Journal of Mathematical Extension
Vol. 18, No. 5, (2024) (6)1-24
URL: https://doi.org/10.30495/JME.2024.3107
ISSN: 1735-8299
Original Research Paper

Twisted (α, β)-Contractive Mapping with Preserving Orthogonality and HU Stability

Z. Heydarpour*
Payame Noor University

J. Izadi
Payame Noor University

Kh. Ghafarnezhad
Azarbaijan Shahid Madani University

In this article, we define the concepts of α-ψ-contractive mappings, twisted (α, β)-admissible mappings, and twisted (α, β)-contractive mappings within orthogonally metric spaces (O-metric spaces). Following this, we explore the fixed point theorem for α-ψ-contractive mappings that preserve orthogonality. For example, we demonstrate the existence of an orthogonally fixed point (O-fixed point) for orthogonally α-ψ-contractive mappings (O-α-ψ-contractive mappings); however, under these conditions, such a point does not exist in the metric space. Next, we establish a common fixed point that preserves orthogonality for the new concepts defined under different conditions in an O-metric space. Furthermore, we illustrate through an example that a fixed point with orthogonality preservation exists for the newly defined concepts varying conditions. Lastly, we show that the concepts we have defined can exhibit stability.

AMS Subject Classification: 47H10, 54H25, 46J10.
Keywords and Phrases: O-fixed point Theorem, O-set, O-twisted (α, β)-admissible mapping.

Received: February 2024; Accepted: August 2024
*Corresponding Author

1 Introduction

The principle of Banach's contraction is a fundamental concept in fixed point theory, with a wide range of practical and theoretical applications across various disciplines. It is vital for understanding the stability and convergence of iterative processes and has proven to be an invaluable tool in mathematical analysis, optimization, and engineering. Relevant fields include linear and variational inequalities, nonlinear analysis, approximation theory, differential and integral equations, fractal mathematics, dynamical systems theory, equilibrium problems, game theory, and mathematical modeling. For example, fixed point theory is crucial for examining the fixed points of specific important single-valued mappings. These mappings are frequently used in practical scenarios across diverse areas such as computer science, engineering, economics, and physics. This analysis enhances our understanding of the stability and behavior of iterative processes, providing valuable insights into the complex relationships and patterns that emerge within these disciplines. For more information see [3, 4, 23, 28, 28].

In 2012, Samet and his research team introduced the concept of α-ψ-contractive mappings within fully developed metric spaces, thereby laying the groundwork for a new dimension of analysis and exploration in the field of mathematical structures. They also presented several fixed point theorems related to these mappings [30]. In the same year, Hasanzadeh and his colleagues made significant contributions by introducing α-ψ-contractive multivalued operators, demonstrating how to derive fixed point results for this new class of operators, as detailed in [5]. Subsequently, Salimi and his team, referenced in [29], expanded upon the foundational work of Samet and others by introducing (α, β)-ψ-contractive mappings. This enhancement deepened the analysis and understanding of spatial structures and the properties of mappings within this context. Their extensive work in 2013 further investigated fixed point theorems associated with these mappings. In 2015, Afshari and Sajjadmanesh [2] built upon the findings of Samet et al. by establishing a common fixed point for two α-ψ-contractive maps, showcasing a practical application of fixed point theory. More recently, research has shifted toward a new operator type known as the α-ψ-contractive operator, with existing literature addressing conditions related to w-distance. Additionally, a study by Guran and Bota [25] in 2015 explored the uniqueness, existence, and HU stability aspects for such mappings, further contributing to the rich tapestry of research surrounding α-ψ-contractive mappings. Several authors have conducted research on various functional equations ($\mathbb{FE}s$) in the field of fixed point theory (see [6, 12, 15, 17, 26]). The question of stability in $\mathbb{FE}$, first raised by Ulam in 1940 [35], pertains to the stability exhibited by group homomorphisms. In 1941, Hyers [14] provided a partial solution to Ulam's inquiry concerning Banach spaces. This notion of stability has since become known as HU stability. Over the years, this concept has been further explored and expanded by numerous researchers in the fields of fixed point problems, $\mathbb{FE}s$, and differential equations ($\mathbb{DE}s$), ultimately leading to the development of what is now referred to as HU stability. For further insights and examples, please see [13, 16, 18, 22, 24].

In 2017, Eshaghi and collaborators [11] first introduced the notion of an orthogonal set, a pivotal concept that has played a crucial role in realms such as fixed point theory, fractional calculus, and $\mathbb{DE}s$ within the broader field of mathematics. This concept has been noted by various authors, as exemplified in [1, 19, 20, 21, 27].

Following this, we will proceed to the specific definition of an orthogonal set, elucidating its properties in detail, and examining various examples that illustrate this foundational concept.

Definition 1.1. [11] Let $\mathcal{Y}$ be a nonempty set, and let u_0 and v be elements in $\mathcal{Y}$. As a result, $\perp \subseteq \mathcal{Y} \times \mathcal{Y}$ is said to have property O-set if there exists $u_0 \in \mathcal{Y}$ such that $[(\forall\ v \in \mathcal{Y})\colon\ v \perp u_0 \quad \text{or} \quad (\forall v \in \mathcal{Y}; u_0 \perp v)]$.

We denote this orthogonally structured set as $(\mathcal{Y}, \perp)$. For each such structure $(\mathcal{Y}, \perp)$, a mapping h from $\mathcal{Y}$ to $\mathcal{Y}$ with a Lipschitz constant $0 < L < 1$ exhibits certain properties. We say that h possesses the properties of an orthogonally generalized metric space (O-generalized metric space), orthogonally sequence (O-sequence), $\perp$-continuous, orthogonally complete (O-complete), and orthogonally preserving (O-preserving) if the following conditions are met:

(1) $(\mathcal{Y}, d)$ be a generalized metric space, then $(\mathcal{Y}, \perp, d)$.

(2) $\{u_n\}_{n\in\mathbb{N}}$ be a sequence, then for all $n; u_n \perp u_{n+1}$ or for all $n; u_{n+1} \perp u_n$.

(3) For each O-sequence $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{Y}$ with $u_n \to u$.

(4) Every Cauchy O-sequence is convergent to a point in $\mathcal{Y}$.
(5) For all $u \perp v$, then $h(u) \perp h(v)$.
(6) A mapping h is $\perp-$contraction mapping if $d(u(u), h(v)) \leq L\, d(u, v)$.

Example 1.2. Considering the set $\mathcal{Y} = [0, +\infty)$ under the condition $u \perp v$ where $u < v$, if the inequality $uv \leq \min\{u, v\}$ holds, then substituting u with either 0 or 1 allows us to classify $\mathcal{Y}$ as an O-set.

Example 1.3. Consider the set $\mathcal{Y} = [0, 1)$ equipped with the standard metric d. Now, consider the mapping $f(u) = \sqrt{u}$ and define $u \perp v$ as $uv \leq u$. It is apparent that $\mathcal{Y}$ exhibits $\perp-$contraction properties.

To gain deeper insights and illustrative examples regarding the concept of orthogonal sets, please refer to sources such as [8, 10, 31, 33, 34]. Now, consider the set Ψ, which consists of all non-decreasing functions ψ that map $[0, +\infty)$ to $[0, +\infty)$. This set meets the requirement that the series $\sum_{n=1}^{+\infty} \psi^n(t)$ converges for every $t > 0$, where ψ^n denotes the n-th iteration of the function ψ.

Next, based on the concepts we've discussed, we will explore the definition and provide some examples of $\perp$-α-admissible mappings. We will demonstrate that these mappings are not α-admissible.

Definition 1.4. [32] In the context of an O-complete metric space $(\mathcal{Y}, \perp, d)$ and a function $\alpha : \mathcal{Y} \times \mathcal{Y} \to [0, +\infty)$, a mapping $H : \mathcal{Y} \to \mathcal{Y}$ is referred to as a preserver of O-α-admissible mappings (also known as $\perp$-α-admissible mappings) if, for any $u, v \in \mathcal{Y}$ such that $u \perp v$ and $\alpha(u, v) \geq 1$, the condition $\alpha(Hu, Hv) \geq 1$ holds true.

In the following, we investigate H is a $\perp$-α-admissible.

Example 1.5. Consider the set $\mathcal{Y} = [0, +\infty)$, where the mapping $H : \mathcal{Y} \to \mathcal{Y}$ is defined by $Hu = u^2$. Let the function $\alpha : \mathcal{Y} \times \mathcal{Y} \to [0, +\infty)$ be defined as follows:

$$\alpha(u, v) = \begin{cases} 1, & if \quad u \leq v \\ 0, & if \quad u > v, \end{cases} \tag{1}$$

for all $u, v \in \mathcal{Y}$, with $u \perp v$.
By Definition 1.1, putting $u = 0$ satisfies the condition in $\alpha(u, v)$. Therefore, H is $\perp$-α-admissible.

In the following example, we will illustrate that while H qualifies as a $\perp$-α-admissible mapping, it fails to meet the criteria required to be considered α-admissible.

Example 1.6. Consider $\mathcal{Y} = \mathbb{R}$ with d representing a standard metric on $\mathcal{Y}$. The orthogonality relation $u \perp v$ is characterized by $uv \leq \{u, v\}$. Let $H : \mathcal{Y} \to \mathcal{Y}$ be a function defined as:

$$Hu = \begin{cases} 2u, & if \quad u > 1 \\ \frac{u}{2}, & if \quad u \leq 1. \end{cases} \tag{2}$$

Moreover, let us introduce the function $\alpha : \mathcal{Y} \times \mathcal{Y} \to [0, +\infty)$ in the following manner:

$$\alpha(u, v) = \begin{cases} 1, & if \quad uv \leq min\{u, v\} \\ 0, & if \quad uv > min\{u, v\}. \end{cases} \tag{3}$$

If $\alpha(u, v) \geq 1$, then $uv \leq u$ and $uv \leq v$. We putting $u = v = 1$, we have

$$\alpha(1, 1) = 1 \ implies \ \alpha(T1, T1) = \alpha(\frac{1}{2}, \frac{1}{2}) = 1, \tag{4}$$

thus, H satisfies the conditions of being a $\perp$-α-admissible mapping. But isn't α-admissible because, we putting $u = 2$ and $v = \frac{1}{4}$, we have

$$\alpha(2, \frac{1}{4}) = 1 \ implies \ \alpha(T2, T\frac{1}{4}) = 0. \tag{5}$$

In the upcoming section, our first task is to define the concepts of O-α-ψ-contractive type mappings, O-twisted (α, β)-admissible mappings, and O-twisted (α, β)-contractive mappings within the context of an O-complete metric space. These definitions will provide a foundation for our subsequent exploration and analysis in the following two sections.

2 Preliminaries

In this section, we will begin by introducing a new definition and providing an illustrative example that highlights the concepts of $\perp$-α-ψ-contractive mappings, twisted $\perp$-(α, β)-admissible mappings, and $\perp$-(α, β)-ψ-contractive mappings within the context of orthogonally structured sets. This will lay the groundwork for a thorough understanding and exploration of these important concepts.

Definition 2.1. Consider an O-complete metric space denoted as $(\mathcal{Y}, \perp, d)$, alongside a mapping H operating from $\mathcal{Y}$ to $\mathcal{Y}$. We define H as an O-α-ψ-contractive mapping (abbreviated as $\perp$-α-ψ-contractive mapping) under the condition that there exist functions $\alpha : \mathcal{Y} \times \mathcal{Y} \to [0, +\infty)$, ψ, and the maintenance of orthogonality. This definition holds for all $u, v \in \mathcal{Y}$, with the additional constraint of $u \perp v$.

$$\alpha(u, v)d(Hu, Hv) \leq \psi(d(u, v)). \tag{6}$$

In the following discussion, we will demonstrate that H is indeed a $\perp$-α-ψ-contractive mapping. However, it is important to emphasize that H does not meet the criteria necessary to be classified as an α-ψ-contraction mapping within the scope of our study.

Example 2.2. Consider the set $\mathcal{Y}$ to be equivalent to the real numbers denoted as $\mathbb{R}$, and let d represent the standard metric defined on $\mathcal{Y}$. We define $u \perp v$ if $uv \leq u^2 \vee v^2$. Let H be the mapping from $\mathcal{Y}$ to $\mathcal{Y}$ defined as:

$$Hu = \begin{cases} \frac{u}{2} & if \quad u \leq 2 \\ \\ 0 & if \quad u > 2. \end{cases} \tag{7}$$

Moreover, let us establish the function $\alpha : \mathcal{Y} \times \mathcal{Y} \to [0, +\infty)$ in the following manner:

$$\alpha(u, v) = \begin{cases} 1 & if \quad uv \leq u^2 \\ \\ 0 & otherwise. \end{cases} \tag{8}$$

In fact, we have 4 cases

- In the case where $u = 0$ and $v \leq 2$, the mapping results show $H(u) = 0$ and $H(v) = \frac{v}{2}$.
- If $u = 0$ and $v > 2$, the function yields $H(u) = H(v) = 0$.
- For $v \leq 1$ and $u \leq 2$, we have $H(v) = \frac{v}{2}$ and $H(u) = \frac{u}{2}$.
- When $v \leq 1$ and $u > 2$, the conditions lead to $u - v > v, H(v) = \frac{v}{2}$, and $H(u) = 0$.

Clearly, H is a $\perp - \alpha - \psi-$contractive mapping with $\psi(t) = \frac{1}{2}$ for all $t \geq 0$, we get

$$\alpha(u, v)d(u, v) \leq \frac{1}{2}d(u, v). \tag{9}$$

Consequently, it follows that H represents a mapping that is $\perp - \alpha - \psi-$contractive in nature. But H isn't a $\alpha - \psi-$contraction mapping, because we putting $u = 3,\ v = 2$ and $\psi(t) < 1$, we have

$$\alpha(3, 2)d(H3, H2) = 1 > \psi(t)d(3, 2). \tag{10}$$

Definition 2.3. Consider $(\mathcal{Y}, \perp, d)$ as an O-complete metric space, where $H : \mathcal{Y} \to \mathcal{Y}$ is identified as an O-twisted (α, β)-admissible mapping (briefly; twisted $\perp$-(α, β)-admissible mapping) if for each $u, v \in \mathcal{Y}$, with $u \perp v$ such that

$$\begin{cases} \alpha(u, v) \geq 1, \\ \beta(u, v) \geq 1, \end{cases} \textit{implies} \begin{cases} \alpha(Hu, Hv) \geq 1, \\ \beta(Hu, Hv) \geq 1. \end{cases} \tag{11}$$

In the following example, we will utilize the definition provided above to explore the physical system that relates force and displacement through the equation $F = k \cdot d^2$. We will incorporate the functions $\alpha(F, d)$ and $\beta(F, d)$ based on this relationship.

Example 2.4. Consider a physical system where the force (F) is related to displacement (d) by the equation $F = k \cdot d^2$, where k is a constant. We define the functions $\alpha(F, d)$ and $\beta(F, d)$ based on the properties of this relationship:

$$\alpha(F, d) = \begin{cases} 1 & \text{if } F \leq k \cdot d^2 \\ 0 & \text{if } F > k \cdot d^2, \end{cases} \tag{12}$$

$$\beta(F, d) = \begin{cases} 1 & \text{if } F \leq k \cdot d^2 \\ 0 & \text{if } F > k \cdot d^2. \end{cases} \tag{13}$$

Now, we show that this system satisfies the conditions for an O-twisted (α, β)-admissible mapping. For every F, d with $F \perp d$ and $\alpha(F, d) \geq 1$, $\beta(F, d) \geq 1$, the conditions $\alpha(k \cdot d^2, k \cdot d^2) \geq 1$ and $\beta(k \cdot d^2, k \cdot d^2) \geq 1$ hold.

For example, we putting $F = k$ and $d = 1$:

- $\alpha(k, 1) = 1$ and $\beta(k, 1) = 1$.
- Using the mapping $F = k \cdot d^2$, we have $F = k \cdot 1^2 = k$.

Now, let's check the conditions:

- $\alpha(k, k) = \alpha(k, k) = 1 \geq 1$.
- $\beta(k, k) = \beta(k, k) = 1 \geq 1$.

Consequently, we have demonstrated that the system where force is related to displacement by the equation $F = k \cdot d^2$ meets the criteria for an O-twisted (α, β)-admissible mapping.

Definition 2.5. Given an O-complete metric space $(\mathcal{Y}, \perp, d)$, let $H : \mathcal{Y} \to \mathcal{Y}$ be an O-twisted (α, β)-contractive mapping, commonly known as $\perp$-(α, β)-ψ-contractive mapping. This mapping is characterized by the existence of functions $\alpha, \beta : \mathcal{Y} \times \mathcal{Y} \to [0, +\infty)$, maintaining orthogonality, such that for all $u, v \in \mathcal{Y}$, where $u \perp v$ or $v \perp u$ is satisfied, the following inequality holds:

$$\alpha(u, v)\beta(u, v)d(Hu, Hv) \leq \psi(d(u, v)). \tag{14}$$

In the following example, we will use the definition provided earlier to examine the mapping $H : \mathcal{Y} \to \mathcal{Y}$ for O-twisted (α, β)-admissibility. Here, $(\mathcal{Y}, \perp, d)$ represents an O-complete metric space, and we apply the defined functions α and β. Our evaluation with specific values shows that H is O-twisted (α, β)-admissible, but it does not satisfy the criteria to be classified as α-admissible.

Example 2.6. Consider an O-complete metric space denoted as $(Y, \perp, d)$. We introduce a specific mapping given by $H : Y \to Y$ defined as follows:

Consider the setting where Y is set as $\mathbb{R}$ and d signifies the standard metric applied to Y. We define the concept where $u \perp v$ holds valid if the condition $uv \leq \min\{u, v\}$ is satisfied. The mapping $H : Y \to Y$ is defined as:

$$H(u) = \begin{cases} 2u & \text{if } u > 1 \\ u^2 & \text{if } u \leq 1. \end{cases} \tag{15}$$

Furthermore, we consider the mapping denoted by $\alpha : Y \times Y \to [0, +\infty)$, which is articulated as:

$$\alpha(u, v) = \begin{cases} 1 & \text{if } uv \leq \min\{u, v\} \\ 0 & \text{if } uv > \min\{u, v\}. \end{cases} \tag{16}$$

Now, we will check if H is O-twisted (α, β)-admissible. By setting $u = 2$ and $v = \frac{1}{4}$, we have:

$$\alpha(2, \frac{1}{4}) = 1, \tag{17}$$

$$\alpha(H(2), H(\frac{1}{4})) = \alpha(4, \frac{1}{16}) = 0. \tag{18}$$

Since $\alpha(2, \frac{1}{4}) \geq 1$ but $\alpha(H(2), H(\frac{1}{4})) < 1$, H is O-twisted (α, β)-admissible but not α-admissible.

In the main results section, we will focus on exploring the properties of O-α-ψ-contractive and O-(α, β)-ψ-contractive mappings within O-complete metric spaces. This investigation builds on the research contributions of Samet et al. [30], Salimi et al. [29], and Afshari et al. [2]. Throughout our study, we will examine fixed point theorems while maintaining a strong emphasis on the fundamental concept of orthogonality, combined with the intriguing ideas of α-ψ-contractive and (α, β)-ψ-contractive mappings.
In the final section, we will conduct a thorough analysis of the HU stability related to the fixed point problem associated with α-ψ-contractive mappings, as well as O-twisted (α, β)-contractive mappings.

3 Main Results

In this section, we will build upon the concepts of $\perp$-α-ψ-contractive mappings and $\perp$-(α, β)-admissible mappings, presenting a series of theorems that reveal the intricate relationships among these ideas within the context of orthogonal sets.

Theorem 3.1. *Assume the presence of an O-complete metric space given by $(\mathcal{Y}, \perp, d)$ alongside a mapping $H : \mathcal{Y} \to \mathcal{Y}$ that exhibits $\perp-\alpha-\psi$-contractivity and fulfills the ensuing requirements:*

(a) H is $\perp$-α-ψ-admissible;
(b) there exists $u_0 \in \mathcal{Y}$ such that $\alpha(u_0, hu_0) \geq 1$;
(c) H is $\perp$-continuous.
In such a case, it follows that H possesses a fixed point, denoted as $u^* \in \mathcal{Y}$, for which the equation $Hu^* = u^*$ holds.

Proof. Based on the orthogonality concept, there is the existence of an element $u_0 \in \mathcal{Y}$ such that

$$(\forall\ v \in \mathcal{Y}, u_0 \perp v) \quad or \quad (\forall\ v \in \mathcal{Y}, v \perp u_0), \tag{19}$$

and

$$\alpha(u_0, Hu_0) \geq 1. \tag{20}$$

Define

$$u_1 := Hu_0 \quad with \quad u_1 \perp Hu_0 \quad or \quad Hu_0 \perp u_1. \tag{21}$$

It follows that

$$u_2 = Hu_1, \cdots, u_{n+1} = Hu_n \quad for\ all\ n \in \mathbb{N}, \tag{22}$$

or

$$u_1 = Hu_0,\ u_2 = H^2 u_0, \cdots, u_{n+1} = H^n u_0 \quad for\ all\ n \in \mathbb{N}. \tag{23}$$

Given that H maintains $\perp$-preservation, the sequence $\{u_n\}_{n\in\mathbb{N}}$ can be identified as an O-sequence. Moreover, with the characteristic of H being $\perp$-α-admissible, it is stipulated that for any $u_0, u_1, u_2 \in \mathcal{Y}$ and $n \in \mathbb{N}$, under the conditions of $u_0 \perp u_1$, $u_0 \perp u_2$, and $u_1 \perp u_2$, the following emerges:

$$\alpha(u_0, u_1) = \alpha(u_0, Hu_0) \geq 1\ implies\ \alpha(Hu_0, Hu_1) = \alpha(u_1, u_2) \geq 1. \tag{24}$$

Applying the method of induction, we can establish that

$$\alpha(u_n, u_{n+1}) \geq 1, \qquad for\ all\ n \in\ \mathbb{N} \tag{25}$$

where $u_n, u_{n+1} \in \mathcal{Y}$ and with $u_n \perp u_{n+1}$.
By Definition 2.1, H is an $\perp$-α-ψ-contractive , then we get

$$\alpha(u_n, u_{n+1}) d(Hu_{n-1}, Hu_n) \leq \psi(d(u_n, u_{n+1})). \tag{26}$$

As a result $d(u_n, u_{n+1}) \leq \psi^n(d(u_0, u_1))$ for all $n \in \mathbb{N}$. Now choose an arbitrary $\varepsilon > 0$. Let $n(\varepsilon) \in \mathbb{N}$ be so that $\sum_{j=n(\varepsilon}^{+\infty} \psi^n(d(u_0, u_1)) < \varepsilon$. Subsequently, in the scenario where m and n belong to the set of natural numbers and m is greater than or equal to n, the result gathered would be

$$d(u_n, u_m) \leq \Big(d(u_n, u_{n+1}) + \cdots + d(u_{m-1}, u_m)\Big). \tag{27}$$

In particular, $d(u_n, u_m) \leq \varepsilon$
Thus, $d(u_n, u_m) \to 0$ as $m, n \to +\infty$. Hence, the sequence $\{u_n\}_{n\in\mathbb{N}}$ is identified as a Cauchy O-sequence, leading to the conclusion that $\mathcal{Y}$ is O-complete. Consequently, there is a presence of $u^* \in \mathcal{Y}$ such that u_n converges to u^*. Furthermore, it is known that H exhibits $\perp$-continuous, so

$$H(u_n) \to H(u^*), \tag{28}$$

and

$$H(u^*) = (\lim_{n\to+\infty} H(u_n)) = \lim_{n\to+\infty} u_{n+1} = u^*. \tag{29}$$

Consequently, the element u^* is identified as a fixed point of the operator H.
To verify the exclusivity aspect of the fixed point, let's take $v^* \in \mathcal{Y}$ to be another fixed point of H. This implies that $H^n(u^*) = u^*$ and $H^n(v^*) = v^*$. By the designation of u_0 at the outset of the demonstration, we derive that

$$\Big[u_0 \perp u^* \quad and \quad u_0 \perp v^*\Big] \quad or \quad \Big[u^* \perp u_0 \quad and \quad v^* \perp u_0\Big]. \tag{30}$$

As H preserves $\perp$, we get

$$\Big[H^n(u_0) \perp H^n(u^*) \quad and \quad H^n(u_0) \perp H^(v^*)\Big], \tag{31}$$

or

$$\Big[H^n(u^*) \perp H^n(u_0) \quad and \quad H^n(v^*) \perp H^n(u_0)\Big], \tag{32}$$

or

$$\Big[H^n(u^*) \perp H^n(u_0) \quad and \quad H^n(v^*) \perp H^n(u_0)\Big], \tag{33}$$

or

$$\Big[H^n(u_0) \perp H^n(u^*) \quad and \quad H^n(u_0) \perp H^n(v^*)\Big] \tag{34}$$

for all $n \in \mathbb{N}$. So, we have

$$\begin{aligned} d(u^*, v^*) &= d(H^n(u^*), H^n(v^*)) \\ &\leq d(H^n(u^*), H^n(u_0)) + d(H^n(u_0), H^n(v^*) \\ &\leq \psi^n(d(u^*, u_0) + \psi^n(d(u_0, v^*)) \\ &\to 0 \quad as \quad n \to +\infty. \end{aligned} \tag{35}$$

So, it follows that $u^* = v^*$. $\square$

Example 3.2. Consider the set $\mathcal{Y} = [0, +\infty)$ equipped with the standard metric denoted as $d(u, v) = |u - v|$. The operator $H : \mathcal{Y} \to \mathcal{Y}$ is defined as $Hu = \frac{u}{2}$ for all $u \in \mathcal{Y}$. Additionally, introduce $\alpha : \mathcal{Y} \times \mathcal{Y} \to [0, +\infty)$ where $\alpha(u, v) = 1$ for all $u, v \in Y$. To characterize the function ψ, let $\psi(t) = \frac{t}{2}$ for all $t \geq 0$. Now, we show H is $\perp -\alpha - \psi$-contractive, we have

$$\alpha(u, v)d(Hu, Hv) = d\left(\frac{u}{2}, \frac{v}{2}\right) = \frac{1}{2}|u - v| = \psi(|u - v|) = \psi(d(u, v)). \tag{36}$$

Thus, H is $\perp -\alpha - \psi$-contractive.
Since $\alpha(u, v) = 1$ for all u, v, and for any $u_0 \in \mathcal{Y}$, $\alpha(u_0, Hu_0) = 1 \geq 1$, H is $\perp -\alpha - \psi$-admissible. In the next, we investigate existence of u_0 such that $\alpha(u_0, Hu_0) \geq 1$:
Take $u_0 = 1$. Then, $\alpha(1, H1) = \alpha(1, \frac{1}{2}) = 1 \geq 1$.
Lastly, we demonstrate the existence of a fixed point for H:
Consider the sequence $\{u_n\}$ that is characterized as $u_n = H^n(u_0)$ for $n \geq 0$, where u_0 is selected as previously explained, and H^n represents the n-th iteration of H. It can be seen that $u_n = \frac{1}{2^n}$.
As $n \to \infty$, we have $u_n \to 0$. Since $\mathcal{Y}$ is complete, $0 \in \mathcal{Y}$, we find that $H0 = \frac{0}{2} = 0$. Therefore, 0 is a fixed point of H.

Theorem 3.3. *Let $(\mathcal{Y}, \perp, d)$ be an O-complete metric space and let $H : \mathcal{Y} \to \mathcal{Y}$ be an O-$(\alpha, \beta) - \psi$-contractive mapping. If they satisfy the following*
(a) H is O-(α, β)-ψ-admissible;
(b) there exists $u_0 \in \mathcal{Y}$ so that $\alpha(u_0, Hu_0) \geq 1$ and $\beta(u_0, Hu_0) \geq 1$;
(c) H is $\perp$-continuous.
In fact, H has a fixed point.

Proof. Let $u_0 \in \mathcal{Y}$ and according to Theorem 3.1, for $u_0 \perp Hu_0$, then $\alpha(u_0, Hu_0) \geq 1$, $\alpha(u_0, Hu_0) \geq 1$ and define an O-sequence $\{u_n\}$ by $u_n = Hu_{n-1} = H^n u_0$ for all $n \in \mathbb{N}$. Since H is an O-twisted (α, β)-contractive mapping, such that for $u_0, u_1, u_2 \in \mathcal{Y}$, with $u_0 \perp u_1$, $u_0 \perp u_2$ and $u_1 \perp u_2$, we get

$$\begin{aligned} &\alpha(u_0, u_1) = \alpha(Hu_0, Hu_1) \geq 1 \textit{ implies } \alpha(u_1, u_2) \geq 1 \\ &\textit{and} \\ &\beta(u_0, u_1) = \beta(u_0, Hu_0) \geq 1 \textit{ implies } \beta(u_1, u_2) \geq 1. \end{aligned} \tag{37}$$

By continuing this process, we have

$$\begin{aligned} &\alpha(u_{2n}, u_{2n+1}) \quad \alpha(u_{2n}, u_{2n-1}) \geq 1 \\ &\textit{and} \\ &\beta(u_{2n}, u_{2n+1}) \quad \beta(u_{2n}, u_{2n-1}) \geq 1. \end{aligned} \tag{38}$$

For all $n \in \mathbb{N}$, we can observe that $u_{2n-1}, u_{2n}, u_{2n+1} \in \mathcal{Y}$, satisfying $u_{2n} \perp u_{2n-1}$ as well as $u_{2n} \perp u_{2n+1}$, where the orthogonality condition is preserved. Let H be an O-twisted (α, β)-contractive mapping. With regard to Definition 2.5, putting $u_0 = u_{2n}$ and $v = u_{2n+1}$, we get

$$\alpha(u_{2n}, u_{2n+1})\beta(u_{2n}, u_{2n+1})d(Hu_{2n}, u_{2n+1}) \leq \psi(d(u_{2n}, u_{2n+1}). \tag{39}$$

As a result, $d(u_{2n}, u_{2n+1}) \leq \psi(d(u_{2n-1}, u_{2n})$.By continuing this process, for all $n \in \mathbb{N}$ and $u_n, u_{n+1} \in \mathcal{Y}$, with $u_n \perp u_{n+1}$, we have

$$d(u_n, u_{n+1}) \leq \psi^n(d(u_0, u_1)). \tag{40}$$

The subsequent segment of the demonstration adheres to a parallel format reminiscent of the proof of Theorem 3.1. □

Theorem 3.4. *Let's assume that $(\mathcal{Y}, \perp, d)$ is an O-complete metric space and that $H : \mathcal{Y} \to \mathcal{Y}$ is a mapping that is $\perp$-α-ψ-contractive. We will further consider that it satisfies conditions (a) and (b) outlined in Theorem 3.1, along with the following conditions:*
(i) if $\{u_n\}$ is an O-sequence in $\mathcal{Y}$ such that $\alpha(u_n, u_{n+1}) \geq 1$, with $u_n \perp u_{n+1}$ and $u_n \to u \in \mathcal{Y}$ as $n \to +\infty$, then $\alpha(u_n, u) \geq 1$ for all

$n \in \mathbb{N}$;
(ii) Given any $u, v \in \mathcal{Y}$ with the property $u \perp v$, we can find $w \in \mathcal{Y}$ such that $u \perp w$, $v \perp w$, $\alpha(u, v) \geq 1$, and $\alpha(v, w) \geq 1$, marking the conditions specified;
subsequently, H will possess a solitary fixed point.

Proof. From the proof of Theorem 3.1, we can conclude that the sequence $\{u_n\}$ forms a Cauchy O-sequence in the O-complete metric space $(\mathcal{Y}, \perp, d)$. Consequently, there exists an element $u^* \in \mathcal{Y}$ such that u_n converges to u^* as n approaches infinity. Additionally, leveraging Theorem 3.1 along with assumption (i), we can establish the following:

$$\alpha(u_n, u^*) \geq 1 \qquad for\ all\ n \in \mathbb{N}. \tag{41}$$

Now, by applying the triangle inequality (a) and (i), we obtain

$$\begin{aligned} d(Hu^*, u^*) &\leq d(Hu^*, Hu_n) + d(u_{n+1}, u^*) \\ &\leq \alpha(u_n, u^*)d(Hu_n, Hu^*) + d(u_{n+1}, u^*) \\ &\leq \psi(d(u_n, u^*)) + d(u_{n+1}, u^*) \end{aligned} \tag{42}$$

As n approaches positive infinity and considering the continuity of ψ, we derive that $d(Hu^*, u^*) = 0$, which implies $Hu^* = u^*$. We will then show that, if condition (ii) is satisfied, the mapping T will have a unique fixed point. Let us assume that u^* and v^* are both fixed points of H. Given condition (ii), we can determine the existence of $w \in \mathcal{Y}$ such that $u^* \perp w$ and $v^* \perp w$, where.

$$\alpha(u^*, w) \geq 1 \quad and \quad \alpha(v^*, w) \geq 1 \tag{43}$$

on the other hand, considering that H is deemed α-admissible, we can deduce from Equation (43) that

$$\alpha(u^*, H^n w) \geq 1 \quad and \quad \alpha(v^*, H^* nw) \geq 1 \qquad for\ all\ n \in \mathbb{N}, \tag{44}$$

Employing Equations (6) and (44), we arrive at the conclusion that

$$\begin{aligned} d(u^*, H^n w) &= d(Hu^*, H(H^{n-1}w)) \\ &\leq \alpha(u^*, H^{n-1}w)d(Hu^*, H(H^{n-1}w)) \\ &\leq \psi(d(u^*, H^{n-1}w)). \end{aligned} \tag{45}$$

This implies that

$$d(u^*, H^n w) \leq \psi^n(d(u^*, w)), \qquad for\ all\ n \in \mathbb{N}. \tag{46}$$

As a result, letting $n \to +\infty$, we get

$$H^n w \to u^*, \tag{47}$$

similarly,

$$H^n w \to v^* \quad as \quad n \to +\infty. \tag{48}$$

By utilizing Equations (47) and (48), the singular nature of the limit results in $u^* = v^*$, thereby concluding the proof. □

Theorem 3.5. *Let's assume that $(\mathcal{Y}, \perp, d)$ is an O-complete metric space and that S and H are two mappings from $\mathcal{Y}$ to itself, both of which are $\perp$-α-ψ-contractive and satisfy the following conditions:*
(a) for all $u, v \in \mathcal{Y}$ with $u \perp v$, $\alpha(u, v) \geq 1$, $\alpha(Su, Hv) \geq 1$ or $\alpha(Hu, Sv) \geq 1$;
(b) there exists $u_0 \in \mathcal{Y}$ such that $\alpha(u_0, Hu_0) \geq 1$;
(c) S and H are $\perp$-continuous;
(d) Given any $u, v \in \mathcal{Y}$ where $u \perp v$ holds true, if $\alpha(u, v)d(Su, Hv) \leq \psi(d(u, v))$ and $\alpha(v, u)d(Su, Hv) \leq \psi(d(u, v))$, then it can be concluded that S and H possess a commen fixed point.

Proof. Let Ω be the set of all mappings $S, H : \mathcal{Y} \to \mathcal{Y}$ such that $Hu \perp Su$ or $Su \perp Hu$. We define the O-sequences $\{u_{2n}\}$ and $\{u_{2n+1}\}$ in $\mathcal{Y}$ by

$$u_{2n} = Su_{2n-1} \quad , \quad u_{2n+1} = Hu_{u_{2n}} \qquad n = 1, 2, ... \tag{49}$$

By Theorem 3.1, we get

$$\begin{aligned} &\alpha(u_0, u_1) = \alpha(u_0, Su_0) \geq 1, \\ &\alpha(u_1, u_2) = \alpha(Su_0, HSu_0) \geq 1. \\ &\alpha(u_1, u_2) \geq 1\ implies\ \alpha(u_2, u_3) = \alpha(Su_1, Hu_2) \geq 1. \end{aligned} \tag{50}$$

Using induction, we can show

$$\alpha(u_n, u_{n+1}) \geq 1 \qquad for\ all\ n \in \mathbb{N}. \tag{51}$$

Clearly, Ω is $\perp$-preserving, for all $S, H \in \Omega$, with $Su \perp Hu$ for all $u \in \mathcal{Y}$, we get

$$\begin{aligned} d(u_{2n}, u_{2n+1}) &= d(Su_{2n-1}, Hu_{2n}) \\ &\leq \alpha(u_{2n-1}, u_{2n}) d(Su_{2n-1}, Hu_{2n}) \\ &\leq \psi d(Su_{2n-1}, Hu_{2n}). \end{aligned} \tag{52}$$

By continuing this process, we have

$$\begin{aligned} d(u_{2n-1}, u_{2n}) &\leq \psi^{2n} d(Su_0, Hu_1) \\ &\leq \psi^{2n} d(u_0, u_1), \qquad for\ all\ n \in \mathbb{N}. \end{aligned} \tag{53}$$

As a result, we get

$$d(u_n, u_{n+1}) \leq \psi^n d(u_0, u_1), \qquad for\ all\ n \in \mathbb{N}. \tag{54}$$

The proof for the Cauchy O-sequence follows a similar structure to that presented in Theorem 3.1. Now, we will show that u serves as the common fixed point needed for both mappings S and H. It suffices to consider that for all $n \in \mathbb{N}$, $u \in \mathcal{Y}$, and $Su \perp Hu$, then

$$\begin{aligned} d(Su, u_{2n+1}) &= d(Su, Hu_{2n}) \\ &\leq \alpha(u, u_{2n}) d(Su, Hu_{2n}) \\ &\leq \psi(d(u, u_{2n}), \end{aligned} \tag{55}$$

when $n \to +\infty$, we get $d(Su, u) \leq \psi(d(u, u)) = \psi(0) = 0$ as gives $Su = u$. Similarly, $Hu = u$. So u is the common fixed point of S and H. $\square$

Example 3.6. Consider the set $\mathcal{Y} = \mathbb{R}$ with the standard metric $d(u, v) = |u - v|$ for any $u, v \in \mathcal{Y}$, assuming $u \perp v$. Let S and $H : \mathcal{Y} \to \mathcal{Y}$ be defined by:

$$Su = \frac{u^2}{3} \quad and \quad Hu = \frac{u^2}{2}. \tag{56}$$

And we define the mapping $\alpha : \mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}$ by

$$\alpha(u, v) = \begin{cases} 1, & if \quad v \leq \frac{5u}{6}, \quad \forall\, u, v \in [-1, +1], \quad u \perp v \\ 0, & if \quad u < v, \quad \forall\, u, v \in [-1, +1], \quad u \perp v. \end{cases} \tag{57}$$

Then, S and H are $\perp$-α-admissible.

Remark 3.7. Suppose $(\mathcal{Y}, \perp, d)$ is an O-complete metric space and $S, H : \mathcal{Y} \to \mathcal{Y}$ are two mappings that are O-$(\alpha, \beta) - \psi$-contractive, satisfying the following
(a) for all $u, v \in \mathcal{Y}$ with $u \perp v$, $\alpha(u, v) \geq 1$ and $\beta(u, v) \geq 1$ then $\alpha(Su, Hv) \geq 1$, $\beta(Su, Hv) \geq 1$ or $\alpha(Hu, Sv) \geq 1$, $\beta(Hu, Sv) \geq 1$;
(b) there exists $u_0 \in \mathcal{Y}$ such that $\alpha(u_0, Hu_0) \geq 1$ and $\beta(u_0, Hu_0) \geq 1$;
(c) S and H is $\perp$-continuous;
(d) if we have u and v belonging to $\mathcal{Y}$ with u being orthogonal to v, and further, if

$$\alpha(u, v)\beta(u, v)d(Su, Hv) \leq \psi(d(u, v)), \tag{58}$$

as well as

$$\alpha(v, u)\beta(u, v)d(Su, Hv) \leq \psi(d(u, v)), \tag{59}$$

then it follows that S and H exhibit a common fixed point.

4 HU Stability

In this section, we establish the HU stability in relation to the conditions outlined in Theorem 3.1 and Theorem 3.3 within an O-metric space. To begin, we will explore the fundamental principles of HU stability. Consider an O-metric space denoted as $(\mathcal{Y}, \perp, d)$, along with the mapping $H : \mathcal{Y} \to \mathcal{Y}$ that we are examining. Our goal is to address the fixed point equation given by:

$$u = Hu, \tag{60}$$

and the inequality (for $\varepsilon > 0$)

$$d(Hv, v) < \varepsilon. \tag{61}$$

We define the fixed point problem (60) to be HU stable in the context of an O-metric space if there exists $k > 0$ such that for every $\varepsilon > 0$ and an ε-solution $v^* \in \mathcal{Y}$—meaning v^* satisfies (61), there exists a solution $u^* \in \mathcal{Y}$ of the fixed point equation (60) such that

$$d\left(u^*, v^*\right) < k\varepsilon. \tag{62}$$

In the upcoming theorem, we will embark on a thorough exploration of the principle of HU Stability as established in Theorem 3.1.

Theorem 4.1. *Assuming the fulfillment of all conditions stated in Theorem 3.1 and with $\alpha(u,v) \geq 1$, let's consider $v^* \in \mathcal{Y}$ as an ε-solution with $d(Hv^*, v^*) < \varepsilon$. Consequently, the fixed point equation (60) is established as HU stable.*

Proof. Given that H exhibits $\perp - \alpha - \psi$-contractive properties, we duly note that

$$d(Hu, Hv) \leq \alpha(u,v) + \psi(d(u,v)), \tag{63}$$

for all $u, v \in \mathcal{Y}$. Using the $\perp$-α-ψ-admissibility of H, we have

$$d(Hu, Hv) \leq \alpha(u,v) + \psi(d(u,v)) \leq \alpha(u,v) + \psi(\alpha(u,v)) \leq \alpha(u,v), \tag{64}$$

for all $u, v \in \mathcal{Y}$. Now, using the fact that H is $\perp$-continuous, we can find a $\delta > 0$ such that if $d(u,v) < \delta$, then $d(Hu, Hv) < \varepsilon$. Choose $\delta = \varepsilon$. Since v^* is an ε-solution, we have $d(Hv^*, v^*) < \varepsilon$. By setting $u^* = v^*$, we get $d(u^*, v^*) = 0 < k\varepsilon$ for any $k \geq 1$.
Therefore, the fixed point problem defined in Theorem 3.1 is HU stable in $O-$metric space. □

In the upcoming theorem, we delve into an in-depth analysis of the HU stability concept in relation to the conditions specified in Theorem 3.3.

Theorem 4.2. *Let's assume that all the conditions of Theorem 3.1 are satisfied, $\alpha(u,v) \geq 1$ and $\beta(u,v) \geq 1$. Suppose $v^* \in \mathcal{Y}$ is an ε-solution and $d(Hv^*, v^*) < \varepsilon$. As a result, the fixed point equation (60) is HU stable.*

Proof. We need to show that for any $\varepsilon > 0$ and an ε-solution $v^* \in \mathcal{Y}$ (i.e., $d(Hv^*, v^*) < \varepsilon$), there exists a solution $u^* \in \mathcal{Y}$ of the fixed point equation (60) such that

$$d(u^*, v^*) < k\varepsilon, \tag{65}$$

for some constant $k > 0$. Given $v^* \in \mathcal{Y}$ such that $d(Hv^*, v^*) < \varepsilon$, we can construct a sequence (v_n) in $\mathcal{Y}$ defined by

$$v_0 = v^*, \quad v_{n+1} = Hv_n, \tag{66}$$

for $n \geq 0$. By the property of HU stability, there exists a fixed point u^* of (60) such that

$$d(u^*, v^*) \leq \limsup_{n \to \infty} d(u^*, v_n). \tag{67}$$

Since H is O-$(\alpha, \beta) - \psi$-contractive, we have

$$d(Hv, v) \leq \psi(d(v, Hv)) \leq \alpha(d(v, Hv)) + \beta(d(v, Hv)), \tag{68}$$

for all v in $\mathcal{Y}$. Using this inequality, we can show that

$$d(u^*, v_n) \leq \alpha(d(u^*, v_{n-1})) + \beta(d(u^*, v_{n-1})), \tag{69}$$

for all $n \geq 1$. By induction, we can prove that

$$d(u^*, v_n) \leq (\alpha^n + \beta^n) d(u^*, v^*), \tag{70}$$

for all $n \geq 1$. Finally, we can choose $k = \max\{\alpha, \beta\}$ to satisfy (65). As a result, one can establish that the fixed point dilemma associated with H is deemed to be HU stable. □

5 Conclusion

Given the significance of O-sets, we defined O-α-ψ-contractive mappings, O-twisted (α, β)-admissible mappings, and O-twisted (α, β)-contractive mappings within the framework of O-metric spaces. We then proved the orthogonal fixed point theorem for these mappings in O-metric spaces. Following this, we illustrated that an orthogonal fixed point exists for O-α-ψ-contractive mappings through examples; however, in these conditions, such a fixed point does not exist in the metric space. Furthermore, while orthogonal fixed points do exist for α-ψ-contractive mappings under varying conditions, this is not the case for O-(α, β)-ψ-contractive mappings within the specified metric space. Finally, we established the HU stability of the fixed point problem concerning α-ψ-contractive mappings and O-twisted (α, β)-contractive mappings.

Acknowledgements
The authors express their gratitude to dear unknown referees for their helpful suggestions which improved the final version of this paper.

References

[1] A. A. N. Abdou, Solving a nonlinear fractional differential equation using fixed point results in orthogonal metric spaces, *Fractal and Fractional*, 7(11) (817) (2023).

[2] H. Afshari and M. Sajjadmanesh, Fixed point theorems for α-ψ-contractive type mappings, *Sahand Communications in Mathematical Analysis*, 2 (2015), 65–72.

[3] T.O. Alakoya, V.A. Uzor and O.T. Mewomo, A new projection and contraction method for solving split monotone variational inclusion, pseudomonotone variational inequality, and common fixed point problems, *Computational and Applied Mathematics*, 42(3) (2023).

[4] M. Alfuraidan, and Q. Ansari, Fixed point theory and graph theory: Foundations and integrative approaches, London, England: Academic Press-Elsevier (2016).

[5] J. Asl Hasanzade, S. Rezapour and N. Shahzad, On fixed points of $\alpha - \psi$-contractive multifunctions, *Fixed Point Theory and Applications*, 212 (2012).

[6] H. Baghani, A new contractive condition related to Rhoades's open question, *Indian Journal of Pure and Applied Mathematics*, 51 (2020), 565–578.

[7] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integrals, *Fundamenta Mathematicae*, 3 (1922), 133–181.

[8] S. Chandok and S. Radenoviĉ, Existence of solution for orthogonal $\mathcal{F}$-contraction mappings via Picard–Jungck sequences, *The Journal of Analysis*, 30 (2022), 677–690.

[9] S. Chandok, L. Paunoviĉ and S. Radenoviĉ, Incomplete Metric With The Application For Integral Equations of Volterra Type, *Mathematical Problems in Engineering*, Article ID 5959903 (2022).

[10] M. Eshaghi Gordji and H. Habibi, Fixed point theory in generalized orthogonal metric space, *Journal of Linear and Topological Algebra*, 6 (2017), 251–260.

[11] M. Eshaghi Gordji, M. Ramezani, M. De La Sen and Y.J. Cho, On orthogonal sets and Banach fixed point theorem, *Fixed Point Theory*, 18 (2017), 569–578.

[12] R.H. Haghi, S. Rezapour and N. Shahzad, On fixed points of quasi-contraction type multifunctions, *Applied Mathematics Letters*, 25 (2012), 843–846.

[13] M. Haji Shaabani and M. Farzi Haromi, Hyers-Ulam stability of some linear operators on a Hilbert space, *Journal of Mathematical Extension*, 16(11) (2022), 1–9.

[14] D.H. Hyers, On the stability of the linear functional equation, *Proceedings of the National Academy of Sciences of the United States of America*, 27 (1941), 222–224.

[15] S. Jafari and M. Shams, Fixed point theorems for ϕ-contraction mappings in probabilistic generalized Menger space, *Indian Journal of Pure and Applied Mathematics*, 51 (2020), 519–532.

[16] S. Jahedi and V. Keshavarz, Approximate generalized additive-quadratic functional equations on ternary Banach algebras, *Journal of Mathematical Extension*, 16 (2022), 1-11.

[17] E. Karapinar and B. Samet, Generalized α-ψ-contractive type mappings and related fixed point theorems with applications, *Abstract and Applied Analysis*, Article ID 793486 (2012).

[18] V. Keshavarz, M. T. Heydari and D. R. Anderson, Hyers-Ulam stabilities for mth differential operators on H^2_β, *Chaos, Solitons & Fractals*, 179 (2024).

[19] V. Keshavarz and S. Jahedi, Hyperstability of orthogonally 3-Lie homomorphism: an orthogonally fixed point approach, Approximation and Computation in Science and Engineering, In: Daras, N.J., Rassias, Th. M. (eds) Approximation and Computation in Science

and Engineering. Springer Optimization and Its Applications, 180, Springer, Cham.

[20] V. Keshavarz and S. Jahedi, Orthogonally C*-ternary Jordan homomorphisms and Jordan derivations: solution and stability, *Journal of Mathematics*, Article ID 3482254 (2022).

[21] V. Keshavarz, S. Jahedi, M. Eshaghi Gordji and S. Bazeghi, Hyperstability of orthogonally Pexider Lie functional equation: An orthogonally fixed point approach, *Thai Journal of Mathematics*, 20 (2022), 235-243.

[22] A. Kheiryan and S. Rezapour, A multi-singular fractional integro-differential equation and its Hyers-Ulam stability, *Journal of Mathematical Extension*, 16(11) (2022), 1-19.

[23] P. Krupa, O. Inverso, M. Tribastone and A. Bemporad, Certification of the proximal gradient method under fixed-point arithmetic for box-constrained QP problems, *Automatica*, 159 (2024).

[24] M. Kumar, P. Kumar, A. Mutlu, R. Ramaswamy, O.A.A. Abdelnaby and S.Radenović, Ulam-Hyers stability and well-posedness of fixed point problems in C*-algebra valued bipolar b-metric spaces, *Mathematics*, 11 (2023).

[25] A. Latif, L. Guran and M.-F. Bota, Ulam-Hyers stability problems and fixed points for contractive type operators on KST-spaces, *Linear and Nonlinear Analysis*, 5(3) (2019), 379-390.

[26] M. Nazam, M. Arshad, and M. Postolache, Coincidence and common fixed point theorems for four mappings satisfying $(\alpha s, F)$-contraction, *Nonlinear Analysis: Modelling and Control*, 23 (2018), 664–690.

[27] M. Nazam, H. Aydi and A. Hussain, Existence theorems for (Ψ, Φ)-orthogonal interpolative contractions and an application to fractional differential equations, *Optimization*, 72(7) (2023), 1899–1929.

[28] C. Nwaigwe and D. N.Benedict, Generalized Banach fixed-point theorem and numerical discretization for nonlinear Volterra-Fredholm equations, *Journal of Computational and Applied Mathematics*, 425 (2023).

[29] P. Salimi, C. Vetro and P. Vetro, Fixed point theorems for twisted $(\alpha, \beta) - \psi$-contractive type mappings and applications, *Filomat*, 27 (2013), 605–615.

[30] B. Samet, C. Vetro and P. Vetro, Fixed point theorems for $\alpha - \psi$-contractive type mappings, *Nonlinear Analysis*, 75 (2012), 2154–2165.

[31] T. Senapati, L. K. Dey, B. Damjanovic and A. Chanda, New fixed point results In orthogonal metric spaces with an application, *Kragujevac Journal of Mathematics*, 42 (2018), 505–516.

[32] M. Ramezani, Orthogonal metric space and convex contractions, *International Journal Of Nonlinear Analysis And Applications*, 6 (2) (2015), 127–132.

[33] M. Ramezani and H. Baghani, Contractive gauge functions in strongly orthogonal metric spaces, *International Journal Of Nonlinear Analysis And Applications*, 8 (2017) 23–28.

[34] Y. Touail and D. El Moutawakil, Fixed point theorems on orthogonal complete metric spaces with an application, *International Journal Of Nonlinear Analysis And Applications*, 12 (2021), 1801–1809.

[35] S. M. Ulam , Problems in modern mathematics, Chapter IV, Science edn. Wiley, New York, 1960.

Zohreh Heydarpour
Department of Mathematics
Assistant Professor of Mathematics
Department of Mathematics, Payame Noor University (PNU), P. O. Box 19395-4697.
Tehran, Iran.
E-mail: zohreh.heydarpour@pnu.ac.ir

Javad Izadi
Department of Mathematics
Assistant Professor of Mathematics
Department of Mathematics, Payame Noor University (PNU), P. O. Box 19395-4697.
Tehran, Iran.
E-mail: j.izadi@pnu.ac.ir

Khadijeh Ghafarnezhad
Department of Mathematics
Azarbaijan Shahid Madani University
Tabriz, Iran
E-mail: ghafarnezhad.kh@gmail.com