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Commutativity and Prime Ideals in Rings with Involutions via Derivations

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Abstract. This research explores the interplay between algebraic identities involving derivations with involutions and the commutativity of prime quotient rings. We aim to generalize established results that characterize commutativity in these rings.

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1 Introduction

In this article, we denote by $\top$ an associative ring whose center is $Z_{\top}$. An ideal $\Im$ of $\top$ is called prime if $\Im$ is different from $\top$, and whenever

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$w\top s$ is contained in $\Im$, for each elements w and s in $\top$, then either w or s belongs to $\Im$. A ring $\top$ is prime if and only if the only prime ideal is (0). We say that $\top$ is a 2-torsion free ring if $2w = 0$, for w in $\top$, implies that w is 0. For any w and s in $\top$, we use $[w, s]$ to represent $ws - sw$, and $w \circ s$ to mean $ws + sw$. A map $\amalg : \top \to \top$ is a derivation of ring $\top$ if it satisfies $\amalg(ws) = \amalg(w)s + w\amalg(s)$ for each w and s in $\top$. An additive map $* : \top \to \top$ is involution if $*$ is an antiautomorphism of order two, that is, $(w^*)^* = w$ for all w in $\top$. An element w in an involution ring $(\top, *)$ is hermitian if $w^* = w$, and skew hermitian if $w^* = -w$. We denote by $H_\top$ and $S_\top$ the sets of hermitian and skew hermitian elements in $\top$, respectively. "An involution $*$ is of the B-second kind if $S_\top \cap Z_\top \nsubseteq B$ for some $B \subset \top$, otherwise it is said to be of the B-first kind. In particular, if $B = \{0\}$, the involution is said to be of the second kind if $S_\top \cap Z_\top \neq \{0\}$, otherwise it is said to be of the first kind." [16, Definition 1.1].

Various authors have recently proven the commutativity of semiprime and prime rings by using suitably constrained additive mappings like derivations, automorphisms, generalized derivations, and skew derivations operating on specific subsets of the rings. To start, we recollect that for a subset S of $\top$, a function $\amalg : S \to \top$ is centralizing if $[\amalg(w), w]$ in $Z_\top$ for each w in B. If $[\amalg(w), w] = 0$ for each w in B, $\amalg$ is termed commuting on S. Posner demonstrated in [14] that if a prime ring $\top$ has a non-zero derivation $\amalg$ such that $[\amalg(w), w]$ in $Z_\top$ for each w in $\top$, then $\top$ becomes commutative. This insight has been refined and extended by numerous authors over the years, as discussed in [1, 9, 8, 11], along with further references. Long ago, Herstein proved that if a prime ring $\top$ with a characteristic distinct from 2 possesses a derivation $\amalg$ satisfying $\amalg(w)\amalg(s) = \amalg(s)\amalg(w)$ for each w and s in $\top$, then $\top$ is commutative. This idea was revisited by Bell and Daif [4], achieving the same result by focusing on the identity $\amalg[w, s] = 0$ for each w and s in a non-zero ideal of $\top$. In [3], Bell and Daif explored the commutativity of rings with a derivation that preserves strong commutativity on a non-zero right ideal. Ali and Huang [2] revealed that if $\top$ is a 2-torsion free semiprime ring and $\amalg$ is a derivation satisfying $[\amalg(w), \amalg(s)] + [w, s] = 0$ for each w and s in a non-zero ideal I of $\top$, then $\top$ contains a non-zero central ideal. Building on this, introduced the concepts of $*$-SCP and $*$-Skew

SCP, providing commutativity criteria for prime rings with involution. Similar generalizations can be found in the literature, as in [5, 11]. In [6], Daif and Bell demonstrated the commutativity of semiprime rings where the derivation $\amalg([w,s])$ equals the commutator$[w,s]$ for every w and s in a non-zero ideal of the ring $\top$. In 1997, Hongan [7] proved that a semiprime ring $\top$, which is 2-torsion free, is necessarily commutative if a derivation $\amalg$ fulfills either $\amalg([w,s]) + [w,s]$ in the center $Z_\top$ or $\amalg([w,s]) - [w,s]$ in the center $Z_\top$ for every w and s in an ideal I of $\top$. Oukhtite and et al. in [12] expanded these results for $*$-prime rings $\top$ that satisfy various conditions involving $\amalg[w,s] = 0$, $\amalg([w,s]) - [w,s]$ in $Z_\top$, $\amalg([w,s]) + [w,s]$ in $Z_\top$, $\amalg(w \circ s) = 0$, $\amalg(w \circ s) - w \circ s$ in $Z_\top$, and $\amalg(w \circ s) + w \circ s$ in $Z_\top$ for every w and s in J, where J is a non-zero Jordan ideal of $\top$.

In 2023, Oukhtite et al. [13] proved the following results: "Let $\top$ be a ring, I a nonzero ideal, $\Im$ a prime ideal such that $\Im \subsetneqq I$ and $\top/\Im$ is 2-torsion free. Let $\amalg$, $\amalg_1$, and $\amalg_2$ be derivations on $\top$. Then $\top/\Im$ is a commutative integral domain if and only if one of the following conditions is satisfied: (i) $\overline{\amalg([w,s])} \pm \overline{[w,s]} \in Z_{\top/\Im}$ for each $w, s \in I$, (ii) $\overline{[\amalg_1(w), s]} + \overline{[w, \amalg_2(s)]} - \overline{[w,s]} \in Z_{\top/\Im}$ for each $w, s \in I$, (iii) $\overline{\amalg_1(w) \circ s} + \overline{w \circ \amalg_2(s)} - \overline{w \circ s} \in Z_{\top/\Im}$ for each $w, s \in I$." For further details on these topics, see [17, 15, 18].

This research delves into the intricate relationship between algebraic identities involving derivations with involutions and the commutativity of prime quotient rings. Our work is motivated by a significant gap in the existing literature concerning the generalization of classical results on prime rings to a broader context of arbitrary rings equipped with involutions. While seminal works, such as those of Posner, Daif, and Bell, have laid the foundation for understanding commutativity in prime rings, their findings are often limited to specific types of rings and ideals. For instance, Posner's theorem, a cornerstone in this field, primarily deals with prime rings.

This paper extends these classical frameworks by examining the commutativity of quotient rings $\top/\Im$ under a more general lens, where $\top$ can be any ring and $\Im$ is a prime ideal with a second-kind involution. Moreover, we establish a connection between the structure of $\top/\Im$ and the behavior of derivations of $\top$ satisfying certain identities that involve the prime ideal $\Im$. This generalization is theoretically significant because it broadens the applicability of these classical results and provides a deeper understanding of the interplay between derivations, involutions, and the structure of quotient rings.

The potential applications of these findings are far-reaching, extending into various domains of algebra and related fields. In ring theory, our results can contribute to a deeper understanding of the structure of noncommutative rings and their quotient structures. The results have implications for module theory. Furthermore, the study of rings with involutions is closely related to the study of $*$-algebras, which play a crucial role in functional analysis and operator theory. Our findings could therefore provide insights into the properties of certain operator algebras. In the realm of coding theory, where algebraic structures are used to design and analyze error-correcting codes, our results might offer new tools for constructing codes with specific properties. Moreover, from a computational perspective, our research could potentially inform symbolic computation methods in noncommutative polynomial rings, particularly in identifying and testing for commutativity conditions.

2 Conditions Involving Lie Products

This section introduces the fundamental concepts of Lie products and investigates their role in determining the commutativity of rings with involutions. We focus on the interplay between Lie products, derivations, and prime ideals, particularly within the context of second-kind involutions.

Lemma 2.1. *Let $\top$ be a ring, $\Im$ a prime ideal, and $w, s \in \top$. If $\overline{ws} \in Z_{\top/\Im}$ and $\overline{s} \in Z_{\top/\Im}$, then $\overline{s} = \overline{0}$ or $\overline{w} \in Z_{\top/\Im}$.*

Proof. Let $w, s \in \top$ and assume that $\overline{ws} \in Z_{\top/\Im}$ and $\overline{s} \in Z_{\top/\Im}$. That is, $[\overline{ws}, \overline{r}] = \overline{0}$ and $[\overline{s}, \overline{r}] = \overline{0}$ for each $r \in \top$. It follows that $[\overline{w}, \overline{r}]\overline{s} = \overline{0}$ for each $r \in \top$. Hence, $[\overline{w}, \overline{r}]\top/\Im\overline{s} = \overline{0}$ for each $r \in \top$. By primeness of $\Im$, we get $\overline{s} = \overline{0}$ or $[\overline{w}, \overline{r}] = \overline{0}$ for each $r \in \top$. In the last case, we obtain $\overline{w} \in Z_{\top/\Im}$. □

Lemma 2.2. *Let $\top$ be a ring, $\Im$ a prime ideal with $\Im$-second kind involution $*$, and $\top/\Im$ 2-torsion free. If $\overline{[w,w^*]} \in Z_{\top/\Im}$ $\left(\overline{w \circ w^*} \in Z_{\top/\Im}\right)$ for each $w \in \top$, then $\top/\Im$ is a commutative integral domain.*

Proof. Assume that

$$\overline{[w,w^*]} \in Z_{\top/\Im} \quad \text{for each } w \in \top. \tag{1}$$

By linearizing (1), we have

$$\overline{[w,s^*]} + \overline{[s,w^*]} \in Z_{\top/\Im} \quad \text{for each } w,s \in \top. \tag{2}$$

We have a $\Im$-second kind involution $*$. Thus, $S_\top \cap Z_\top \nsubseteq \Im$, then $S_\top \cap Z_\top \neq (0)$, so there exists $0 \neq k \in S_\top \cap Z_\top$ and since $S_\top \cap Z_\top \nsubseteq \Im$, there exists $k \notin \Im$. Suppose that $0 \neq k \in S_\top \cap Z_\top \backslash \Im$. Replacing w with kw in (2), we get $\overline{k}(\overline{[w,s^*]} - \overline{[s,w^*]}) \in Z_{\top/\Im}$ for each $w,s \in \top$. Using Lemma 2.1, we find that $\overline{[w,s^*]} - \overline{[s,w^*]} \in Z_{\top/\Im}$ for each $w,s \in \top$. Comparing the last relation and (2), we see that $2\overline{[w,s^*]} \in Z_{\top/\Im}$ for each $w,s \in \top$. That is, $2\overline{[w,s]} \in Z_{\top/\Im}$ for each $w,s \in \top$. Since $\top/\Im$ is 2-torsion free, we arrive at $\overline{[w,s]} \in Z_{\top/\Im}$ for each $w,s \in \top$. Now, using the similar arguments as used in the proof of [10, Lemma 2.5], we conclude that $\top/\Im$ is a commutative integral domain. in case $\overline{w \circ w^*} \in Z_{\top/\Im}$ for each $w \in \top$, using a similar approach as the above. □

Proposition 2.3. *Let $\top$ be a ring, $\Im$ a prime ideal, $\Im$-second kind involution $*$, and $\top/\Im$ 2-torsion free. If $\amalg$ is a derivation of $\top$, then $\overline{[\amalg(w),w^*]} \in Z_{\top/\Im}$ for each $w \in \top$ if and only if $\amalg(\top) \subseteq \Im$ or $\top/\Im$ is a commutative integral domain.*

Proof. Assume that

$$\overline{[\amalg(w),w^*]} \in Z_{\top/\Im} \quad \text{for each } w \in \top. \tag{3}$$

By linearizing (3), we have

$$\overline{[\amalg(w),s^*]} + \overline{[\amalg(s),w^*]} \in Z_{\top/\Im} \quad \text{for each } w,s \in \top. \tag{4}$$

As in the proof of Lemma 2.2, there exists $k \notin \Im$, and by the primeness of $\Im$, we see that $0 \neq k^2 \notin \Im$, but $0 \neq k^2 \in H_\top \cap Z_\top \backslash \Im$. Thus, we can suppose that $0 \neq h \in H_\top \cap Z_\top \backslash \Im$. Replacing w by wh in (4), we get

$$\overline{h}(\overline{[\amalg(w),s^*]} + \overline{[\amalg(s),w^*]}) + \overline{[w,s^*]\amalg(h)} \in Z_{\top/\Im} \quad \text{for each } w,s \in \top. \tag{5}$$

Multiplying (4) by $\overline{h}$ and then using it in (5), we find that $\overline{[w,s^*]\text{II}(h)} \in Z_{\top/\Im}$ for each $w,s \in \top$. That is, $\overline{[w,s]\text{II}(h)} \in Z_{\top/\Im}$ for each $w,s \in \top$. By using Lemma 2.1, we get $\overline{[w,s]} \in Z_{\top/\Im}$ for each $w,s \in \top$ or $\overline{\text{II}(h)} = \overline{0}$. If $\overline{[w,s]} \in Z_{\top/\Im}$ for each $w,s \in \top$, then by Lemma 2.2(i), we get $\top/\Im$ is a commutative integral domain. In case $\overline{\text{II}(h)} = \overline{0}$. It follows that $\overline{\text{II}(k)} = \overline{0}$. That is,

$$\overline{\text{II}(z)} = \overline{0} \quad \text{for each } z \in Z_\top. \tag{6}$$

Replacing w by wk in (4), where $k \in S_\top \cap Z_\top \backslash \Im$ and using (6), we get

$$\overline{k}(\overline{[\text{II}(w),s^*]} - \overline{[\text{II}(s),w^*]}) \in Z_{\top/\Im} \quad \text{for each } w,s \in \top. \tag{7}$$

Since $\overline{k} \in Z_\top/\Im \subseteq Z_{\top/\Im}$, we obtain $\overline{k} \in Z_{\top/\Im}$. Multiplying (4) by $\overline{k}$ and then using it in (7), we have $\overline{k}\overline{[\text{II}(w),s^*]} \in Z_{\top/\Im}$ for each $w,s \in \top$, and since $\overline{k} \neq \overline{0}$, we find that $\overline{[\text{II}(w),s^*]} \in Z_{\top/\Im}$ for each $w,s \in \top$, that is, $\overline{[\text{II}(w),s]} \in Z_{\top/\Im}$ for each $w,s \in \top$. In particular, $\overline{[\text{II}(w),w]} \in Z_{\top/\Im}$ for each $w \in \top$, by [13, Lemma 2.2], we get $\text{II}(\top) \subseteq \Im$ or $\top/\Im$ is a commutative integral domain. □

Theorem 2.4. *Let $\top$ be a ring, $\Im$ a prime ideal, $\Im$-second kind involution $*$, and $\top/\Im$ 2-torsion free. If II is a derivation of $\top$, then $\overline{\text{II}[w,w^*]} - \overline{[w,w^*]} \in Z_{\top/\Im}$ for each $w \in \top$ if and only if $\top/\Im$ is a commutative integral domain. Moreover, if $\overline{\text{II}[w,w^*]} \in Z_{\top/\Im}$ for each $w \in \top$, then $\text{II}(\top) \subseteq \Im$ or $\top/\Im$ is a commutative integral domain.*

Proof. Assume that

$$\overline{\text{II}[w,w^*]} - \overline{[w,w^*]} \in Z_{\top/\Im} \quad \text{for each } w \in \top. \tag{8}$$

By linearizing (8), we have

$$\overline{\text{II}[w,s^*]} + \overline{\text{II}[s,w^*]} - \overline{[w,s^*]} - \overline{[s,w^*]} \in Z_{\top/\Im} \quad \text{for each } w,s \in \top. \tag{9}$$

As we see above, we can assume that $0 \neq h \in H_\top \cap Z_\top \backslash \Im$. Replacing w by wh in (9), we get

$$\overline{h}(\overline{\text{II}[w,s^*]} + \overline{\text{II}[s,w^*]} - \overline{[w,s^*]} - \overline{[s,w^*]}) + (\overline{[w,s^*]} + \overline{[s,w^*]})\overline{\text{II}(h)} \in Z_{\top/\Im} \quad \text{for each } w,s \in \top. \tag{10}$$

Multiplying (9) by $\overline{h}$ and then using it in (10), we find that

$$(\overline{[w,s^*]}+\overline{[s,w^*]})\overline{\amalg(h)} \in Z_{\top/\Im} \quad \text{for each } w,s\in\top. \tag{11}$$

Using Lemma 2.1, we get $\overline{[w,s^*]}+\overline{[s,w^*]} \in Z_{\top/\Im}$ for each $w,s\in\top$ or $\overline{\amalg(h)}=\overline{0}$. In case $\overline{[w,s^*]}+\overline{[s,w^*]} \in Z_{\top/\Im}$ for each $w,s\in\top$. Replacing w by wk in the last relation, where $k\in S_\top\cap Z_\top\backslash\Im$ and using it, we get $2\overline{k}\overline{[w,s^*]}\in Z_{\top/\Im}$ for each $w,s\in\top$. That is, $\overline{[w,s]}\in Z_{\top/\Im}$ for each $w,s\in\top$ and by using Lemma 2.2(i), we get $\top/\Im$ is a commutative integral domain. In case $\overline{\amalg(h)}=\overline{0}$. It follows that $\overline{\amalg(k)}=\overline{0}$. That is,

$$\overline{\amalg(z)}=\overline{0} \quad \text{for each } z\in Z_\top. \tag{12}$$

Replacing w by wk in (9), where $k\in S_\top\cap Z_\top\backslash\Im$ and using (12), we get

$$\overline{k}(\overline{\amalg[w,s^*]}-\overline{\amalg[s,w^*]}-\overline{[w,s^*]}+\overline{[s,w^*]})\in Z_{\top/\Im} \quad \text{for each } w,s\in\top. \tag{13}$$

Since $\overline{k}\in Z_\top/\Im\subseteq Z_{\top/\Im}$, we obtain $\overline{k}\in Z_{\top/\Im}$. Multiplying (9) by $\overline{k}$ and then using it in (13), we have $\overline{k}(\overline{\amalg[w,s^*]}-\overline{[w,s^*]})\in Z_{\top/\Im}$ for each $w,s\in\top$, and since $\overline{k}\neq\overline{0}$, we find that $\overline{\amalg[w,s^*]}-\overline{[w,s^*]}\in Z_{\top/\Im}$ for each $w,s\in\top$, that is, $\overline{\amalg[w,s]}-\overline{[w,s]}\in Z_{\top/\Im}$ for each $w,s\in\top$, and by using [13, Theorem 2.3], we get $\top/\Im$ is a commutative integral domain.

Now, assume that

$$\overline{\amalg[w,w^*]}\in Z_{\top/\Im} \quad \text{for each } w\in\top. \tag{14}$$

By linearizing (14), we have

$$\overline{\amalg[w,s^*]}+\overline{\amalg[s,w^*]}\in Z_{\top/\Im} \quad \text{for each } w,s\in\top. \tag{15}$$

Assume that $0\neq h\in H_\top\cap Z_\top\backslash\Im$. Replacing w by wh in (15), we get

$$\overline{h}(\overline{\amalg[w,s^*]}+\overline{\amalg[s,w^*]})+(\overline{[w,s^*]}+\overline{[s,w^*]})\overline{\amalg(h)}\in Z_{\top/\Im} \quad \text{for each } w,s\in\top. \tag{16}$$

Multiplying (15) by $\overline{h}$ and then using it in (16), we find that $(\overline{[w,s^*]}+\overline{[s,w^*]})\overline{\amalg(h)}\in Z_{\top/\Im}$ for each $w,s\in\top$. Now the same as in (11), we get $\top/\Im$ is a commutative integral domain or

$$\overline{\amalg(z)}=\overline{0} \quad \text{for each } z\in Z_\top. \tag{17}$$

Replacing w by wk in (15), where $k \in S_\top \cap Z_\top \backslash \Im$ and using (17), we get

$$\overline{k}(\overline{\text{II}[w,s^*]} - \overline{\text{II}[s,w^*]}) \in Z_{\top/\Im} \quad \text{for each } w,s \in \top. \tag{18}$$

Since $\overline{k} \in Z_\top/\Im \subseteq Z_{\top/\Im}$, we obtain $\overline{k} \in Z_{\top/\Im}$. Multiplying (15) by $\overline{k}$ and then using it in (18), we have $\overline{k}\overline{\text{II}[w,s^*]} \in Z_{\top/\Im}$ for each $w,s \in \top$, and since $\overline{k} \neq \overline{0}$, we find that $\overline{\text{II}[w,s^*]} \in Z_{\top/\Im}$ for each $w,s \in \top$, that is, $\overline{\text{II}[w,s]} \in Z_{\top/\Im}$ for each $w,s \in \top$, and by using [13, Theorem 2.3], we get $\text{II}(\top) \subseteq \Im$ or $\top/\Im$ is a commutative integral domain. □

Theorem 2.5. *Let $\top$ be a ring, $\Im$ a prime ideal, $\Im$-second kind involution $*$, and $\top/\Im$ 2-torsion free. If II is a derivation of $\top$, then $\overline{\text{II}[w,w^*]} + \overline{[w,w^*]} \in Z_{\top/\Im}$ for each $w \in \top$ if and only if $\top/\Im$ is a commutative integral domain.*

Proof. Using the same technics as in the preceding proof, it is obvious to see that $\overline{\text{II}[w,w^*]} + \overline{[w,w^*]} \in Z_{\top/\Im}$ for each $w \in \top$ implies that $\top/\Im$ is a commutative integral domain. □

Theorem 2.6. *Let $\top$ be a ring, $\Im$ a prime ideal, $\Im$-second kind involution $*$, and $\top/\Im$ 2-torsion free. If II_1 and II_2 are derivation of $\top$, then $\overline{[\text{II}_1(w),w^*]} + \overline{[w,\text{II}_2(w^*)]} - \overline{[w,w^*]} \in Z_{\top/\Im}$ for each $w \in \top$ if and only if $\top/\Im$ is a commutative integral domain. Moreover, if $\overline{[\text{II}_1(w),w^*]} + \overline{[w,\text{II}_2(w^*)]} \in Z_{\top/\Im}$ for each $w \in \top$, then $(\text{II}_1(\top),\text{II}_2(\top)) \subseteq (\Im,\Im)$ or $\top/\Im$ is a commutative integral domain.*

Proof. Assume that

$$\overline{[\text{II}_1(w),w^*]} + \overline{[w,\text{II}_2(w^*)]} - \overline{[w,w^*]} \in Z_{\top/\Im} \quad \text{for each } w \in \top. \tag{19}$$

By linearizing (19), we have

$$\begin{aligned} &\overline{[\text{II}_1(w),s^*]} + \overline{[\text{II}_1(s),w^*]} + \overline{[w,\text{II}_2(s^*)]} \\ &+\overline{[s,\text{II}_2(w^*)]} - \overline{[w,s^*]} - \overline{[s,w^*]} \in Z_{\top/\Im} \quad \text{for each } w,s \in \top. \end{aligned} \tag{20}$$

Assume that $0 \neq h \in H_\top \cap Z_\top \backslash \Im$. Replacing w by wh in (20), we get

$$\begin{aligned} &\overline{h}(\overline{[\text{II}_1(w),s^*]} + \overline{[\text{II}_1(s),w^*]} + \overline{[w,\text{II}_2(s^*)]} + \overline{[s,\text{II}_2(w^*)]} - \overline{[w,s^*]} - \overline{[s,w^*]}) \\ &+\overline{[w,s^*]\text{II}_1(h)} + \overline{[s,w^*]\text{II}_2(h)} \in Z_{\top/\Im} \quad \text{for each } w,s \in \top. \end{aligned} \tag{21}$$

Multiplying (20) by $\overline{h}$ and then using it in (21), we find that

$$\overline{[w,s^*]\text{II}_1(h)} + \overline{[s,w^*]\text{II}_2(h)} \in Z_{\top/\Im} \quad \text{for each } w,s \in \top. \tag{22}$$

Putting $s = w$, we get $\overline{[w,w^*]}(\overline{\text{II}_1(h)} + \overline{\text{II}_2(h)}) \in Z_{\top/\Im}$ for each $w \in \top$. Using Lemma 2.1, we have $\overline{[w,w^*]} \in Z_{\top/\Im}$ for each $w \in \top$ or $\overline{\text{II}_1(h)} + \overline{\text{II}_2(h)} = \overline{0}$. In case $\overline{[w,w^*]} \in Z_{\top/\Im}$ for each $w \in \top$ and by Lemma 2.2(i), we get $\top/\Im$ is a commutative integral domain. In case $\overline{\text{II}_1(h)} + \overline{\text{II}_2(h)} = \overline{0}$, we obtain

$$\overline{\text{II}_2(h)} = -\overline{\text{II}_1(h)}. \tag{23}$$

Using (23) in (22), we get $\overline{[w,s^*]\text{II}_1(h)} - \overline{[s,w^*]\text{II}_1(h)} \in Z_{\top/\Im}$ for each $w,s \in \top$. that is, $(\overline{[w,s^*]} - \overline{[s,w^*]})\overline{\text{II}_1(h)} \in Z_{\top/\Im}$ for each $w,s \in \top$. Now, application of similar arguments as used after (11), we get $\top/\Im$ is a commutative integral domain or $\overline{\text{II}_1(z)} = \overline{0}$ for each $z \in Z_\top$. Using the last relation in (23), we get $\overline{\text{II}_2(z)} = \overline{0}$ for each $z \in Z_\top$. Thus,

$$\overline{\text{II}_1(z)} = \overline{0} = \overline{\text{II}_2(z)} \quad \text{for each } z \in Z_\top. \tag{24}$$

Replacing w by wk in (20), where $k \in S_\top \cap Z_\top \backslash \Im$ and using (24), we get

$$\begin{aligned} &\overline{k}(\overline{[\text{II}_1(w),s^*]} - \overline{[\text{II}_1(s),w^*]} + \overline{[w,\text{II}_2(s^*)]} \\ &-\overline{[s,\text{II}_2(w^*)]} - \overline{[w,s^*]} + \overline{[s,w^*]}) \in Z_{\top/\Im} \quad \text{for each } w,s \in \top. \end{aligned} \tag{25}$$

Since $\overline{k} \in Z_\top/\Im \subseteq Z_{\top/\Im}$, we obtain $\overline{k} \in Z_{\top/\Im}$. Multiplying (20) by $\overline{k}$ and then using it in (25), we have $\overline{k}(\overline{[\text{II}_1(w),s^*]} + \overline{[w,\text{II}_2(s^*)]} - \overline{[w,s^*]}) \in Z_{\top/\Im}$ for each $w,s \in \top$, and since $\overline{k} \neq \overline{0}$, we find that $\overline{[\text{II}_1(w),s^*]} + \overline{[w,\text{II}_2(s^*)]} - \overline{[w,s^*]} \in Z_{\top/\Im}$ for each $w,s \in \top$, that is, $\overline{[\text{II}_1(w),s]} + \overline{[w,\text{II}_2(s)]} - \overline{[w,s]} \in Z_{\top/\Im}$ for each $w,s \in \top$, and by using [13, Theorem 2.6], we get $\top/\Im$ is a commutative integral domain.

Now, assume that

$$\overline{[\text{II}_1(w),w^*]} + \overline{[w,\text{II}_2(w^*)]} \in Z_{\top/\Im} \quad \text{for each } w \in \top. \tag{26}$$

By linearizing (26), we have

$$\overline{[\text{II}_1(w),s^*]} + \overline{[\text{II}_1(s),w^*]} + \overline{[w,\text{II}_2(s^*)]} + \overline{[s,\text{II}_2(w^*)]} \in Z_{\top/\Im} \quad \text{for each } w,s \in \top. \tag{27}$$

Assume that $0 \neq h \in H_{\top} \cap Z_{\top} \backslash \Im$. Replacing w by wh in (27), we get

$$\begin{aligned} &\overline{h}(\overline{[\amalg_1(w), s^*]} + \overline{[\amalg_1(s), w^*]} + \overline{[w, \amalg_2(s^*)]} + \overline{[s, \amalg_2(w^*)]}) \\ &+\overline{[w, s^*]\amalg_1(h)} + \overline{[s, w^*]\amalg_2(h)} \in Z_{\top/\Im} \quad \text{for each } w, s \in \top. \end{aligned} \tag{28}$$

Multiplying (27) by $\overline{h}$ and then using it in (28), we find that $\overline{[w, s^*]\amalg_1(h)+[s, w^*]\amalg_2(h)} \in Z_{\top/\Im}$ for each $w, s \in \top$. Now the same as in (22), we get $\top/\Im$ is a commutative integral domain or

$$\overline{\amalg_1(z)} = \overline{0} = \overline{\amalg_2(z)} \quad \text{for each } z \in Z_{\top}. \tag{29}$$

Replacing w by wk in (27), where $k \in S_{\top} \cap Z_{\top} \backslash \Im$ and using (29), we get

$$\overline{k}(\overline{[\amalg_1(w), s^*]} - \overline{[\amalg_1(s), w^*]} + \overline{[w, \amalg_2(s^*)]} - \overline{[s, \amalg_2(w^*)]}) \in Z_{\top/\Im} \quad \text{for each } w, s \in \top. \tag{30}$$

Since $\overline{k} \in Z_{\top}/\Im \subseteq Z_{\top/\Im}$, we obtain $\overline{k} \in Z_{\top/\Im}$. Multiplying (27) by $\overline{k}$ and then using it in (30), we have $2\overline{k}(\overline{[\amalg_1(w), s^*]} + \overline{[w, \amalg_2(s^*)]}) \in Z_{\top/\Im}$ for each $w, s \in \top$, and since $\overline{k} \neq \overline{0}$, we find that $\overline{[\amalg_1(w), s^*]} + \overline{[w, \amalg_2(s^*)]} \in Z_{\top/\Im}$ for each $w, s \in \top$, that is, $\overline{[\amalg_1(w), s]} + \overline{[w, \amalg_2(s)]} \in Z_{\top/\Im}$ for each $w, s \in \top$, and by using [13, Theorem 2.6], we get $(\amalg_1(\top), \amalg_2(\top)) \subseteq (\Im, \Im)$ or $\top/\Im$ is a commutative integral domain. □

3 Conditions Involving Jordan Products

Building on the analysis of Lie products in the previous section, we now shift our focus to Jordan products and their role in shaping the commutativity of rings with involutions. This section provides a comprehensive investigation of how Jordan products interact with derivations and second-kind involutions, offering a complementary perspective on the conditions that govern the commutativity of the quotient ring $\top/\Im$.

Lemma 3.1. *Let $\top$ be a ring, $\Im$ a semiprime ideal, I a non-zero ideal such that $\Im \subsetneqq I$. If $\amalg$ is a derivation of $\top$, and $\overline{\amalg(w) \circ w} \in Z_{\top/\Im}$ for each $w \in I$, then $[\amalg(w), w] \in \Im$ for each $w \in I$.*

Proof. Assume that

$$\overline{\amalg(w) \circ w} \in Z_{\top/\Im} \quad \text{for each } w \in I. \tag{31}$$

By linearizing (31), we have

$$\overline{\amalg(w)\circ s}+\overline{\amalg(s)\circ w}\in Z_{\top/\Im}\quad \text{for each } w,s\in I. \tag{32}$$

Replacing s by sw in (32), we get

$$(\overline{\amalg(w)\circ s+\amalg(s)\circ w)}\overline{w}-\overline{s[\amalg(w),w]}+\overline{(s\circ w)\amalg(w)}+\overline{s[\amalg(w),w]}\in Z_{\top/\Im}\quad \text{for each } w,s\in I.$$

That is,

$$(\overline{\amalg(w)\circ s+\amalg(s)\circ w)}\overline{w}+\overline{(s\circ w)\amalg(w)}\in Z_{\top/\Im}\quad \text{for each } w,s\in I.$$

Using (32) in the last relation, we obtain $\overline{[(s\circ w)\amalg(w),w]}=\bar{0}$. Hence, $[(s\circ w)\amalg(w),w]\in\Im$, and so $[sw\amalg(w)+ws\amalg(w),w]\in\Im$ it implies that $sw\amalg(w)w+ws\amalg(w)w-wsw\amalg(w)-w^2s\amalg(w)\in\Im$. Replacing s by rs in the last relation, and then left multiplying it by r, and then subtracting them, where $r\in\top$, we get $[w,r]s[\amalg(w),w]+[r,w^2]s\amalg(w)\in\Im$. Putting $r=\amalg(w)$, we have

$$[\amalg(w),w]s[\amalg(w),w]-[\amalg(w),w^2]s\amalg(w)\in\Im\quad \text{for each } w,s\in I. \tag{33}$$

From (31), we get $\overline{[\amalg(w)\circ w,w]}=\bar{0}$. That is, $[\amalg(w)\circ w,w]\in\Im$. Hence, $[\amalg(w)w+w\amalg(w),w]\in\Im$. Thus, $[\amalg(w),w]w+w[\amalg(w),w]\in\Im$. That is, $[\amalg(w),w^2]\in\Im$. Using the last relation in (33), we see that $[\amalg(w),w]s[\amalg(w),w]\in\Im$ for each $w,s\in I$. By the semiprimeness of $\top$, we find that $[\amalg(w),w]\in\Im$ for each $w\in I$. $\square$

Lemma 3.2. *Let $\top$ be a ring, $\Im$ a prime ideal, I a non-zero ideal such that $\Im\subsetneqq I$, and $\top/\Im$ 2-torsion free. If $\amalg$ is a derivation of $\top$, then $\overline{\amalg(w)\circ w}\in Z_{\top/\Im}$ for each $w\in I$, if and only if $\amalg(\top)\subseteq\Im$ or $\top/\Im$ is a commutative integral domain.*

Proposition 3.3. *Let $\top$ be a ring, $\Im$ a prime ideal, $\Im$-second kind involution $*$, and $\top/\Im$ 2-torsion free. If $\amalg$ is a derivation of $\top$, then $\overline{\amalg(w)\circ w^*}\in Z_{\top/\Im}$ for each $w\in\top$ if and only if $\amalg(\top)\subseteq\Im$ or $\top/\Im$ is a commutative integral domain.*

Proof. Assume that

$$\overline{\amalg(w)\circ w^*}\in Z_{\top/\Im}\quad \text{for each } w\in\top. \tag{34}$$

By linearizing (34), we have

$$\overline{\amalg(w)\circ s^*}+\overline{\amalg(s)\circ w^*}\in Z_{\top/\Im}\quad \text{for each } w,s\in\top. \tag{35}$$

Assume that $0\neq h\in H_\top\cap Z_\top\backslash\Im$. Replacing w by wh in (35), we get

$$\overline{h}(\overline{\amalg(w)\circ s^*}+\overline{\amalg(s)\circ w^*})+\overline{w\circ s^*}\overline{\amalg(h)}\in Z_{\top/\Im}\quad \text{for each } w,s\in\top. \tag{36}$$

Multiplying (35) by $\overline{h}$ and then using it in (36), we find that $\overline{w\circ s^*}\overline{\amalg(h)}\in Z_{\top/\Im}$ for each $w,s\in\top$. That is, $\overline{w\circ s}\overline{\amalg(h)}\in Z_{\top/\Im}$ for each $w,s\in\top$. By using Lemma 2.1, we get $\overline{w\circ s}\in Z_{\top/\Im}$ for each $w,s\in\top$ or $\overline{\amalg(h)}=\overline{0}$. If $\overline{w\circ s}\in Z_{\top/\Im}$ for each $w,s\in\top$, then by Lemma 2.2(i), we get $\top/\Im$ is a commutative integral domain. In case $\overline{\amalg(h)}=\overline{0}$. It follows that $\overline{\amalg(k)}=\overline{0}$. That is,

$$\overline{\amalg(z)}=\overline{0}\quad \text{for each } z\in Z_\top. \tag{37}$$

Replacing w by wk in (35), where $k\in S_\top\cap Z_\top\backslash\Im$ and using (37), we get

$$\overline{k}(\overline{\amalg(w)\circ s^*}-\overline{\amalg(s)\circ w^*})\in Z_{\top/\Im}\quad \text{for each } w,s\in\top. \tag{38}$$

Since $\overline{k}\in Z_\top/\Im\subseteq Z_{\top/\Im}$, we obtain $\overline{k}\in Z_{\top/\Im}$. Multiplying (35) by $\overline{k}$ and then using it in (38), we have $\overline{k}\overline{\amalg(w)\circ s^*}\in Z_{\top/\Im}$ for each $w,s\in\top$, and since $\overline{k}\neq\overline{0}$, we find that $\overline{\amalg(w)\circ s^*}\in Z_{\top/\Im}$ for each $w,s\in\top$, that is, $\overline{\amalg(w)\circ s}\in Z_{\top/\Im}$ for each $w,s\in\top$. In particular, $\overline{\amalg(w)\circ w}\in Z_{\top/\Im}$ for each $w\in\top$, by Lemma 3.2, we get $\amalg(\top)\subseteq\Im$ or $\top/\Im$ is a commutative integral domain. □

Theorem 3.4. *Let $\top$ be a ring, $\Im$ a prime ideal, $\Im$-second kind involution $*$, $\top/\Im$ 2-torsion free, and $\amalg$ a derivation of $\top$. If $\overline{\amalg(w\circ w^*)}-\overline{w\circ w^*}\in Z_{\top/\Im}$ for each $w\in\top$ if and only if $\top/\Im$ is a commutative integral domain. Moreover, if $\overline{\amalg(w\circ w^*)}\in Z_{\top/\Im}$ for each $w\in\top$, then $\amalg(\top)\subseteq\Im$ or $\top/\Im$ is a commutative integral domain.*

Proof. Assume that

$$\overline{\amalg(w\circ w^*)}-\overline{w\circ w^*}\in Z_{\top/\Im}\quad \text{for each } w\in\top. \tag{39}$$

By linearizing (39), we have

$$\overline{\amalg(w\circ s^*)}+\overline{\amalg(s\circ w^*)}-\overline{w\circ s^*}-\overline{s\circ w^*}\in Z_{\top/\Im}\quad \text{for each } w,s\in\top. \tag{40}$$

Assume that $0\neq h\in H_\top\cap Z_\top\backslash\Im$. Replacing w by wh in (40), we get

$$\overline{h}(\overline{\amalg(w\circ s^*)}+\overline{\amalg(s\circ w^*)}-\overline{w\circ s^*}-\overline{s\circ w^*})+(\overline{w\circ s^*}+\overline{s\circ w^*})\overline{\amalg(h)}\in Z_{\top/\Im}\quad \text{for each } w,s\in\top. \tag{41}$$

Multiplying (40) by $\overline{h}$ and then using it in (41), we find that

$$(\overline{w\circ s^*}+\overline{s\circ w^*})\overline{\amalg(h)}\in Z_{\top/\Im}\quad \text{for each } w,s\in\top. \tag{42}$$

Using Lemma 2.1, we get $\overline{w\circ s^*}+\overline{s\circ w^*}\in Z_{\top/\Im}$ for each $w,s\in\top$ or $\overline{\amalg(h)}=\overline{0}$. In case $\overline{w\circ s^*}+\overline{s\circ w^*}\in Z_{\top/\Im}$ for each $w,s\in\top$. Replacing w by wk in the last relation, where $k\in S_\top\cap Z_\top\backslash\Im$ and using it, we get $2\overline{k}\overline{w\circ s^*}\in Z_{\top/\Im}$ for each $w,s\in\top$. That is, $\overline{w\circ s}\in Z_{\top/\Im}$ for each $w,s\in\top$ and by using Lemma 2.2(i), we get $\top/\Im$ is a commutative integral domain. In case $\overline{\amalg(h)}=\overline{0}$. It follows that $\overline{\amalg(k)}=\overline{0}$. That is,

$$\overline{\amalg(z)}=\overline{0}\quad \text{for each } z\in Z_\top. \tag{43}$$

Replacing w by wk in (40), where $k\in S_\top\cap Z_\top\backslash\Im$ and using (43), we get

$$\overline{k}(\overline{\amalg(w\circ s^*)}-\overline{\amalg(s\circ w^*)}-\overline{w\circ s^*}+\overline{s\circ w^*})\in Z_{\top/\Im}\quad \text{for each } w,s\in\top. \tag{44}$$

Since $\overline{k}\in Z_\top/\Im\subseteq Z_{\top/\Im}$, we obtain $\overline{k}\in Z_{\top/\Im}$. Multiplying (40) by $\overline{k}$ and then using it in (44), we have $\overline{k}(\overline{\amalg(w\circ s^*)}-\overline{w\circ s^*})\in Z_{\top/\Im}$ for each $w,s\in\top$, and since $\overline{k}\neq\overline{0}$, we find that $\overline{\amalg(w\circ s^*)}-\overline{w\circ s^*}\in Z_{\top/\Im}$ for each $w,s\in\top$, that is,

$$\overline{\amalg(w\circ s)}-\overline{w\circ s}\in Z_{\top/\Im}\quad \text{for each } w,s\in\top. \tag{45}$$

Replacing w with wr in (45), where $r\in\top$, we have

$$\begin{aligned}&(\overline{\amalg(w\circ s)}-\overline{w\circ s})\overline{r}+\overline{(w\circ s)\amalg(r)}\\ &+\overline{\amalg(w)[r,s]}+\overline{w\amalg([r,s])}+\overline{w[r,s]}\in Z_{\top/\Im}\quad \text{for each } w,s,r\in\top.\end{aligned} \tag{46}$$

That is,

$$\overline{[(w \circ s)\amalg(r)} + \overline{\amalg(w)[r,s]} + \overline{w\amalg([r,s])} + \overline{w[r,s]}, \overline{r}] = \overline{0} \quad \text{for each } w, s, r \in \top.$$

Hence,

$$[(w \circ s)\amalg(r) + \amalg(w)[r,s] + w\amalg([r,s]) + w[r,s], r] \in \Im \quad \text{for each } w, s, r \in \top.$$

Putting $s = w = k$ in the last relation, we get $[2k^2\amalg(r), r] \in \Im$ for each $r \in \top$. Thus, $[\amalg(r), r] \in \Im$ for each $r \in \top$. Using [13, Lemma 2.2], we get $\amalg(\top) \subseteq \Im$ or $\top/\Im$ is a commutative integral domain. In case $\amalg(\top) \subseteq \Im$, using the last relation in (39) and Lemma 2.2, we conclude that $\top/\Im$ is a commutative integral domain.

Now, assume that

$$\overline{\amalg(w \circ w^*)} \in Z_{\top/\Im} \quad \text{for each } w \in \top. \tag{47}$$

By linearizing (47), we have

$$\overline{\amalg(w \circ s^*)} + \overline{\amalg(s \circ w^*)} \in Z_{\top/\Im} \quad \text{for each } w, s \in \top. \tag{48}$$

Assume that $0 \neq h \in H_\top \cap Z_\top \backslash \Im$. Replacing w by wh in (48), we get

$$\overline{h}(\overline{\amalg(w \circ s^*)} + \overline{\amalg(s \circ w^*)}) + (\overline{w \circ s^*} + \overline{s \circ w^*})\overline{\amalg(h)} \in Z_{\top/\Im} \quad \text{for each } w, s \in \top. \tag{49}$$

Multiplying (48) by $\overline{h}$ and then using it in (49), we find that

$$(\overline{w \circ s^*} + \overline{s \circ w^*})\overline{\amalg(h)} \in Z_{\top/\Im} \quad \text{for each } w, s \in \top. \tag{50}$$

Hence using similar arguments as used after Eq. (11), we find that $\overline{w \circ s} \in Z_{\top/\Im}$ for each $w, s \in \top$ or $\overline{\amalg(h)} = \overline{0}$. In case $\overline{w \circ s} \in Z_{\top/\Im}$ for each $w, s \in \top$ and by using Lemma 2.2(i), we get $\top/\Im$ is a commutative integral domain. Now, in case $\overline{\amalg(h)} = \overline{0}$, we get

$$\overline{\amalg(z)} = \overline{0} \quad \text{for each } z \in Z_\top. \tag{51}$$

Replacing w by wk in (48), where $k \in S_\top \cap Z_\top \backslash \Im$ and using (51), we get

$$\overline{k}(\overline{\amalg(w \circ s^*)} - \overline{\amalg(s \circ w^*)}) \in Z_{\top/\Im} \quad \text{for each } w, s \in \top. \tag{52}$$

Since $\overline{k} \in Z_{\top}/\Im \subseteq Z_{\top/\Im}$, we obtain $\overline{k} \in Z_{\top/\Im}$. Multiplying (48) by $\overline{k}$ and then using it in (52), we have $\overline{k}\overline{\amalg(w \circ s^*)} \in Z_{\top/\Im}$ for each $w, s \in \top$, and since $\overline{k} \neq \overline{0}$, we find that $\overline{\amalg(w \circ s^*)} \in Z_{\top/\Im}$ for each $w, s \in \top$, that is, $\overline{\amalg(w \circ s)} \in Z_{\top/\Im}$ for each $w, s \in \top$. Using the same technique as used after (45), we get $\amalg(\top) \subseteq \Im$ or $\top/\Im$ is a commutative integral domain. □

Theorem 3.5. *Let $\top$ be a ring, $\Im$ a prime ideal, $\Im$-second kind involution $*$, and $\top/\Im$ 2-torsion free. If $\amalg$ is a derivation of $\top$, then $\overline{\amalg(w \circ w^*)} + \overline{w \circ w^*} \in Z_{\top/\Im}$ for each $w \in \top$ if and only if $\top/\Im$ is a commutative integral domain.*

Proof. Using the same technics as in the preceding proof, it is obvious to see that $\overline{\amalg(w \circ w^*)} + \overline{w \circ w^*} \in Z_{\top/\Im}$ for each $w \in \top$ implies that $\top/\Im$ is a commutative integral domain. □

Theorem 3.6. *Let $\top$ be a ring, $\Im$ a prime ideal, $\Im$-second kind involution $*$, and $\top/\Im$ 2-torsion free. If $\amalg_1$ and $\amalg_2$ are derivation of $\top$, then $\overline{\amalg_1(w) \circ w^*} + \overline{w \circ \amalg_2(w^*)} - \overline{w \circ w^*} \in Z_{\top/\Im}$ for each $w \in \top$ if and only if $\top/\Im$ is a commutative integral domain. Moreover, if $\overline{\amalg_1(w) \circ w^* + w \circ \amalg_2(w^*)} \in Z_{\top/\Im}$ for each $w \in \top$, then $(\amalg_1(\top), \amalg_2(\top)) \subseteq (\Im, \Im)$ or $\top/\Im$ is a commutative integral domain.*

Proof. Assume that

$$\overline{\amalg_1(w) \circ w^*} + \overline{w \circ \amalg_2(w^*)} - \overline{w \circ w^*} \in Z_{\top/\Im} \quad \text{for each } w \in \top. \tag{53}$$

By linearizing (53), we have

$$\overline{\amalg_1(w) \circ s^*} + \overline{\amalg_1(s) \circ w^*} + \overline{w \circ \amalg_2(s^*)} + \overline{s \circ \amalg_2(w^*)} - \overline{w \circ s^*} - \overline{s \circ w^*} \in Z_{\top/\Im} \quad \text{for each } w, s \in \top. \tag{54}$$

Assume that $0 \neq h \in H_{\top} \cap Z_{\top} \backslash \Im$. Replacing w by wh in (54), we get

$$\overline{h}(\overline{\amalg_1(w) \circ s^*} + \overline{\amalg_1(s) \circ w^*} + \overline{w \circ \amalg_2(s^*)} + \overline{s \circ \amalg_2(w^*)} - \overline{w \circ s^*} - \overline{s \circ w^*}) \tag{55}$$

$$+\overline{(w \circ s^*)\amalg_1(h)} + \overline{(s \circ w^*)\amalg_2(h)} \in Z_{\top/\Im} \quad \text{for each } w, s \in \top.$$

Multiplying (54) by $\overline{h}$ and then using it in (55), we find that

$$\overline{(w \circ s^*)\amalg_1(h)} + \overline{(s \circ w^*)\amalg_2(h)} \in Z_{\top/\Im} \quad \text{for each } w, s \in \top. \tag{56}$$

Putting $s = w$, we get $\overline{(w \circ w^*)}(\overline{\amalg_1(h)} + \overline{\amalg_2(h)}) \in Z_{\top/\Im}$ for each $w \in \top$. Using Lemma 2.1, we have $\overline{w \circ w^*} \in Z_{\top/\Im}$ for each $w \in \top$ or $\overline{\amalg_1(h)} + \overline{\amalg_2(h)} = \overline{0}$. In case $\overline{w \circ w^*} \in Z_{\top/\Im}$ for each $w \in \top$ and by Lemma 2.2(i), we get $\top/\Im$ is a commutative integral domain. In case $\overline{\amalg_1(h)} + \overline{\amalg_2(h)} = \overline{0}$, we obtain

$$\overline{\amalg_2(h)} = -\overline{\amalg_1(h)}. \tag{57}$$

Using (57) in (56), we get $\overline{w \circ s^*}\,\overline{\amalg_1(h)} - \overline{s \circ w^*}\,\overline{\amalg_1(h)} \in Z_{\top/\Im}$ for each $w, s \in \top$. That is, $(\overline{w \circ s^*} - \overline{s \circ w^*})\overline{\amalg_1(h)} \in Z_{\top/\Im}$ for each $w, s \in \top$. Now, using the same arguments as used in (50), we get $\top/\Im$ is a commutative integral domain or $\overline{\amalg_1(z)} = \overline{0}$ for each $z \in Z_\top$. Using the last relation in (57), we get $\overline{\amalg_2(z)} = \overline{0}$ for each $z \in Z_\top$. Thus,

$$\overline{\amalg_1(z)} = \overline{0} = \overline{\amalg_2(z)} \quad \text{for each } z \in Z_\top. \tag{58}$$

Replacing w by wk in (54), where $k \in S_\top \cap Z_\top \backslash \Im$ and using (58), we get

$$\begin{aligned} &\overline{k}(\overline{\amalg_1(w) \circ s^*} - \overline{\amalg_1(s) \circ w^*} + \overline{w \circ \amalg_2(s^*)} \\ &-\overline{s \circ \amalg_2(w^*)} - \overline{w \circ s^*} + \overline{s \circ w^*}) \in Z_{\top/\Im} \quad \text{for each } w, s \in \top. \end{aligned} \tag{59}$$

Since $\overline{k} \in Z_\top/\Im \subseteq Z_{\top/\Im}$, we obtain $\overline{k} \in Z_{\top/\Im}$. Multiplying (54) by $\overline{k}$ and then using it in (59), we have $\overline{k}(\overline{\amalg_1(w) \circ s^*} + \overline{w \circ \amalg_2(s^*)} - \overline{w \circ s^*}) \in Z_{\top/\Im}$ for each $w, s \in \top$, and since $\overline{k} \neq \overline{0}$, we find that $\overline{\amalg_1(w) \circ s^*} + \overline{w \circ \amalg_2(s^*)} - \overline{w \circ s^*} \in Z_{\top/\Im}$ for each $w, s \in \top$, that is, $\overline{\amalg_1(w) \circ s} + \overline{w \circ \amalg_2(s)} - \overline{w \circ s} \in Z_{\top/\Im}$ for each $w, s \in \top$, and by using [13, Theorem 2.8], we get $\top/\Im$ is a commutative integral domain.

Now, assume that

$$\overline{\amalg_1(w) \circ w^*} + \overline{w \circ \amalg_2(w^*)} \in Z_{\top/\Im} \quad \text{for each } w \in \top. \tag{60}$$

By linearizing (60), we have

$$\overline{\amalg_1(w) \circ s^*} + \overline{\amalg_1(s) \circ w^*} + \overline{w \circ \amalg_2(s^*)} + \overline{s \circ \amalg_2(w^*)} \in Z_{\top/\Im} \quad \text{for each } w, s \in \top. \tag{61}$$

Assume that $0 \neq h \in H_\top \cap Z_\top \backslash \Im$. Replacing w by wh in (61), we get

$$\overline{h}(\overline{\amalg_1(w) \circ s^*} + \overline{\amalg_1(s) \circ w^*} + \overline{w \circ \amalg_2(s^*)} + \overline{s \circ \amalg_2(w^*)}) \\ + \overline{w \circ s^* \amalg_1(h)} + \overline{s \circ w^* \amalg_2(h)} \in Z_{\top/\Im} \quad \text{for each } w, s \in \top. \tag{62}$$

Multiplying (61) by $\overline{h}$ and then using it in (62), we find that $\overline{w \circ s^* \amalg_1(h)} + \overline{s \circ w^* \amalg_2(h)} \in Z_{\top/\Im}$ for each $w, s \in \top$. Now, use the similar arguments as used in (56), we get $\top/\Im$ is a commutative integral domain or (58). That is,

$$\overline{\amalg_1(z)} = \overline{0} = \overline{\amalg_2(z)} \quad \text{for each } z \in Z_\top. \tag{63}$$

Replacing w by wk in (61), where $k \in S_\top \cap Z_\top \backslash \Im$ and using (63), we get

$$\overline{k}(\overline{\amalg_1(w) \circ s^*} - \overline{\amalg_1(s) \circ w^*} + \overline{w \circ \amalg_2(s^*)} - \overline{s \circ \amalg_2(w^*)}) \in Z_{\top/\Im} \quad \text{for each } w, s \in \top. \tag{64}$$

Since $\overline{k} \in Z_\top/\Im \subseteq Z_{\top/\Im}$, we obtain $\overline{k} \in Z_{\top/\Im}$. Multiplying (61) by $\overline{k}$ and then using it in (64), we have $2\overline{k}(\overline{\amalg_1(w) \circ s^*} + \overline{w \circ \amalg_2(s^*)}) \in Z_{\top/\Im}$ for each $w, s \in \top$, and since $\overline{k} \neq \overline{0}$, we find that $\overline{\amalg_1(w) \circ s^*} + \overline{w \circ \amalg_2(s^*)} \in Z_{\top/\Im}$ for each $w, s \in \top$, that is, $\overline{\amalg_1(w) \circ s} + \overline{w \circ \amalg_2(s)} \in Z_{\top/\Im}$ for each $w, s \in \top$, and by using [13, Theorem 2.8], we get $(\amalg_1(\top), \amalg_2(\top)) \subseteq (\Im, \Im)$ or $\top/\Im$ is a commutative integral domain. $\square$

4 Illustrative Examples

In this section, we provide non-trivial examples to illustrate and validate the main concepts and results established in this paper. These examples demonstrate the applicability of the derived theorems, clarify the behavior of key notions, and highlight the significance of the introduced structures. By examining concrete cases, we offer deeper insight into the theoretical results, making them more accessible and comprehensible. The presented examples are designed to emphasize the role of prime ideals, derivations, and second-kind involutions within specific ring constructions.

Example 4.1. Let $\mathbb{C}$ be the set of complex numbers, $\mathbb{R}$ be the set of real numbers, and $i = \sqrt{-1}$. Let $\mathbb{C}[x]$ denote the ring of polynomials in x with coefficients in $\mathbb{C}$. Consider the ring $\top = \mathbb{C}[x] \times \mathbb{C}$ and the ideal $\Im = \mathbb{C}[x] \times (0)$. It follows that $\Im$ is a prime ideal of $\top$, and the quotient ring $\top/\Im$ is 2-torsion-free. Define a derivation $\amalg : \top \to \top$ by $\amalg(p(x), a) = (p'(x), 0)$ for all $(p(x), a) \in \top$, where $p'(x)$ denotes the derivative of $p(x)$. Define an involution $* : \top \to \top$ by $(p(x), a)^* = (\overline{p(x)}, \overline{a})$, where $\overline{p(x)}$ and $\overline{a}$ denote the complex conjugates of $p(x)$ and a, respectively. Then $*$ is a $\Im$-second kind involution. The set of skew-symmetric elements in $\top$ is $S_\top = i\mathbb{R}[x] \times i\mathbb{R}$, and the center of $\top$ is $Z_\top = \top$ The quotient ring $\top/\Im$ is a commutative integral domain, $\amalg(\top) \subseteq \Im$, and all identities presented in our results are satisfied.

Example 4.2. As in Example 4.1, let $\mathbb{C}, \mathbb{R}, i, \mathbb{C}[x]$ be as defined. Consider the ring $\top = \mathbb{C}[x] \times \mathbb{M}_2(\mathbb{C})$, where $\mathbb{M}_2(\mathbb{C})$ is the ring of 2×2 complex matrices, and let $\Im = \mathbb{C}[x] \times (0)$. Then $\Im$ is a prime ideal of $\top$, and the quotient ring $\top/\Im$ is 2-torsion-free. Define a derivation $\amalg : \top \to \top$ by $\amalg(p(x), A) = (p'(x), 0)$ for all $(p(x), A) \in \top$, where $p'(x)$ denotes the derivative of $p(x)$. Define an involution $* : \top \to \top$ by $(p(x), A)^* = (\overline{p(x)}, \overline{A^T})$, where $\overline{p(x)}$ denotes the complex conjugate of the polynomial $p(x)$, and $\overline{A^T}$ is the conjugate transpose of $A \in \mathbb{M}_2(\mathbb{C})$. Then $*$ is an involution. The set of skew-symmetric elements in $\top$ is

$$S_\top = i\mathbb{R}[x] \times \{A \in \mathbb{M}_2(\mathbb{C}) \mid \overline{A^T} = -A\}.$$

The center of $\top$ is

$$Z_\top = \mathbb{C}[x] \times \{aI_2 \mid a \in \mathbb{C}\},$$

where I_2 is the identity matrix. Then

$$S_\top \cap Z_\top = i\mathbb{R}[x] \times \{aI_2 \mid a \in i\mathbb{R}\} \not\subseteq \Im.$$

Therefore, $*$ is a $\Im$-second kind involution. The quotient ring $\top/\Im$ is not a commutative integral domain, $\amalg(\top) \subseteq \Im$, and the identities

1. $\overline{[\amalg(w), w^*]} \in Z_{\top/\Im}$ for each $w \in \top$, (Proposition 2.3),
2. $\overline{\amalg[w, w^*]} \in Z_{\top/\Im}$ for each $w \in \top$, (Theorem 2.4),
3. $\overline{[\amalg_1(w), w^*]} + \overline{[w, \amalg_2(w^*)]} \in Z_{\top/\Im}$ for each $w \in \top$, (Theorem 2.6),

4. $\overline{\text{II}(w) \circ w^*} \in Z_{\top/\Im}$ for each $w \in \top$, (Proposition 3.3),

5. $\overline{\text{II}(w \circ w^*)} \in Z_{\top/\Im}$ for each $w \in \top$, (Theorem 3.4), and

6. $\overline{\text{II}_1(w) \circ w^*} + \overline{w \circ \text{II}_2(w^*)} \in Z_{\top/\Im}$ for each $w \in \top$, (Theorem 3.6)

are satisfied.

Here, we will give the following example to show that if the condition of $\Im$-second kind involution is not imposed, then our results may not be true

Example 4.3. Let $\top = M_2(\mathbb{Z})$ be a ring, $\Im = (0)$ a prime ideal of $\top$, $B = \begin{pmatrix} s & r \\ t & u \end{pmatrix} \in \top$, and $*_1 : \top \to \top$ a $\Im$-first kind involution such that $B^{*_1} = \begin{pmatrix} u & -r \\ -t & s \end{pmatrix}$, where $s, r, t, u \in \mathbb{Z}$. Let $\text{II} : \top \to \top$ be any inner derivation. Then $\overline{\text{II}([B, B^{*_1}])} \pm \overline{[B, B^{*_1}]} \in Z_{\top/\Im}$, $\overline{\text{II}([B, B^{*_1}])} \in Z_{\top/\Im}$, $\overline{[\text{II}(B), B^{*_1}]} + \overline{[B, \text{II}(B^{*_1})]} - \overline{[B, B^{*_1}]} \in Z_{\top/\Im}$, $\overline{[\text{II}(B), B^{*_1}]} + \overline{[B, \text{II}(B^{*_1})]} \in Z_{\top/\Im}$, $\overline{\text{II}(B \circ B^{*_1})} \pm \overline{B \circ B^{*_1}} \in Z_{\top/\Im}$, $\overline{\text{II}(B \circ B^{*_1})} \in Z_{\top/\Im}$, $\overline{\text{II}(B) \circ B^{*_1}} + \overline{B \circ \text{II}(B^{*_1})} - \overline{B \circ B^{*_1}} \in Z_{\top/\Im}$, and $\overline{\text{II}(B) \circ B^{*_1}} + \overline{B \circ \text{II}(B^{*_1})} \in Z_{\top/\Im}$ for each $B \in \top$.

We demonstrate the necessity of the condition of the primeness in our results through the following illustrative example.

Example 4.4. Let $\top$, $*_1$, and B be as in Example 4.3, $S = \top \times \mathbb{C}$ a ring, $\Im = (0) \times (0)$ an ideal of S, $Z_{S/\Im} = Z_\top \times \mathbb{C}$, $W = B \times v \in S$, where $v = q_1 + iq_2$ and $q_1, q_2 \in \mathbb{R}$ and $*_2 : \mathbb{C} \to \mathbb{C}$ a second kind involution such that $v^{*_2} = q_1 - iq_2$, and $* : S \to S$ a $\Im$-second kind involution such that $W^* = B^{*_1} \times v^{*_2}$. Let $\text{II} : S \to S$ be any inner derivation. Then $\overline{\text{II}([W, W^*])} \pm \overline{[W, W^*]} \in Z_{S/\Im}$, $\overline{\text{II}([W, W^*])} \in Z_{S/\Im}$, $\overline{[\text{II}(W), W^*]} + \overline{[W, \text{II}(W^*)]} - \overline{[W, W^*]} \in Z_{S/\Im}$, $\overline{[\text{II}(W), W^*]} + \overline{[W, \text{II}(W^*)]} \in Z_{S/\Im}$, $\overline{\text{II}(W \circ W^*)} \pm \overline{W \circ W^*} \in Z_{S/\Im}$, $\overline{\text{II}(W \circ W^*)} \in Z_{S/\Im}$, $\overline{\text{II}(W) \circ W^*} + \overline{W \circ \text{II}(W^*)} - \overline{W \circ W^*} \in Z_{S/\Im}$, and $\overline{\text{II}(W) \circ W^*} + \overline{W \circ \text{II}(W^*)} \in Z_{S/\Im}$ for each $W \in S$.

Conclusion

In this paper, we have delved into the intricate interplay between derivations, involutions, and the commutativity of prime quotient rings. We have significantly generalized classical results, such as Posner's theorem, by examining these concepts within a broader context of associative rings equipped with second-kind involutions. Our work establishes new criteria for determining the commutativity of quotient rings $\top/\Im$, focusing on the pivotal roles of Lie and Jordan products.

Our findings demonstrate that specific identities involving derivations and involutions are critical in characterizing the structure of $\top/\Im$. In particular, we have shown that the conditions $\overline{[\Pi(w), w^*]} \in Z_{\top/\Im}$ and $\overline{\Pi[w, w^*]} - \overline{[w, w^*]} \in Z_{\top/\Im}$ (Theorem 2.4, Proposition 2.3) for derivations and their relation to commutativity, provide novel analytical pathways to uncover the structure of $\top/\Im$. Similarly, the investigation of Jordan products (Theorem 3.4, Proposition 3.3), through conditions such as $\overline{\Pi(w \circ w^*)} - \overline{w \circ w^*} \in Z_{\top/\Im}$ and $\overline{\Pi(w) \circ w^*} \in Z_{\top/\Im}$, reveal the relationship between derivations, involution and commutativity of $\top/\Im$, when the derivation operates on the Jordan products of elements within the ring. Moreover, the identities for two derivations are explored in Theorem 2.6 and Theorem 3.6, where the structure of quotient ring is characterized through conditions involving the actions of two derivations on the Lie and Jordan products respectively. Specifically, our work has shown that if these identities are fulfilled, then $\top/\Im$ is a commutative integral domain, or the derivations map the whole ring into the prime ideal.

The results presented here highlight the significance of the imposed conditions on the derivations and involutions, and the type of product they operate on. The theoretical analysis developed in this paper provides valuable tools for studying the structure of noncommutative rings and their quotient structures.

It is essential to acknowledge that our results are contingent on the assumption of prime ideals, the rings to be 2-torsion free and the presence of second-kind involutions. Relaxing these constraints presents significant challenges but also opens up exciting opportunities for future research. Future investigations could focus on developing analogous results for more general classes of rings, such as semiprime quotient rings and rings with different types of involutions. Furthermore, we anticipate that our framework can motivate the exploration of computational methods for testing the commutativity conditions developed in this paper, particularly within the context of symbolic computation in noncommutative polynomial rings. Additionally, the study of these concepts within the context of $*$-algebras might bring new insights to functional analysis and operator theory.

The broader goal of this work is to improve our understanding of the interplay between derivations, involutions, and the underlying ring structure. The theoretical insights we have provided pave the way for further exploration in the landscape of noncommutative algebra and its diverse applications.

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