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On θ-centralizing θ-generalized Derivations on Convolution Algebras

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Abstract. Let θ be an isomorphism on $L_0^\infty(w)^*$. In this paper, we investigate θ-generalized derivations on $L_0^\infty(w)^*$. We show that every θ-centralizing θ-generalized derivation on $L_0^\infty(w)^*$ is a θ-right centralizer. We also prove that this result is true for θ-skew centralizing θ-generalized derivations.

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1 Introduction

Let $w : [0, \infty) \to [1, \infty)$ be a continuous function such that $w(0) = 1$ and for every $x, y \geq 0$

$$w(x + y) \leq w(x)w(y).$$

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Let $L^1(w)$ be the Banach algebra of all Lebesgue measurable functions on $[0,\infty)$. Let also $M(w)$ be the Banach algebra of all complex regular Borel measure on $[0,\infty)$; for study of these Banach algebras see [5, 13]. We denote by $L_0^\infty(w)$ the Banach space of all Lebesgue measurable functions f on $[0,\infty)$ such that

$$\text{ess sup}\{f(y)\chi_{(x,\infty)}(y)/w(y):\ y\geq 0\}\to 0$$

as $x\to+\infty$, where $\chi_{(x,\infty)}$ is the characteristic function on (x,∞). It is proved that the dual of $L_0^\infty(w)$, represented by $L_0^\infty(w)^*$, is a Banach algebra with the first Arens product defined by

$$mn(f)=m(nf),$$

where the functional nf is defined by $nf(\varphi)=n(f\varphi)$, in which

$$f\varphi(x)=\int_0^\infty f(x+y)\varphi(y)dy$$

for all $m,n\in L_0^\infty(w)^*$, $f\in L_0^\infty(w)$, $\varphi\in L^1(w)$ and $x\geq 0$; for more details see [8, 9]. By the usual way, $L^1(w)$ may be regarded as a subspace of $L_0^\infty(w)^*$. In this case, $L^1(w)$ is a closed ideal of $L_0^\infty(w)^*$. Note that the sequence

$$\{i\chi_{(0,1/i)}\}_{i\in\mathbb{N}}$$

is a bounded approximate identity for $L^1(w)$. The set of all weak*-cluster points of an approximate identity of $L^1(w)$ bounded by one is denoted by $\Lambda(L_0^\infty(w)^*)$. It is easy to see that $u\in\Lambda(L_0^\infty(w)^*)$ if and only if u is a right identity for $L_0^\infty(w)^*$.

Let θ be a homomorphism on $L_0^\infty(w)^*$ and T be a linear map on $L_0^\infty(w)^*$. Then T is called a *θ-right centralizer* if for every $m,n\in L_0^\infty(w)^*$

$$T(mn)=\theta(m)T(n).$$

A linear map d on $L_0^\infty(w)^*$ is called a *θ-derivation* if

$$d(mn)=d(m)\theta(n)+\theta(m)d(n)$$

for all $m, n \in L_0^\infty(w)^*$. Also, a linear map D on $L_0^\infty(w)^*$ is called a *θ-generalized derivation with associated to derivation* d, if

$$D(mn) = \theta(m)D(n) + d(m)\theta(n)$$

for all $m, n \in L_0^\infty(w)^*$. We denote this concept with (D, d). Note that if $D = d$, then D is a θ-derivation. In the case where $d = 0$, then for every $m, n \in L_0^\infty(w)^*$

$$D(mn) = \theta(m)D(n).$$

This type of θ-generalized derivation is called a *θ-right centralizer*. A linear map T on $L_0^\infty(w)^*$ is called *θ-commuting* if for every $m \in L_0^\infty(w)^*$

$$T(m)\theta(m) = \theta(m)T(m),$$

and T is called *θ-centralizing* if

$$[T(m), \theta(m)] := T(m)\theta(m) - \theta(m)T(m) \in Z(L_0^\infty(w)^*)$$

for all $m \in L_0^\infty(w)^*$, where $Z(L_0^\infty(w)^*)$ is the center of $L_0^\infty(w)^*$.

Some authors studied the Banach algebra $L_0^\infty(w)^*$ [1, 2, 10, 11, 12]. For example, Ahmadi Gandomani and the second author [1] studied generalized derivations on $L_0^\infty(w)^*$. They showed that every centralizing generalized derivation on $L_0^\infty(w)^*$ is a right centralizer. They proved that this result holds for skew centralizing generalized derivations; see also [3, 6, 7]. In this paper, we investigate these facts for θ-generalized derivations on $L_0^\infty(w)^*$.

2 Main Results

In the following, let $\mathrm{ran}(L_0^\infty(w)^*)$ be the right annihilator of $L_0^\infty(w)^*$, that is, the set of all $r \in L_0^\infty(w)^*$ such that $mr = 0$ for all $m \in L_0^\infty(w)^*$.

Theorem 2.1. *Let θ be a homomorphism on $L_0^\infty(w)^*$ and (D, d) be a θ-generalized derivation on $L_0^\infty(w)^*$. Then the following statements hold.*

(i) *D maps* $\mathrm{ran}(L_0^\infty(w)^*)$ *into* $\mathrm{ran}(L_0^\infty(w)^*)$.

(ii) *D is θ-centralizing if and only if D is θ-commuting.*

(iii) *If θ is an isomorphism and D is θ-centralizing, then D is a θ-right centralizer.*

Proof. (i) Let $k \in L_0^\infty(w)^*$, $u \in \Lambda(L_0^\infty(w)^*)$ and $r \in \text{ran}(L_0^\infty(w)^*)$. Then

$$kD(r) = k\theta(u)D(r) = k[D(ur) - d(u)\theta(r)] = 0,$$

where $u \in \Lambda(L_0^\infty(w)^*)$. Hence $D(r) \in \text{ran}(L_0^\infty(w)^*)$.

(ii) Let $m \in L_0^\infty(w)^*$ and

$$[D(m), \theta(m)] \in Z(L_0^\infty(w)^*).$$

So if $u \in \Lambda(L_0^\infty(w)^*)$, then

$$\begin{aligned}
[D(m), \theta(m)] &= [D(m), \theta(m)]u \\
&= u[D(m), \theta(m)] \\
&= u(D(m)\theta(m) - \theta(m)D(m)) \\
&= uD(m)\theta(m) - uD(m)\theta(m) \\
&= 0.
\end{aligned}$$

It follows that D is θ-commuting.

(iii) Let θ be an isomorphism and D be θ-centralizing. Then for every $m \in L_0^\infty(w)^*$

$$D(m)\theta(m) = \theta(m)D(m).$$

Choose $u \in \Lambda(L_0^\infty(w)^*)$. Then

$$D(u) = D(u)\theta(u) = \theta(u)D(u). \tag{1}$$

On the other hand,

$$D(u) = D(uu) = \theta(u)D(u) + d(u)\theta(u) = \theta(u)D(u) + d(u). \tag{2}$$

From this and (1), we have $d(u) = 0$. Note that if $r \in \text{ran}(L_0^\infty(w)^*)$, then

$$(r + u)^2 = r + u$$

and $\theta(u + r)$ is a right identity for $L_0^\infty(w)^*$. So

$$\begin{aligned}
D(u + r) &= D(u + r)\theta(u + r) \\
&= D((u + r)^2) - d(u + r)\theta(u + r) \\
&= D(u + r) - d(r).
\end{aligned}$$

Hence $d(r) = 0$. This shows that for every $m \in L_0^\infty(w)^*$,

$$\begin{aligned} d(m) &= d(um) \\ &= d(u)\theta(m) + \theta(u)d(m) \\ &= \theta(u)d(m) \\ &= 0, \end{aligned}$$

because d maps $L_0^\infty(w)^*$ into $\operatorname{ran}(L_0^\infty(w)^*)$; see [12]. Therefore,

$$\begin{aligned} D(m) &= D(mu) \\ &= \theta(m)D(u) + d(m)\theta(u) \\ &= \theta(m)D(u) \end{aligned}$$

for all $m \in L_0^\infty(w)^*$. That is, D is a θ-right centralizer. □

Let θ be a homomorphism on $L_0^\infty(w)^*$. Then a map

$$T : L_0^\infty(w)^* \to L_0^\infty(w)^*$$

is called *θ-skew commuting* if for every $m \in L_0^\infty(w)^*$

$$\langle T(m), \theta(m)\rangle := T(m)\theta(m) + \theta(m)T(m) = 0,$$

if for every $m \in L_0^\infty(w)^*$,

$$\langle T(m), \theta(m)\rangle \in Z(L_0^\infty(w)^*),$$

then T is called *θ-skew cenralizing*.

Theorem 2.2. *Let θ be an isomorphism on $L_0^\infty(w)^*$ and (D, d) be a θ-generalized derivation on $L_0^\infty(w)^*$. Then the following statements hold.*

(i) *If D is θ-skew centralizing, then there exists $n \in Z(L_0^\infty(w)^*)$ such that $D(m) = mn$ for all $m \in L_0^\infty(w)^*$.*

(ii) *If D is θ-skew commuting, then $D = 0$ on $L_0^\infty(w)^*$.*

Proof. (i). Let $m \in L_0^\infty(w)^*$ and

$$\langle D(m), \theta(m)\rangle \in Z(L_0^\infty(w)^*).$$

Then

$$0 = [\langle D(m), \theta(m)\rangle, \theta(m)] = [D(m), \theta(m)^2].$$

It follows that

$$D(u) = D(u)\theta(u)^2 = \theta(u)^2 D(u) = \theta(u)D(u).$$

On the other hand,

$$D(u) = D(uu) = \theta(u)D(u) + d(u).$$

Hence $d(u) = 0$. An argument similar to the proof of Theorem 1, shows that $D(m) = \theta(m)D(u)$ for all $m \in L_0^\infty(w)^*$. But,

$$\begin{aligned}\langle D(u), \theta(u)\rangle &= D(u)\theta(u) + \theta(u)D(u)\\ &= \theta(u)D(u)\theta(u) + \theta(u)D(u)\\ &= 2\theta(u)D(u)\end{aligned}$$

is an elemnt of $Z(L_0^\infty(w)^*)$. Since

$$D(m) = \theta(m)D(u) = \theta(m)\theta(u)D(u),$$

the statement (i) holds.

(ii) Let D be θ-skew commuting. Then there exists $n \in Z(L_0^\infty(w)^*)$ such that

$$D(m) = mn$$

for all $m \in L_0^\infty(w)^*$. So, if $u \in \Lambda(L_0^\infty(w)^*)$, then

$$\begin{aligned}0 &= D(u)\theta(u) + \theta(u)D(u)\\ &= un\theta(u) + \theta(u)un\\ &= u\theta(u)n + \theta(u)n\\ &= n\theta(u) + n\theta(u)\\ &= 2n\theta(u)\\ &= 2n.\end{aligned}$$

Hence $n = 0$ and therefore, $D = 0$. □

Theorem 2.3. *Let θ be a homomorphism on $L_0^\infty(w)^*$ and (D, d) be a θ-generalized derivation on $L_0^\infty(w)^*$. Then D is a θ-derivation if and only if $D = d$.*

Proof. Let D be a θ-derivation. Then for every $m \in L_0^\infty(w)^*$, we have

$$\begin{aligned} D(m) = D(m.u) &= D(m)\theta(u) + \theta(m)D(u) \\ &= D(m) + \theta(m)D(u) \end{aligned}$$

and

$$D(m) = \theta(m)D(u) + d(m)\theta(u) = \theta(m)D(u) + d(m).$$

Hence $d = D$. $\square$

We denote by $\mathrm{rad}(L_0^\infty(w)^*)$ the radical of $L_0^\infty(w)^*$. Before, we give the next result, let us recall that a map $T : L_0^\infty(w)^* \to L_0^\infty(w)^*$ is called *spectrally infinitesimal* if

$$r(T(m)) = 0$$

for all $m \in L_0^\infty(w)^*$, where $r(.)$ is the spectral radius.

Corollary 2.4. *Let θ be a homomorphism on $L_0^\infty(w)^*$ and (D, d) be a θ-generalized derivation on $L_0^\infty(w)^*$. If D maps $L_0^\infty(w)^*$ into $\mathrm{rad}(L_0^\infty(w)^*)$ or D is spectrally infinitesimal, then D is a θ-derivation.*

Proof. First, note that if D maps $L_0^\infty(w)^*$ into

$$\mathrm{rad}(L_0^\infty(w)^*) = \mathrm{ran}(L_0^\infty(w)^*),$$

then D is spectrally infinitesimal. Hence $r(D(m)) = 0$ for all $m \in L_0^\infty(w)^*$. One can prove that

$$\frac{L_0^\infty(w)^*}{\mathrm{ran}(L_0^\infty(w)^*)}$$

is isomorphic to the commutative Banach algebra $M(w)$; see [9]. In view of [14], there exists $c > 0$ such that

$$r(mD(u)) \leq c\, r(m)\, r(D(u)) = 0$$

for all $m \in L_0^\infty(w)^*$ and $u \in \Lambda(L_0^\infty(w)^*)$. It follows from Proposition 25.1 (ii) of [4] and that

$$D(u) \in \mathrm{rad}(L_0^\infty(w)^*) = \mathrm{ran}(L_0^\infty(w)^*).$$

Therefore,

$$D(m) = \theta(m)D(u) + d(m) = d(m)$$

for all $m \in L_0^\infty(w)^*$. $\square$

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References

[1] M. H. Ahmadi Gandomani and M. J. Mehdipour, Generalized derivations on some convolution algebras, *Aequationes Math.*, 92 (2) (2018), 223–241.

[2] M. H. Ahmadi Gandomani and M. J. Mehdipour, Symmetric bi-derivations and their generalizations on group algebras, *Filomat*, 35 (4) (2021), 1233–1240.

[3] E. Albas and N. Argac, Generalized derivations of prime rings, *Algebra Colloq.*, 11 (3) (2004), 399–410.

[4] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer, Berlin, 1973.

[5] H. G. Dales and A. T. Lau, The second duals of Beurling algebras, *Mem. Amer. Math. Soc.*, 177 (836) (2005), 1–191.

[6] V. De Filippis, Generalized derivations in prime rings and noncommutative Banach algebras, *Bull. Korean Math. Soc.*, 45 (4) (2008), 621–629.

[7] B. Hvala, Generalized derivations in rings, Comm. Algebra, 26 (4) (1998), 1147–1166.

[8] A. T. Lau and J. Pym, Concerning the second dual of the group algebra of a locally compact group, *J. London Math. Soc.*, 41 (3) (1990), 445–460.

[9] S. Maghsoudi, R. Nasr-Isfahani and A. Rejali, Strong Arens irregularity of Beurling algebras with a locally convex topology, *Arch. Math.* (*Basel*), 86 (5) (2006), 437–448.

[10] M. J. Mehdipour and R. Nasr-Isfahani, Completely continuous elements of Banach algebras related to locally compact groups., *Bull. Austral. Math. Soc.*, 76 (1) (2007), 49–54.

[11] M. J. Mehdipour and R. Nasr-Isfahani, Weakly compact multipliers on Banach algebras related to a locally compact group, *Acta Math. Hungar.*, 127 (3) (2010), 195–206.

[12] M. J. Mehdipour and Z. Saeedi, Derivations on convolution algebras, *Bull. Korean Math. Soc.*, 52 (4) (2015), 1123–1132.

[13] H. Reiter and J. D. Stegeman, *Classical Harmonic Analysis and Locally Compact Groups*, Second edition, London Mathematical Society Monographs, New Series, 22, The Clarendon Press, Oxford University Press, New York, 2000.

[14] J. Zemanek, Spectral radius characterizations of commutativity in Banach algebras, *Studia Math.*, 61 (3) (1977), 257-–268.

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