

A Power Mapping and Its Fixed Points

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Abstract. In this paper, we show that for a nonempty finite set U and a power mapping $T : U \rightarrow \mathcal{P}^*(U)$, where $\mathcal{P}^*(U)$ is set of all nonempty subsets of U , there exists a nonempty subset F of U such that $T'(F) = F$ and for any $K \subseteq U$, $T'(K) = \bigcup_{x \in K} T(x)$. Also for a power mapping $T : U \rightarrow \mathcal{P}(U)$, we get an equivalent condition for having a nonempty fixed points set. Finally, we present a method to obtain all of fixed points of T .

AMS Subject Classification: 05C09, 05C15, 05C76.

Keywords and Phrases: Power mapping, Fixed point, Fower set

1 Introduction

In mathematics, a fixed-point theorem is a result saying that a function F will have at least one fixed point (a point x for which $F(x) = x$), under some conditions on F that can be stated in general terms. In the other words, a fixed point, also known as an invariant point, is a value that does not change under a given transformation. Any set of fixed points of a transformation is also an invariant set. There exist various types of fixed point theorems, such as Brouwer fixed point theorem, Atiyah-Bott

Received: July 2024; Accepted: February 2025

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fixed point theorem, Banach fixed point theorem, etc [3, 2, 11, 9, 10].

A power mapping is a mathematical function that maps elements from one set, the domain of the function, to subsets of another set. Power mappings are used in a variety of mathematical fields, including optimization, control theory and game theory [1, 8, 14, 15].

Let U is a nonempty finite set and $T : U \rightarrow \mathcal{P}^*(U)$ is a power mapping, then there exists a nonempty subset F of U such that $T'(F) = F$. Also for a power mapping $T : U \rightarrow \mathcal{P}(U)$, we get a necessary and sufficient condition for having a nonempty fixed point. We know that it is very hard to calculate fixed points by direct method, for example if $|U| = 100$, then to determine all the fixed points necessary to check the number of $|\mathcal{P}^*(U)| = 2^{100} - 1 \approx 1.26765 \times 10^{30}$ subsets of U . In this paper, we present a new method to obtain all fixed points of the power mapping $T : U \rightarrow \mathcal{P}(U)$, where U is a finite set. In addition, we state and investigate the three main fixed point theorems on the power mappings.

2 Basic Concepts

In this paper, we introduce a new concept of a power mapping. Let U and W be sets and $T : U \rightarrow \mathcal{P}(W)$ be a mapping, then we call T a *power mapping*, where $\mathcal{P}(W)$ is set of all subsets of W . We denote the set of all power mapping from U in to $\mathcal{P}(W)$ by $\langle U, W \rangle_0 = \text{Map}(U, \mathcal{P}(W))$. Also, we set $\langle U, W \rangle = \text{Map}(U, \mathcal{P}^*(W))$, where $\mathcal{P}^*(W)$ is set of all non-empty subsets of W . We write for simplicity $\langle U, U \rangle_0 = \langle U \rangle_0$ and $\langle U, U \rangle = \langle U \rangle$. This type of mappings, appear in many mathematical theories, such as algebraic hyperstructures theory, T-rough sets theory numbers theory and graph theory [6, 5, 12, 13, 7, 4, 5].

In this section, we present several examples about fixed points of finite algebraic hyperstructures.

- Example 2.1.**
1. Let G be a graph and g is a vertex of G , then $T : G \rightarrow \mathcal{P}(G)$ is a power mapping, where $T(g) = V_g$ and $V_g = \{x : g \text{ and } x \text{ are adjacent vertices}\}$.
 2. Let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the set of natural numbers. Then the mapping $T : \mathbb{N} \rightarrow \mathcal{P}^*(\mathbb{N})$, where $T(n) = \{d \in \mathbb{N} : d|n\}$ is a power mapping.
 3. The mapping $\Phi : \mathbb{N} \rightarrow \mathcal{P}^*(\mathbb{N})$, where $\Phi(n) = \{m \in \mathbb{N} : m \leq n \text{ and } (m, n) = 1\}$ is a power mapping.
 4. Let (A, \cdot) be a finite algebraic structure. For any $u \in A$ define $T_u(a) = \{x \in A : a \cdot x = u\}$. Then T is a power mapping.
 5. Let (H, \star) be a finite hyperstructure and $h \in H$. Then $T_h : H \rightarrow \mathcal{P}^*(H)$ is a power mapping, where $T_h(a) = a \star h$.

6. Let (X, τ) be a topological space and $u \in X$. Suppose that for any $x \in X$, $T_u(x)$ is an open subset, such as $O_x \in \tau$, such that $x \in O_x$ and $y \notin O_x$. Then T_u is a power mapping.

Now, we show, if U is a nonempty finite set and $T : U \rightarrow \mathcal{P}^*(U)$, where $\mathcal{P}^*(U)$ is set of all non-empty subsets of U is a power mapping, then there exists a nonempty subset F of U such that $T'(F) = F$.

Definition 2.2. Let U and W be sets, and $T : U \rightarrow \mathcal{P}(W)$ be a power mapping. We define $T' : \mathcal{P}(U) \rightarrow \mathcal{P}(W)$, such that $T'(K) = \bigcup_{x \in K} T(x)$.

It is clear that $T'(\emptyset) = \emptyset$, and $T_1 = T_2$ if and only if $T'_1 = T'_2$.

Definition 2.3. Let U be a set, $T \in \langle U \rangle_0$ and K be a subset of U .

1. A subset F of U is called a fixed point of T , if and only if $T'(F) = F$.
2. $fix_K(T) = \{K_0 \subseteq K \mid K_0 \neq \emptyset \text{ and } T'(K_0) = K_0\}$.
3. $Fix_K(T) = \{K_0 \subseteq K \mid T'(K_0) = K_0\}$.
4. $fix(T) = fix_U(T)$.
5. $Fix(T) = Fix_U(T)$.
6. If $F \subseteq U$ is a fixed point of T and for any $x \in F$, $T(x) \neq \emptyset$, then F is called a *normal fixed point* of T .
7. If F is not a normal fixed point of T , then F is called an *abnormal fixed point*.

Example 2.4. Let $U = \{x, y, z, u, v, w, t\}$ and

$$T : \begin{cases} x \rightarrow \{y\}, \\ y \rightarrow \{x, w\}, \\ z \rightarrow \{u, w\}, \\ u \rightarrow \{z\}, \\ v \rightarrow \{w\}, \\ w \rightarrow \{y\}, \\ t \rightarrow \{z\}. \end{cases}$$

Then $T'(\{x, y, w\}) = \{x, y, w\}$, so $\{x, y, w\}$ is a fixed point of T .

Definition 2.5. Let $T \in \langle U \rangle$ and F be a fixed point of T .

F is called a *minimal fixed point*, if $F \neq \emptyset$ and if F' is a nonempty fixed point of T and $F' \subseteq F$, then $F' = F$. F is called a *maximal fixed point* of T if there is not a fixed point such as F' such that $F \subset F'$.

Definition 2.6. Let $T : U \rightarrow \mathcal{P}(U)$ be a power mapping and $K \subseteq U$. Then

$$(1) (T')^0(K) = K.$$

- (2) $(T')^1(K) = T'(K)$.
- (3) $(T')^{n+1}(K) = T'((T')^n(K))$ for $n \in \mathbb{N}_0$. We write for simplicity $(T')^n$ by T'^n and $T'^n(\{x_1, x_2, \dots, x_n\})$ by $T'^n(x_1, x_2, \dots, x_n)$ for any $x_1, x_2, \dots, x_n \in U$.

3 Fixed Point Theorem

In this section, we will prove the main theorem 1 on a power mapping.

Lemma 3.1. *Let $T \in \langle U \rangle_0$.*

(1) *If $K \subseteq K'$ and $n \in \mathbb{N}_0$, then $T'^n(K) \subseteq T'^n(K')$.*

(2) *If for any $\alpha \in I$, $K_\alpha \subseteq U$ and $n \in \mathbb{N}_0$, then $T'^n(\bigcup_{\alpha \in I} K_\alpha) = \bigcup_{\alpha \in I} T'^n(K_\alpha)$.*

Proof. It is straitforward. \square

Lemma 3.2. *Let $T \in \langle U \rangle_0$ and $K \subseteq U$. Then the $Fix_K(T)$ is closed under operation \bigcup .*

Proof. It follows from Lemma 3.1. \square

Example 3.3. Suppose that $U = \{x, y, z, t\}$ and $T \in \langle U \rangle$, such that

$$T(u) = \begin{cases} \{x, y\} & u = x, \\ \{z\} & u = y, \\ \{z\} & u = z, \\ \{y, t\} & u = t. \end{cases} \quad (1)$$

Then $\{x, y, z\}$ and $\{y, z, t\}$ are fixed points, so $U = \{x, y, z\} \bigcup \{y, z, t\}$ is a fixed point by Lemma 3.2, but $\{y, z\} = \{z, y, z\} \cap \{y, z, t\}$ is not a fixed point.

Lemma 3.4. *Let U be a nonempty set and $T \in \langle U \rangle$. If there exists a nonempty finite subset K of U such that $K \supseteq T'(K)$, then there exists a nonempty subset K_0 of K such that $T'(K_0) = K_0$.*

Proof. Since $T'(K) \subseteq K$, therefore

$$K \supseteq T'(K) \supseteq (T')^2(K) \supseteq \dots \supseteq (T')^n(K) \supseteq \dots \supseteq \emptyset, \quad (2)$$

by Lemma 3.1. Since K is finite, so in (2), there exists a $n \in \mathbb{N}_0$ such that $(T')^n(K) = (T')^{n+1}(K)$. Set $K_0 = (T')^n(K)$, then K_0 is a nonempty subset of K such that $T'(K_0) = K_0$. \square

Lemma 3.5. *Let U be a nonempty finite set and $T \in \langle U \rangle_0$. If there exists a nonempty subset $K \subseteq U$ such that $K \subseteq T'(K)$, then there exists a subset K_1 of U such that $K_1 \supseteq K$ and $T'(K_1) = K_1$.*

Proof. We have

$$K \subseteq T'(K) \subseteq (T')^2(K) \subseteq \cdots \subseteq U, \quad (3)$$

by Lemma 3.1. Since U is a finite set, hence the chain (3) has a finite length, hence there exists a $n \in \mathbb{N}_0$, such that $(T')^n(K) = (T')^{n+1}(K)$. Set $K_1 = (T')^n(K)$, then $T'(K_1) = K_1$. \square

Theorem 3.6 (Main Theorem 1: Fixed Point Theorem). *If U is a nonempty finite set and $T \in \langle U \rangle$, then there exists a nonempty subset F of U such that $T'(F) = F$.*

Proof. Since $T'(U) \subseteq U$, hence the claim follows from Lemma 3.4. \square

If the conditions of Fixed Point Theorem are not established, then it is not necessary that the assertion to be established. See the following examples.

Example 3.7. (1) Suppose that $U = \{x, y\}$, $T(x) = \{y\}$ and $T(y) = \emptyset$. Then T does not have any nonempty fixed point.

(2) Let \mathbb{N} be the set of natural numbers, and $T : \mathbb{N} \rightarrow \mathcal{P}^*(\mathbb{N})$ be a power mapping, where $T(n) = \{n + 1\}$. Then T does not have any nonempty fixed point.

Example 3.8. Suppose that (H, \star) is a finite algebraic hyperstructure. for any $h \in H$ define $T_h(x) = h \star x$. Then for any $h \in H$, there exists a nonempty subset F_h of H , such that $T'_h(F_h) = F_h$, by Fixed Point Theorem.

4 Obtaining the Fixed Points

Suppose that $T : U \rightarrow \mathcal{P}^*(U)$ is a power mapping, therefore $T' : \mathcal{P}^*(U) \rightarrow \mathcal{P}^*(U)$ is a mapping. If we want to obtain all of fixed points, directly, then it means that we have to test $2^{|U|} - 1$ members of $\mathcal{P}^*(U)$, this is a very difficult task.

Definition 4.1. Let $T \in \langle U \rangle_0$ and K be a subset of U . K is called a *T-small subset* (or a *small subset* of U), if $K \subseteq T'(K)$. K is called a *T-big subset* (or a *big subset* of U), if $K \supseteq T'(K)$. The subset K of U is called a *T-normal subset* (or a *normal subset* of U) if K is a T-small subset or a T-big subset of U .

Example 4.2. Suppose that $U = \{x, y, z, u, v, w, t\}$ and

$$T = \begin{cases} x \rightarrow \{y\}, \\ y \rightarrow \{y, z\}, \\ z \rightarrow \{z\}, \\ u \rightarrow \{u, v\}, \\ v \rightarrow \{u\}, \\ w \rightarrow \{u\}, \\ t \rightarrow \emptyset. \end{cases}$$

We have $T'(\{x, y, z\}) = \{y, z\}$, hence $\{x, y, z\}$ is a big subset of U . Since $T'(\{u\}) = \{u, v\}$, hence $\{u\}$ is a small subset of U . Therefore the sets $\{x, y, z\}$ and $\{u\}$ are normal subsets of U . Also $\{z\}$ is a normal subset of U . The sets $\{v\}$, $\{w\}$, $\{v, w\}$, and $\{v, w, t\}$ are not a normal subset of U .

According to Lemma 3.4, Lemma 3.5, and Definition 4.1, we have:

Corollary 4.3. *Let $T \in \langle U \rangle$, then*

- (1) *any nonempty finite big subset of U , contains a nonempty fixed point;*
- (2) *if U is a finite set, then any nonempty small subset is contained in a nonempty fixed point.*

Definition 4.4. Let $T \in \langle U \rangle_0$ and K be a normal subset of U . If n is the smallest number such that $(T')^n(K) = (T')^{n+1}(K)$, then we write $o_T(K) = n$.

Lemma 4.5. *Let U be a finite set, $T \in \langle U \rangle_0$ and K be a subset of U , then*

- (1) *if K is a big subset of U , then $o_T(K) \leq |K|$;*
- (2) *if K is a small subset of U , then $o_T(K) \leq |K^c|$.*

Proof. It is obvious. \square

Lemma 4.6. *Let U be a nonempty finite set and K be a normal subset of U .*

- (1) *If $T \in \langle U \rangle$, then $T^{o_T(K)}(K)$ is a nonempty fixed point.*
- (2) *If $T \in \langle U \rangle_0$, then $T^{o_T(K)}(K)$ is a fixed point.*

Proof. It is obvious. \square

Example 4.7. Let $U = \{x, y, z, u, v, w\}$. Define

$$T : \begin{cases} x \rightarrow \{y\}, \\ y \rightarrow \{x\}, \\ z \rightarrow \{z, u\}, \\ u \rightarrow \{z\}, \\ v \rightarrow \{v\}, \\ w \rightarrow \{u\}. \end{cases} \quad \Lambda : \begin{cases} x \rightarrow \{x, y, z\}, \\ y \rightarrow \emptyset, \\ z \rightarrow \{x\}, \\ u \rightarrow \{v\}, \\ v \rightarrow \emptyset, \\ w \rightarrow \{v, w\}. \end{cases}$$

Then $\{z\}$ is a T -small subset. So $\{z\} \subset \{z, u\} = T'(z) = (T')^2(z)$. Therefore $o_T(\{z\}) = 1$ and $\{z, u\}$ is a fixed point by Lemma 4.6. Also $\{z.u.w\}$ is a T -big subset and $o_T(\{z, u, w\}) = 1$.

The subset $\{x, y\}$ is a Λ -small subset, and $o_\Lambda(\{x, y\}) = 1$. Thus $\Lambda'\{x, y\} = \{x, y, z\}$ is a nonempty fixed point, by Lemma 4.6. Also $\{u, v, w\}$ is a Λ -big subset and $o_\Lambda(\{u, v, w\}) = 1$. We have $\Lambda'\{u, v, w\} = \{v, w\}$.

Definition 4.8. Let $T \in \langle U \rangle_0$, S is a small subset and B is a big subset of U . We define $(T')^\infty(S) = \bigcup_{n \geq 0} (T')^n(S)$ and $(T')^\infty(B) = \bigcap_{n \geq 0} (T')^n(B)$.

Corollary 4.9. Let $T \in \langle U \rangle_0$.

- (1) If S is a small subset of U and $o_T(S) < \infty$, then $(T')^\infty(S) = (T')^{o_T(S)}(S)$.
- (2) If B is a big subset of U and $o_T(B) < \infty$, then $(T')^\infty(B) = (T')^{o_T(B)}(B)$.

Corollary 4.10. Let U be a finite set and K be a normal subset of U .

- (1) If $T \in \langle U \rangle$, then $(T')^\infty(K)$ is a nonempty fixed point.
- (2) If $T \in \langle U \rangle_0$, then $(T')^\infty(K)$ is a fixed point.

Lemma 4.11. Let U be a finite set and $T \in \langle U \rangle_0$.

- (1) If K is a small subset of U , then $T'^\infty(K)$ is the smallest fixed point, such that contains K .
- (2) If K is a big subset of U , then $T'^\infty(K)$ is the biggest fixed point, such that is contained in K .

Proof.

- (1) If $T'(K) = K$, then it is clear. Suppose that $K \subset T'(K)$ and F is a fixed point, such that $K \subset F \subset T'^\infty(K)$. Then

$$T'^\infty(K) \subseteq T'^\infty(F) = F \subset T'^\infty(K).$$

It is a contradiction.

- (2) It is similar to (1).

□

Definition 4.12. Let $T \in \langle U \rangle_0$ and K be a subset of U . We define $\vec{K} = \bigcup_{n \geq 0} (T')^n(K)$, and $[K] = (T')^\infty(\vec{K})$. For any $x_1, x_2, \dots, x_n \in U$, we show $[\{x_1, x_2, \dots, x_n\}]$ by $[x_1, x_2, \dots, x_n]$.

Theorem 4.13. Let U be a nonempty finite set and K be a nonempty subset of U . Then

- (1) \vec{K} is a big subset of U .
- (2) If $T \in \langle U \rangle$, then $[K]$ is a nonempty fixed point.
- (3) If $T \in \langle U \rangle_0$, then $[K]$ is a fixed point of T .

Proof. By Lemma 3.1, we have

$$\begin{aligned} T'(K \cup T'(K) \cup T'^2(K) \cup \dots) &= T'(K) \cup T'^2(K) \cup T'^3(K) \dots \\ &\subseteq K \cup T'(K) \cup T'^2(K) \cup T'^3(K) \dots \end{aligned}$$

Therefore $\bigcup_{n \geq 0} (T')^n(K)$ is a big subset of U . Now the proof follows from Corollary 4.10. □

Example 4.14. In Example 4.7, we have

$$\vec{\{x\}} = \bigcup_{n \geq 0} (T')^n(x) = \{x\} \cup \{y\} = \{x, y\}.$$

Hence $\{x, y\}$ is a big subset of U by Theorem 4.13. Since $T'^\infty(\{x, y\}) = \{x, y\}$, therefore $\{x, y\}$ is a fixed point of T by Theorem 4.13.

Example 4.15. Assume that $U = \mathbb{N}$ and $T : \mathbb{N} \rightarrow \mathcal{P}^*(\mathbb{N})$ is a power mapping, where

$$T(n) = \begin{cases} \{2\} & n = 1, \\ \{3\} & n = 2, 3, \\ \{1, n+1\} & n = 4, 5, 6, \dots \end{cases}$$

Let $K = \{1, 2, 3, 4\}$, then $\vec{K} = \mathbb{N}$ and $[K] = \{1, 2, 3\}$. It is obvious that $[K]$ is not a fixed point and $[[K]] = \{3\}$ is a fixed point of T .

Lemma 4.16. If $T \in \langle U \rangle_0$ and $K_1 \subseteq K_2$, then $\vec{K}_1 \subseteq \vec{K}_2$ and $[K_1] \subseteq [K_2]$.

Proof. It follows from Lemma 3.1. □

Lemma 4.17. Let U be a nonempty finite set, $T \in \langle U \rangle_0$ and $F \subseteq U$. Then F is a fixed point of T if and only if $\vec{F} = [F] = F$.

Proof. It is obvious. □

Lemma 4.18. Let U be a nonempty finite set, $T \in \langle U \rangle_0$ and $K \subseteq U$. Then $\vec{\vec{K}} = \vec{K}$ and $[\vec{K}] = \vec{[K]} = [[K]] = [K]$.

Proof. It follows from Theorem 4.13 and Lemma 4.17. \square

Corollary 4.19. *Let U be a nonempty finite set, $T \in \langle U \rangle_0$ and $K_1, K_2 \subseteq U$. If $\overrightarrow{K_1} = \overrightarrow{K_2}$, then $[K_1] = [K_2]$.*

Proof. We have $[K_1] = [\overrightarrow{K_1}] = [\overrightarrow{K_2}] = [K_2]$, by Lemma 4.18. \square

Lemma 4.20. *Let U be a nonempty finite set and $T \in \langle U \rangle_0$. If K is a subset of U , then there exists a number $n \geq 0$ such that*

$$\overrightarrow{K} = K \cup T'(K) \cup \cdots \cup (T')^n(K).$$

Proof. Assume that

$$K \subset K \cup T'(K) \subset K \cup T'(K) \cup T'^2(K) \subset \cdots \subset U.$$

This chain can not be strict, forever. Therefore there exists the smallest number $n \geq 0$, such that

$$\begin{aligned} K \subset K \cup T'(K) \subset \cdots \subset K \cup T'(K) \cup \cdots \cup (T')^n(K) \\ = K \cup T'(K) \cup \cdots \cup (T')^n(K) \cup (T')^{n+1}(K). \end{aligned}$$

Then for any $m \geq n$,

$$\bigcup_{i=0}^{i=m \geq n} (T')^i(K) = K \cup T'(K) \cup \cdots \cup (T')^n(K).$$

\square

Definition 4.21. Let U be a nonempty finite set, $T \in \langle U \rangle_0$ and K be a subset of U . We write $O_T(K) = n$, if n is the smallest number, such that

$$K \cup T'(K) \cup \cdots \cup T'^n(K) = K \cup T'(K) \cup \cdots \cup T'^n(K) \cup T'^{n+1}(K).$$

Example 4.22. Let U be a nonempty finite set and $T \in \langle U \rangle_0$. Then

- (1) $O_T(\emptyset) = O_T(U) = 0$.
- (2) If K is a normal subset of U , then $O_T(K) \leq o_T(K)$.

Lemma 4.23. *Let U be a nonempty finite set, $T \in \langle U \rangle_0$ and $K_1, K_2 \subseteq U$, then*

- (1) $\overrightarrow{K_1 \cup K_2} = \overrightarrow{K_1} \cup \overrightarrow{K_2}$.
- (2) $[K_1 \cup K_2] = [K_1] \cup [K_2]$.

Proof.

- (1) It follows from Lemma 3.1 and Lemma 4.20.
- (2) It follows from (1) and Lemma 3.1.

\square

Lemma 4.24. *Let U be a nonempty finite set, $T \in \langle U \rangle_0$ and K be a subset of U . Then the following statements are equivalent.*

- (1) $\vec{K} \subseteq [K]$,
- (2) $\vec{K} = [K]$,
- (3) $\vec{K} \in \text{Fix}(T)$.

Proof.

- (1 \Rightarrow 2) If $K \subseteq [K]$, then $\vec{K} \subseteq \overline{[K]} = [K]$ by Lemma 4.16 and Lemma 4.17.
- (2 \Rightarrow 3) It is obvious.
- (3 \Rightarrow 4) It follows from Theorem 4.13.
- (4 \Rightarrow 1) It is clear that $K \subseteq \vec{K}$. Since \vec{K} is a fixed point of T , hence $[\vec{K}] = \vec{K}$. Therefore $K \subseteq \vec{K} = [\vec{K}] = [K]$ by Lemma 4.18.

Lemma 4.25. *Let U be a nonempty finite set, $T \in \langle U \rangle_0$, $K \subseteq U$ and F be a fixed point of T . If $F \subseteq \vec{K}$ then $F \subseteq [K]$.*

Proof. It follows from Lemma 4.16 and Lemma 4.18. \square

Lemma 4.26. *Let U be a nonempty finite set, $T \in \langle U \rangle$ and F be a minimal fixed point of T . If K is a nonempty subset of F , then $[K] = \vec{K} = F$.*

Proof. It follows from Lemma 4.16 and Theorem 4.13. \square

Corollary 4.27. *Let U be a nonempty finite set and $T \in \langle U \rangle$. If F is a minimal fixed point and $x \in F$, then $F = [x]$.*

Definition 4.28. Let $T \in \langle U \rangle_0$ and F be a fixed point. F is called a *principal fixed point* of T if and only if there exists $x \in U$ such that $F = [x]$.

By Corollary 4.27, any minimal fixed point is a principal fixed point. But it is not necessary that a principal fixed point is a minimal fixed point. For example:

Example 4.29. Let $U = \{x, y, z, t, v, w\}$ and

$$T(u) = \begin{cases} \{x, y\} & u = x, \\ \{y, z\} & u = y, \\ \{z\} & u = z, \\ \{x, z\} & u = t, \\ \{z\} & u = v, \\ \{x, y\} & u = w. \end{cases}$$

We have $[x] = \{x, y, z\}$, $[y] = \{y, z\}$ and $[z] = \{z\}$. Therefore

$$\emptyset \subset [z] \subset [y] \subset [x] = [U].$$

Lemma 4.30. *Let U be a nonempty finite set and $T \in \langle U \rangle_0$. Then \emptyset and $[U]$ are minimum and maximum fixed point, respectively.*

Proof. It is obvious. \square

Lemma 4.31. *Let U be a nonempty finite set and $T \in \langle U \rangle_0$. If F is a fixed point, then there exist $x_1, x_2, \dots, x_n \in [U]$ such that $F = [x_1] \cup [x_2] \cup \dots \cup [x_n]$.*

Proof. There exist $x_1, x_2, \dots, x_n \in [U]$, such that $F = \{x_1, x_2, \dots, x_n\}$ by Lemma 4.16 and Lemma 4.17. Hence $F = \bigcup_{x \in F} \{x\}$. Therefore $F = [F] = [\bigcup_{x \in F} \{x\}] = \bigcup_{x \in F} [x]$ by Lemma 4.17 and Lemma 4.23. \square

Theorem 4.32 (Main Theorem 2). *Let U be a nonempty finite set and $T \in \langle U \rangle_0$. Set $PF(T) = \{[x] \mid x \in [U]\}$, then $Fix(T) = \{\bigcup_{[x] \in \Sigma} [x] \mid \Sigma \subseteq PF(T)\}$.*

Proof. Let $\Omega = \{\bigcup_{[x] \in \Sigma} [x] \mid \Sigma \subseteq PF(T)\}$. Then $\Omega \subseteq Fix(T)$ by Theorem 4.13 and Lemma 3.2. And we have $Fix(T) \subseteq \Omega$ by lemma 4.31. Therefore $\Omega = Fix(T)$. \square

Assume that $[U] = \{x, x', x'', y, y', z, u\}$, $[x] = [x'] = [x'']$ and $[y] = [y']$. Then $PF(T) = \{[x], [y], [z], [u]\} = \{[x'], [y], [z], [u]\} = \{[x'], [y'], [z], [u]\} = \dots$

Proposition 4.33. *Let U be a nonempty finite set and $T \in \langle U \rangle_0$. Then $|Fix(T)| \leq 2^{|PF(T)|}$.*

Proof. It is obvious. \square

Example 4.34. Suppose that $U = \{x, y, z, u, v, w, r, t\}$ and

$$T : \begin{cases} x \rightarrow \{y\}, \\ y \rightarrow \{z\}, \\ z \rightarrow \{u\}, \\ u \rightarrow \{z\}, \\ v \rightarrow \{v, u\}, \\ w \rightarrow \{u\}, \\ r \rightarrow \{r, t\}, \\ t \rightarrow \{r\}. \end{cases}$$

Then $[U] = \{z, u, v, r, t\}$, $[z] = [u] = \{u, z\}$, $[v] = \{u, v, z\}$ and $[r] = [t] = \{r, t\}$. Hence $PF(T) = \{[z], [v], [r]\}$. And $Fix(T) = \{\emptyset, [z], [v], [r], [z] \cup [r], [v] \cup [r], [z] \cup [v] \cup [r]\}$ by Theorem 4.32. Therefore

$$Fix(T) = \{\emptyset, \{u, z\}, \{r, t\}, \{u, v, z\}, \{u, z, r, t\}, \{u, v, z, r, t\}\}$$

and $|Fix(T)| = 6$. Also we have

$$(1) \quad x \notin [x].$$

- (2) $[v]$ is not a minimal fixed point.
- (3) $u \in [u]$.
- (4) $[x] \subset [v]$.
- (5) $[z] \cup [v] = [v]$.

Example 4.35. Assume that $U = \mathbb{Z}_{24}$, where $(\mathbb{Z}_{24}, +)$ is the group of integers modulo 24. Let $T : \mathbb{Z}_{24} \rightarrow \mathcal{P}^*(\mathbb{Z}_{24})$ be a power mapping, such that $T(\bar{n}) = 2 \cdot \bar{n} + \bar{1}$. Since \mathbb{Z}_{24} is a big subset, hence $[\mathbb{Z}_{24}] = T'^{\infty}(\mathbb{Z}_{24}) = \{7, 15, 23\}$. We have $[7] = [15] = \{7, 15\}$ and $[23] = \{23\}$. By Theorem 4.32, we have $PF(T) = \{\{7\}, [23]\}$ and therefore $Fix(T) = \{\emptyset, [7], [23], [7] \cup [23]\} = \{\emptyset, \{7, 15\}, \{23\}, \{7, 15, 23\}\}$.

Example 4.36. Let (H, \star) be an algebraic hyperstructure, where $H = \{a, b, c, d\}$ and:

$$\begin{cases} a \star a = \{b\}, & a \star b = \{a, b\}, & a \star c = \{b, c\}, \\ b \star a = \{a\}, & b \star b = \{b\}, & b \star c = \{a\}, \\ c \star a = \{a, b, c\}, & c \star b = \{b\}, & c \star c = \{c\}, \\ x \star y = \{a\} & \text{if } x = d \text{ or } y = d. \end{cases}$$

For any $h \in H$ define $T_h(x) = h \star x$. Since H is a nonempty finite set and $T_h \in \langle H \rangle$, therefore there exists a nonempty subset F_h of H , such that $T'_h(F_h) = F_h$, by Fixed Point Theorem. For example:

$$T_a = \begin{cases} a \rightarrow a \star a = \{b\}, \\ b \rightarrow a \star b = \{a, b\}, \\ c \rightarrow a \star c = \{b, c\} \\ d \rightarrow a \star d = \{a\}. \end{cases}$$

Use Theorem 4.32. We have $[H] = \{a, b, c\}$. $[a] = [b] = \{a, b\}$ and $[c] = \{a, b, c\}$. Hence $PF(T_a) = \{\{a, b\}, \{a, b, c\}\}$. Therefore $Fix(T_a) = \{\emptyset, \{a, b\}, \{a, b, c\}\}$. The set $\{a, b\}$ is a minimal fixed point.

Theorem 4.37 (Main Theorem 3). *Let U be a nonempty finite set and $T : U \rightarrow \mathcal{P}(U)$ be a power mapping. Then T has a nonempty fixed point, if and only if $[U] \neq \emptyset$.*

Proof. (\Leftarrow) It follows from Theorem 4.13.

(\Rightarrow) Assume that F is a nonempty fixed point. Hence $\emptyset \neq F = [F] \subseteq [U]$ by Lemma 4.16 and Lemma 4.17. \square

Example 4.38. Suppose that $U = \{x, y, z, t\}$ and $T : U \rightarrow \mathcal{P}(U)$ is a power mapping, where

$$T(u) = \begin{cases} \{y, z\} & u = x, \\ \{z\} & u = y, \\ \{t\} & u = z, \\ \emptyset & u = t. \end{cases}$$

Since $[U] = \emptyset$, therefore T does not have a nonempty fixed point by Theorem 4.37.

Example 4.39. In Example 4.36, define $T(x) = c \star x - x \star c$. Hence:

$$T = \begin{cases} a \longrightarrow \{a\}, \\ b \longrightarrow \{b\}, \\ c \longrightarrow \emptyset, \\ d \longrightarrow \emptyset. \end{cases}$$

Since $[H] = \{a, b\} \neq \emptyset$, hence T has a nonempty fixed point, by Theorem 4.37.

By Theorem 4.32, we have $PF(T) = \{[a], [b]\}$, where $[a] = \{a\}$ and $[b] = \{b\}$. Therefore $Fix(T) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.

Assume that Σ is a nonempty set, $|\Sigma| = n$, $A = \{f : \Sigma \rightarrow \Sigma \mid f \text{ is a mapping}\}$, $B = \{f \in A \mid \exists x \in \Sigma : f(x) = x\}$ and $C = A - B$. Then $|A| = n^n$, $|B| = n^n - (n-1)^n$ and $|C| = (n-1)^n$. Therefore $\lim_{|\Sigma| \rightarrow \infty} \left(\frac{A}{C}\right) = e$, $\lim_{|\Sigma| \rightarrow \infty} \frac{A}{B} = \frac{e}{e-1}$ and $\lim_{|\Sigma| \rightarrow \infty} \frac{B}{C} = e-1$, where e is the Napier's number.

5 Conclusion

In this paper, we introduced a new concept called of a power mapping. We showed that if U is a nonempty finite set and $T : U \rightarrow \mathcal{P}^*(U)$ is a power mapping, then there exists a nonempty subset F of U such that $T'(F) = F$ and F is called a fixed point. We proved if U is a nonempty finite set, then the power mapping $T : U \rightarrow \mathcal{P}(U)$ has a nonempty fixed point, if and only if $[U] \neq \emptyset$. We showed that if $PF(T) = \{[x] \mid x \in [U]\}$ then $Fix(T) = \{\bigcup_{[x] \in \Sigma} [x] \mid \Sigma \subseteq PF(T)\}$.

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