_0 of G with an end vertex y_0 of P_{m+1}. It is clear that if the path is glued to a different vertex v_1 of G, then the two graphs $G_{v_1}(m)$ and $G_{v_0}(m)$ may not be isomorphic. Throughout our discussion, this vertex is fixed, then we shall simply use the notation $G(m)$ (if there is no likelihood of confusion).
We have the following result for domination polynomial of $G(m)$.

Theorem 3.1.([7]) *For every $m \geqslant 3$,*

$$D(G(m),x) = x\Big[D(G(m-1),x) + D(G(m-2),x) + D(G(m-3),x)\Big]$$

This recursive formula is for $m \geqslant 3$ and we need to know $D(G(1),x)$, $D(G(2),x)$ and $D(G(3),x)$. Here we illustrate about $D(G(1),x)$.

Let G be a graph and v a vertex in G. Let $G(1)$ be G with an edge vw attached to v (a handle of length 1). This is $G(1)$ from the above definition. Let $A(x)$ be the generating polynomial of the dominating sets of G containing v; let $C(x)$ be the generating polynomial of the subsets of G dominating every vertex of G except v. We have the following theorem:

Theorem 3.2. *$D(G(1),x) = xD(G,x) + A(x) + xC(x)$.*

Proof. We explain the right-hand side: the first term counts the dominating subsets of G with w adjoined; the dominating subsets of G that contain v are dominating subsets also for $G(1)$; the 3rd term counts the subsets counted by $C(x)$ with w adjoined. □

The equation in Theorem 3.2 is useful only for a graph G that we are able to find $A(x)$ and $C(x)$.
Let us see what happens if G is the cycle C_n ($n \geqslant 1$). Note that C_1 is a vertex and C_2 is the path P_2. We show the polynomial $A(x)$ and $C(x)$ for the cycle C_n, by $A_n(x)$ and $C_n(x)$, respectively. The first $A_n(x)$ polynomials are (by straightforward counting):

$A_1(x) = x$;
$A_2(x) = x + x^2$;
$A_3(x) = x + 2x^2 + x^3$;
$A_4(x) = 3x^2 + 3x^3 + x^4$;
$A_5(x) = 2x^2 + 6x^3 + 4x^4 + x^5$;
$A_6(x) = x^2 + 7x^3 + 10x^4 + 5x^5 + x^6$.

The first $C_n(x)$ polynomials are:

$C_1(x) = 1$;
$C_2(x) = 0$;
$C_3(x) = 0$;
$C_4(x) = x$;
$C_5(x) = x^2$;
$C_6(x) = x^2 + x^3$.

Looking at these expressions, we have

$$A_n(x) = x(A_{n-1}(x) + A_{n-2}(x) + A_{n-3}(x));$$

$$C_n(x) = x(C_{n-1}(x) + C_{n-2}(x) + C_{n-3}(x)).$$

So, the domination polynomials of the cycles with a handle follow easily.

4. A New Approach for Domination Polynomialof $H \circ K_1$

In [2] the domination polynomial of graphs of the form $H \circ K_1$ and $H \circ K_2$ has obtained. Here we give another approach to compute the domination polynomial of corona of an arbitrary graphs with K_1.

Theorem 4.1. *Let G be a graph, $u, v \in V(G)$ and $deg(u) = deg(v) = 1$. If $uw, vw' \in E(G)$ and $ww' \notin E(G)$, then $D(G, x) = D(G + ww', x)$.*

Proof. Clearly, every dominating set for G is a dominating set for $G + ww'$. Now, let $S \subseteq V(G)$ be a dominating set for $G + ww'$. If both $w, w' \in S$ or both $w, w' \notin S$, then obviously S is also a dominating set for G. So suppose that $w \in S$ and $w' \notin S$ (or $w \notin S$ and $w' \in S$). Since S is a dominating set for $G + ww'$, we have $v \in S$. This implies that S is a dominating set for G. Therefore we conclude that $D(G, x) = D(G + ww', x)$ and the proof is complete. □

The following result is an immediate consequence of the above theorem. We recall that a quasipendant vertex is a vertex adjacent to a pendant vertex.

Theorem 4.2. *Let H and K be two simple graphs and the vertex b (the vertex c) be a quasipendant vertex of graph H (graph K). Let G be the a graph obtained by joining H and K with the edge bc. Then $D(G, x) = D(H, x)D(K, x)$.*

We need the following theorem to obtain further results:

Theorem 4.3. ([6]) *If a graph G has m components $G_1, \ldots, G_m$, then $D(G, x) = D(G_1, x) \cdots D(G_m, x)$.*

Theorem 4.4. *For any graph G of order n, $D(G \circ K_1, x) = x^n(x + 2)^n$.*

Proof. By Theorems 4.2 and 4.3 we can delete from $G \circ K_1$ all the edges of G (they all connect two quasipendant vertices) and the domination polynomial will not change. So we have

$$D(G \circ K_1, x) = \prod_{i=1}^{n} D(K_2, x).$$

Since $D(K_2, x) = x^2 + 2x$, we have the result. □

Let H be an arbitrary graph with $V(H) = \{v_1, \ldots, v_n\}$. By Theorems 4.2 and 4.3, for any graph G of the form $H \circ K_1$, $D(G, x) = D(\bigcup_{i=1}^{n} K_2, x)$. By similar arguments we have the following theorem:

Theorem 4.5. *Let H_i, $1 \leqslant i \leqslant n$, be a graph containing a vertex v_i of degree $|V(H_i)| - 1$, $|V(H_i)| \geqslant 2$, and H be a graph with vertex set $V(H) = \{u_1, \ldots, u_n\}$. If G is a graph formed by identifying the vertex v_i with u_i for every $1 \leqslant i \leqslant n$, then*

$$D(G, x) = \prod_{i=1}^{n} D(H_i, x).$$

In other words, the domination polynomial of G does not depend on the geometrical structure of H.

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