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On the Parseval Continuous K-g-Frames and Their Duals

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Abstract. In this article, we explore the stability properties of continuous K-g-frames and provide various characterizations related to synthesis and frame operators. Additionally, we extend certain results from continuous g-frames to tight and parseval continuous K-g-frames. Furthermore, we establish a continuous K-g-dual for Parseval continuous K-g-frames in Hilbert spaces, along with an analysis of their dual properties.

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1 Introduction

The concept of frames in Hilbert spaces was first introduced by Duffin and Schaeffer [12], when they studied some problems in nonharmonic Fourier series in 1952. Daubechies [9] reintroduced the frames as a

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generalization of orthonormal bases. Frames are very important and useful in signal processing, image processing, sampling theory and coding theory; for more details see [6, 8, 10].
Note that a sequence $\{f_n\}_{n=1}^{\infty}$ in a separable Hilbert space H is called a Bessel sequence if there exists $B > 0$ such that

$$\sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leq B\|f\|^2, \qquad f \in H.$$

By Theorem 3.2.3 of [7], $\{f_n\}_{n=1}^{\infty}$ is a Bessel sequence if and only if the mapping, definde by

$$T : l^2 \longrightarrow H, \quad T(\{c_n\}_{n=1}^{\infty}) := \sum_{n=1}^{\infty} c_n f_n,$$

is a well-defined bounded linear operator with $\| T \| \leq \sqrt{B}$. Then T is called the synthesis operator of $\{f_n\}_{n=1}^{\infty}$. So, the adjoint operator of T is given by

$$T^*(f) = \{\langle f, f_n\rangle\}_{n=1}^{\infty}, \qquad f \in H,$$

and is called the analysis operator of $\{f_n\}_{n=1}^{\infty}$.
A sequence $F = \{f_n\}_{n=1}^{\infty}$ in H is called a frame for H, if there exist constants A, $B > 0$ such that

$$A\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leq B\|f\|^2, \qquad f \in H.$$

The constants A and B are called the lower and upper frame bounds, respectively. We say that a frame is tight (resp. Parseval) if one can choose $A = B$ (resp. $A = B = 1$). The frame operator of a frame $\{f_n\}_{n=1}^{\infty}$ is defined by

$$S : H \longrightarrow H, \quad Sf = TT^*f = \sum_{n=1}^{\infty} \langle f, f_n\rangle f_n, \qquad f \in H,$$

which is a bounded, positive and invertible operator; see Lemma 5.1.5 of [7].

The important of paying attention to this issue, caused different generalizations of frames to be proposed. In 2006, Sun [23] introduced g-frames as a generalization of discreate frames. He gave a characterization of g-frames and studied g-Riesz bases, g-orthonormal bases and dual g-frames. Continuous frames as another generalization of frames with respect to a family of operators indexed by a locally compact space endowed with a Radon measure, was proposed by Ali et al [3]. Some another authors investigated continuous frame theory and discussed the dual of continuous frames, continuous frame operator and pre-frame operator, etc [14, 16, 22].

In 2012, K-frames was introduced by Găvruta [15], as a new generalization of frames. It is a frame for the range of a bounded linear operator K on a Hilbert space H, that reconstruct the members of range of K. Recently, Abtahi et al [2] introduced and studied the space of all operators $K \in B(H)$, such that for a given Bessel sequence $\{f_n\}_{n=1}^{\infty}$, to be a K-frame. They showed that in a separable Hilbert space H, if K, L_1 and L_2 belong to $B(H)$ such that $R(K) \subseteq R(L_1K) \cap R(L_2K)$ and $F = \{f_n\}_{n=1}^{\infty}$ and $G = \{g_n\}_{n=1}^{\infty}$ are two strongly disjoint K-frames in H, then $L_1F + L_2G$ is a K-frame for H. We refer the reader to [2, 11, 17, 18, 24].

Alizadeh et al in [4] gave the equivalent characterization of continuous K-g-frames (c-K-g-frames) for H and they showed that $\{\Lambda_\omega K^*\}_{\omega\in\Omega}$ is a c-K-g-frame for H, where $K \in B(H)$ and $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$. Moreover, they proved that $\{\Lambda_\omega T^*\}_{\omega\in\Omega}$ is a c-TK-g-frame for H, if $T, K \in B(H)$ and $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$. Also, they introduced the perturbation of c-K-g-frames and proved that if $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K-g-frame for H and $\{\Gamma_\omega\}_{\omega\in\Omega}$ is a family of strongly measurable operators, then $\{\Gamma_\omega\}_{\omega\in\Omega}$ is a c-K-g-frame for H.

In this article, we generalize some results from K-frames and c-g-frames to tight and Parseval c-K-g-frames. Then by remebering the concept of c-K-g-duals, we obtain some results about them. Throughout this article, (Ω, μ) is a measure space, H, H_1 and H_2 are separable complex Hilbert spaces and $B(H_1, H_2)$ is the Banach space of all bounded operators from H_1 into H_2. We use $B(H)$ as the abbreviation of $B(H, H)$. Also, $R(K)$ denotes the range of the operator $K \in B(H)$ and $\{H_\omega\}_{\omega\in\Omega}$

is a family of Hilbert spaces. At first, we review some definitions of generalized frames.

Definition 1.1. (see [21]) *Let $K \in B(H)$ be a bounded operator. A sequence $\{f_n\}_{n=1}^{\infty}$ in H is called a K-frame for H, if there exist two positive constants A and B such that*

$$A\|K^*f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leq B\|f\|^2, \qquad f \in H.$$

If $K = I$ is the identity operator on H, then K-frames are ordinary frames.

Definition 1.2. (see [4]) *A mapping $F = \{f_\omega\}_{\omega\in\Omega}$ from a measure space Ω into $\bigcup_{\omega\in\Omega} H_\omega$ is called strongly measurable, if F as a mapping from Ω to $\bigoplus_{\omega\in\Omega} H_\omega$ to be measurable.*

Definition 1.3. (see [4]) *Let $K \in B(H)$. A family $\{\Lambda_\omega \in B(H, H_\omega) : \omega \in \Omega\}$ is called a continuous K-g-frame (c-K-g-frame) for H with respect to $\{H_\omega\}_{\omega\in\Omega}$, if*

(i) $\{\Lambda_\omega f\}_{\omega\in\Omega}$ is strongly measurable, for each $f \in H$;

(ii) There exist constants $0 < A \leq B < \infty$, such that

$$A\|K^*f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) \leq B\|f\|^2, \qquad f \in H.$$

In particular, if $K = I$ is the identity operator on H, then Λ is called a continuous g-frame (c-g-frame) for H.

As usual, the constants A, B are called the lower and upper c-K-g-frame bounds, respectively. If $A\|K^*f\|^2 = \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega)$, for all $f \in H$, then $\{\Lambda_\omega\}_{\omega\in\Omega}$ is called a tight c-K-g-frame and every tight c-K-g-frame with $A = 1$, is called Parseval c-K-g-frame.
For a c-g-Bessel family (or c-K-g-frame) $\{\Lambda_\omega\}_{\omega\in\Omega}$, the c-g (or frame) operator $S : H \longrightarrow H$ is defined by

$$\langle Sf, g\rangle = \int_\Omega \langle {\Lambda_\omega}^* \Lambda_\omega f, g\rangle d\mu(\omega), \qquad f, g \in H.$$

It is obvious that $\langle Sf, f\rangle = \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega)$, for all $f \in H$, and so S is a positive operator. In [4], the authors characterized a c-K-g-frame by frame operator. Also they showed that, under some conditions, the frame operator of a c-K-g-frame is invertible on the closed subspace range $R(K)$ of K. In the following, we construct some new c-K-g-frames and then we generalize some results from K-frames and c-g-frames to tight and Parseval c-K-g-frames. In particular, according to bounded operators $K_1 \in B(H_1)$, $K_2 \in B(H_2)$ and $T \in B(H_1, H_2)$, if $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K_1-g-frame for H_1, we provide some conditions such that $\{\Lambda_\omega T^*\}_{\omega\in\Omega}$ is a c-K_2-g-frame for H_2. Also we prove that any c-K-g-dual of a Parseval c-K-g-frame $\{\Lambda_\omega\}_{\omega\in\Omega}$ is both a Parseval c-$KK^\dagger$-g-frame and a Parseval c-g-frame, where $K^\dagger$ is the Pseudo-inverse of the operator K [7].

2 Constructing New c-K-g-Frames

In the following, according two bounded operators K_1 and K_2 between Hilbert spaces and a given c-K_1-g-frame, we construct some new c-K_2-g-frames. These extend the results of [2]. First, we give the following theorem.

Theorem 2.1. *Let K_1, $K_2 \in B(H)$, α be a scaler and suppose that $\{\Lambda_\omega\}_{\omega\in\Omega}$ is both a c-K_1-g-frame and c-K_2-g-frame. Then $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-αK_1-g-frame, c-K_1K_2-g-frame and c-$(K_1 + K_2)$-g-frame.*

Proof. For simplicity, it can be assumed that the lower and upper bounds of any two frames are equal. So by assumption, there exist positive constants A and B such that

$$A\|K_1{}^*f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) \leq B\|f\|^2, \tag{1}$$

and

$$A\|K_2{}^*f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) \leq B\|f\|^2, \tag{2}$$

for all $f \in H$. For $\alpha = 0$, the implication is trivial. If $\alpha \neq 0$, then by (1),

$$\frac{A}{|\alpha|^2}\|(\alpha K_1)^* f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega).$$

So $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-αK_1-g-frame. Moreover

$$\begin{aligned}\|(K_1K_2)^*f\|^2 &= \|{K_2}^*({K_1}^*f)\|^2\\ &\leq \|K_2\|^2\|{K_1}^*f\|^2\\ &\leq \frac{\|K_2\|^2}{A}\int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega).\end{aligned}$$

Thus

$$\frac{A}{\|K_2\|^2}\|(K_1K_2)^*f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega),$$

for all $f\in H$, and $\{\Lambda_\omega\}_{\omega\in}$ is a c-K_1K_2-g-frame.
Finally, by (1), (2) and Cauchy-Schwarz inequality we have

$$\begin{aligned}\|(K_1+K_2)^*f\|^2 &= \|{K_1}^*f+{K_2}^*f\|^2\\ &\leq \|{K_1}^*f\|^2+\|{K_2}^*f\|^2+2|\langle {K_1}^*f,{K_2}^*f\rangle|\\ &\leq \|{K_1}^*f\|^2+\|{K_2}^*f\|^2+2\|{K_1}^*f\|\|{K_2}^*f\|\\ &\leq \frac{1}{A}\int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega)+\frac{1}{A}\int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega)\\ &+ \frac{2}{A}\int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega).\end{aligned}$$

Thus we get

$$\frac{A}{4}\|(K_1+K_2)^*f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega),$$

for any $f\in H$. It follows that $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-(K_1+K_2)-g-frame. □

In [19, 20], the authors have proved the construction of K-g-frames in discrete case. They also investigated g-frame representations via a linear operator. Now, we show that for a c-K-g-frame for H, according to any operator $T\in B(H)$, one can construct some new c-K-g-frames.

Theorem 2.2. *Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a c-K-g-frame for H and $T\in B(H)$. If T^*KK^* is a positive operator, then $\{\Lambda_\omega+\Lambda_\omega T\}_{\omega\in\Omega}$ is a c-K-g-frame for H.*

Proof. By assumption, there exist positive constants A and B such that for every f in the space H,

$$A\|K^*f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) \leq B\|f\|^2.$$

So for any $f \in H$, we have

$$\begin{aligned} A\|K^*(I+T)f\|^2 &\leq \int_\Omega \|\Lambda_\omega (I+T)f\|^2 d\mu(\omega) \\ &\leq B\|(I+T)f\|^2 \\ &\leq B\|I+T\|^2\|f\|^2. \end{aligned}$$

Since T^*KK^* is positive,

$$\begin{aligned} A\|K^*(I+T)f\|^2 &= A(\|K^*f\|^2 + 2Re\langle K^*f, K^*Tf\rangle + \|K^*Tf\|^2) \\ &\geq A(\|K^*f\|^2 + \|K^*Tf\|^2) \\ &\geq A\|K^*f\|^2. \end{aligned}$$

So

$$A\|K^*f\|^2 \leq \int_\Omega \|\Lambda_\omega (I+T)f\|^2 d\mu(\omega) \leq B\|I+T\|^2\|f\|^2, \quad f \in H,$$

and $\{\Lambda_\omega + \Lambda_\omega T\}_{\omega\in\Omega}$ is a c-K-g-frame for H. $\square$

Theorem 2.3. *Suppose that $K_1 \in B(H_1)$ and $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K_1-g-frame for H_1. Let $K_2 \in B(H_2)$, $T \in B(H_1, H_2)$ has closed range and $TK_1 = K_2T$. If $R(K_2{}^*) \subset R(T)$, then $\{\Lambda_\omega T^*\}_{\omega\in\Omega}$ is a c-K_2-g-frame for H_2.*

Proof. Since $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K_1-g-frame for H_1, there exist positive constants A, B such that

$$A\|K_1{}^*f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) \leq B\|f\|^2, \quad f \in H_1.$$

Assume that $h \in H_2$ is arbitrary. We obtain

$$\begin{aligned} A\|K_1{}^*T^*h\|^2 &\leq \int_\Omega \|\Lambda_\omega T^*h\|^2 d\mu(\omega) \\ &\leq B\|T^*h\|^2 \\ &\leq B\|T\|^2\|h\|^2. \end{aligned}$$

Since $TK_1 = K_2T$, $R(K_2{}^*) \subset R(T)$ and $R(T)$ is closed, by Lemma 2.2 of [21], for any $h \in H_2$ we have

$$\|K_1{}^*T^*h\|^2 = \|T^*K_2{}^*h\|^2 \geq \|T^\dagger\|^{-2}\|K_2{}^*h\|^2,$$

where $T^\dagger$ is the Pseudo-inverse of the operator T. Thus

$$\begin{aligned} A\|T^\dagger\|^{-2}\|K_2{}^*h\|^2 &\leq \int_\Omega \|\Lambda_\omega T^*h\|^2 d\mu(\omega) \\ &\leq B\|T\|^2\|h\|^2. \end{aligned}$$

That is, $\{\Lambda_\omega T^*\}_{\omega\in\Omega}$ is a c-K_2-g-frame for H_2. □

Corollary 2.4. *Let $K \in B(H)$ and $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a c-K-g-frame for H. Let $T \in B(H)$ with closed range and $TK = KT$. If $R(K^*) \subset R(T)$, then $\{\Lambda_\omega T^*\}_{\omega\in\Omega}$ is a c-K-g-frame for H.*

Recall that an operator $T \in B(H_1, H_2)$ is bounded below, if there exists a constant $c > 0$ such that

$$\| Th \| \geq c \| h \|,$$

for all $h \in H_1$. It is straightforward to see that the operator T is surjective, if and only if, T^* is bounded below, if and only if T^* is injective and has closed range (see for instance, Lemma 2.4.1 from [7]).

Theorem 2.5. *Let $K_1 \in B(H_1)$, $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a tight c-K_1-g-frame for H_1 and $K_2 \in B(H_2)$ be a bounded below operator. Let $T \in B(H_1, H_2)$ such that $TK_1 = K_2T$. Then $\{\Lambda_\omega T^*\}_{\omega\in\Omega}$ is a c-K_2-g-frame for H_2 if and only if T is surjective.*

Proof. If T is a surjective operator, then by Corollary 3.6 of [21], $\{\Lambda_\omega T^*\}_{\omega\in\Omega}$ is a c-K_2-g-frame for H_2.
Conversely, if $\{\Lambda_\omega T^*\}_{\omega\in\Omega}$ is a c-K_2-g-frame for H_2 with lower bound A_1, then for any $h \in H_2$:

$$A_1\|K_2{}^*h\|^2 \leq \int_\Omega \|\Lambda_\omega T^*h\|^2 d\mu(\omega).$$

Since $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a tight c-K_1-g-frame for H_1 with bound A, then

$$A\|K_1{}^*f\|^2 = \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega), \qquad f \in H_1.$$

Moreover since ${K_1}^*T^* = T^*{K_2}^*$, for any $h \in H_2$ we have

$$\begin{aligned} A\|T^*{K_2}^*h\|^2 &= A\|{K_1}^*T^*h\|^2 \\ &= \int_\Omega \|\Lambda_\omega T^*h\|^2 d\mu(\omega) \\ &\geq A_1\|{K_2}^*h\|^2, \end{aligned}$$

and so

$$\|T^*{K_2}^*h\|^2 \geq \tfrac{A_1}{A}\|{K_2}^*h\|^2, \qquad h \in H_2.$$

By the fact that $K_2 \in B(H_2)$ is bounded below, ${K_2}^*$ is surjective. Now, the above inequality shows that T^* is bounded below and so T is surjective. This completes the proof. $\square$

Now, we obtain a new tight c-K-g-frame. We need the following two lemmas:

Lemma 2.6. *Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ and $\{\Gamma_\omega\}_{\omega\in\Omega}$ be two c-g-Bessel families with bounds B_1 and B_2 for H, with respect to $\{H_\omega\}_{\omega\in\Omega}$ and $L_1, L_2 \in B(H, H_1)$. Then $\{\Lambda_\omega {L_1}^* + \Gamma_\omega {L_2}^*\}_{\omega\in\Omega}$ is a c-g-Bessel family for H_1 with respect to $\{H_\omega\}_{\omega\in\Omega}$.*

Proof. $\{\Lambda_\omega {L_1}^* + \Gamma_\omega {L_2}^*\}_{\omega\in\Omega}$ is a c-g-Bessel family for H_1 if for each $f \in H_1$:

$$\begin{aligned} \int_\Omega \|(\Lambda_\omega {L_1}^* + \Gamma_\omega {L_2}^*)f\|^2 d\mu(\omega) &\leq 2\int_\Omega \|(\Lambda_\omega {L_1}^*)f\|^2 d\mu(\omega) \\ &+ 2\int_\Omega \|(\Gamma_\omega {L_2}^*)f\|^2 d\mu(\omega) \\ &\leq 2B_1\|{L_1}^*f\|^2 + 2B_2\|{L_2}^*f\|^2 \\ &\leq 2B_1\|L_1\|^2\|f\|^2 + 2B_2\|L_2\|^2\|f\|^2 \\ &\leq 2(\sqrt{B_1}\|L_1\| + \sqrt{B_2}\|L_2\|)^2\|f\|^2, \end{aligned}$$

and the proof is completed. $\square$

Lemma 2.7. *Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a c-g-Bessel family for H. Then $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a tight c-K-g-frame for H with frame operator S if and only if there exists a constant $A > 0$ such that $S = AKK^*$.*

Proof. The family $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a tight c-K-g-frame if and only if,

$$A\|K^*f\|^2 = \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega),$$

for all $f \in H$ and some $A > 0$. Since

$$A\|K^*f\|^2 = \langle AKK^*f, f\rangle$$

and

$$\int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) = \langle Sf, f\rangle,$$

for all $f \in H$, one can see that $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a tight c-K-g-frame for H, if and only if

$$\langle Sf, f\rangle = \langle AKK^*f, f\rangle,$$

for all $f \in H$. The positivity of the operators S and AKK^* implies that the last assertion is equivalent to $S = AKK^*$. □

For the following theorem, we also need the concept of c-K-g-dual of a c-K-g-frame, which will be introduced in section 3 from [5].

Theorem 2.8. *Suppose that $K \in B(H)$ and $L_1, L_2 \in B(H, H_1)$. If $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K-g-frame with frame operator S_Λ and $\Gamma = \{\Gamma_\omega\}_{\omega\in\Omega}$ is its c-K-g-dual with frame operator S_Γ, then $\{\Lambda_\omega {L_1}^* + \Gamma_\omega {L_2}^*\}_{\omega\in\Omega}$ is a tight c-K-g-frame with respect to $\{H_\omega\}_{\omega\in\Omega}$ for H_1, if and only if there exists constant $A > 0$ such that*

$$AKK^* = L_1 S_\Lambda {L_1}^* + L_1 K {L_2}^* + L_2 K^* {L_1}^* + L_2 S_\Gamma {L_2}^*.$$

Proof. By Lemma 2.6, $\{\Lambda_\omega {L_1}^* + \Gamma_\omega {L_2}^*\}_{\omega\in\Omega}$ is a c-g-Bessel family for H_1. If S is the frame operator of this family, then for any $f \in H_1$:

$$\begin{aligned} Sf &= \int_\Omega (\Lambda_\omega {L_1}^* + \Gamma_\omega {L_2}^*)^*(\Lambda_\omega {L_1}^* + \Gamma_\omega {L_2}^*) f d\mu(\omega) \\ &= \int_\Omega (L_1 {\Lambda_\omega}^* + L_2 {\Gamma_\omega}^*)(\Lambda_\omega {L_1}^* + \Gamma_\omega {L_2}^*) f d\mu(\omega). \end{aligned}$$

By the Lemma 2.7 from [13], we have

$$\begin{aligned} Sf &= L_1 \int_\Omega {\Lambda_\omega}^* \Lambda_\omega {L_1}^* f d\mu(\omega) + L_1 \int_\Omega {\Lambda_\omega}^* \Gamma_\omega {L_2}^* f d\mu(\omega) \\ &+ L_2 \int_\Omega {\Gamma_\omega}^* \Lambda_\omega {L_1}^* f d\mu(\omega) + L_2 \int_\Omega {\Gamma_\omega}^* \Gamma_\omega {L_2}^* f d\mu(\omega) \\ &= L_1 S_\Lambda {L_1}^* f + L_1 K {L_2}^* f + L_2 K^* {L_1}^* f + L_2 S_\Gamma {L_2}^* f. \end{aligned}$$

Then by Lemma 2.7, the family $\{\Lambda_\omega {L_1}^* + \Gamma_\omega {L_2}^*\}_{\omega\in\Omega}$ is a tight c-K-g-frame if and only if, there exists constant $A > 0$ such that $S = AKK^*$, or equivalently,

$$AKK^* = L_1 S_\Lambda {L_1}^* + L_1 K {L_2}^* + L_2 K^* {L_1}^* + L_2 S_\Gamma {L_2}^*,$$

and the proof is completed. □

3 c-K-g-Duals

In this section, it is introduced the c-K-g-dual frame of a given c-K-g-frame and we obtain c-K-g-dual of Parseval c-K-g-frames. Then we obtain synthesis operator related to c-K-g-duals. The following theorem is an important result in the study of the c-K-g-duals. We refer to [4, 5] for details.

Theorem 3.1. *Let $K \in B(H)$. Then the following statements are equivalent.*

(i) $\{\Lambda_\omega\}_{\omega\in\Omega}$ *is a c-K-g-frame for H with respect to* $\{H_\omega\}_{\omega\in\Omega}$.

(ii) $\{\Lambda_\omega\}_{\omega\in\Omega}$ *is a c-g-Bessel family for H with respect to* $\{H_\omega\}_{\omega\in\Omega}$ *and there exists a c-g-Bessel family* $\{\Gamma_\omega\}_{\omega\in\Omega}$ *for H with respect to* $\{H_\omega\}_{\omega\in\Omega}$ *such that,*

$$\langle Kf, h\rangle = \int_\Omega \langle {\Lambda_\omega}^* \Gamma_\omega f, h\rangle d\mu(\omega), \qquad f, h \in H.$$

So, for any c-K-g-frame $\{\Lambda_\omega\}_{\omega\in\Omega}$, we have a c-g-Bessel family

$$\{\Gamma_\omega \in B(H, H_\omega) : \omega \in \Omega\}$$

such that

$$Kf = \int_\Omega {\Lambda_\omega}^* \Gamma_\omega f d\mu(\omega), \qquad f \in H,$$

and

$$K^* f = \int_\Omega {\Gamma_\omega}^* \Lambda_\omega f d\mu(\omega), \qquad f \in H.$$

The family $\{\Gamma_\omega\}_{\omega\in\Omega}$ is called a c-K-g-dual of $\{\Lambda_\omega\}_{\omega\in\Omega}$ and it is a c-K^*-g-frame. It is obvious that, under the above assumptions, $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K^*-g-dual of $\{\Gamma_\omega\}_{\omega\in\Omega}$ and if K is a self adjoint operator, $\{\Lambda_\omega\}_{\omega\in\Omega}$ and $\{\Gamma_\omega\}_{\omega\in\Omega}$ are called c-K-g-dual pairs.

Theorem 3.2. *Suppose that $K \in B(H)$ such that K^* is surjective and $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a Parseval c-K-g-frame for H. Then $\{\Lambda_\omega(K^*)^\dagger\}_{\omega\in\Omega}$ is a c-K-g-dual of $\{\Lambda_\omega\}_{\omega\in\Omega}$. Moreover, $\{\Lambda_\omega(K^*)^\dagger\}_{\omega\in\Omega}$ is both a Parseval c-$K^\dagger K$-g-frame and Parseval c-g-frame.*

Proof. Assume that $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a Parseval c-K-g-frame. Then

$$\langle K^*f, K^*f\rangle = \int_\omega \langle \Lambda_\omega f, \Lambda_\omega f\rangle d\mu(\omega), \qquad f \in H,$$

and we have

$$KK^*f = \int_\Omega {\Lambda_\omega}^* \Lambda_\omega f d\mu(\omega), \qquad f \in H.$$

Since K^* is surjective,

$$f = K^*(K^*)^\dagger f = K^*{(K^\dagger)}^* f, \tag{3}$$

for any $f \in H$, and so

$$Kf = KK^*{(K^\dagger)}^* f = \int_\Omega {\Lambda_\omega}^* \Lambda_\omega {(K^\dagger)}^* f d\mu(\omega). \tag{4}$$

Therefore, $\{\Lambda_\omega{(K^\dagger)}^*\}_{\omega\in\Omega} = \{\Lambda_\omega(K^*)^\dagger\}_{\omega\in\Omega}$ is a c-K-g-dual of $\{\Lambda_\omega\}_{\omega\in\Omega}$.

Now, if we product both sides of the relation (4) by $(K^\dagger)^* f$, then for any $f \in H$ we have

$$\int_\Omega \| \Lambda_\omega{(K^\dagger)}^* f \|^2 \, d\mu(\omega) = \| K^*{(K^\dagger)}^* f \|^2 = \| {(K^\dagger K)}^* f \|^2, \tag{5}$$

and $\{\Lambda_\omega{(K^\dagger)}^*\}_{\omega\in\Omega}$ is a Parseval c-$K^\dagger K$-g-frame for H. Finally, by (3), the operator $K^*(K^*)^\dagger$ and so $K^\dagger K$ is the identity operator on H, and so (5) implies that $\{\Lambda_\omega(K^*)^\dagger\}_{\omega\in\Omega}$ is a Parseval c-g-frame. □

Definition 3.3. *A c-g-Bessel sequence* $\{\Lambda_\omega\}_{\omega\in\Omega}$ *in* H *is independent, if for a c-g-Bessel sequence* $\{\Psi_\omega\}_{\omega\in\Omega}$ *in* H *and all* $f \in H$,

$$\int_\Omega {\Lambda_\omega}^* \Psi_\omega f d\mu(\omega) = 0,$$

implies that $\Psi_\omega = 0$ *a.e.*

In the followig, we inspect the independent relation between a Parseval c-K-g-frame $\{\Lambda_\omega\}_{\omega\in\Omega}$ and $\{\Lambda_\omega (K^\dagger)^*\}_{\omega\in\Omega}$ as its c-K-g-dual.

Theorem 3.4. *Suppose that* $K \in B(H)$ *such that* K^* *is surjective and* $\{\Lambda_\omega\}_{\omega\in\Omega}$ *is a Parseval c-K-g-frame for* H*. Then* $\{\Lambda_\omega\}_{\omega\in\Omega}$ *is independent if and only if its c-K-g-dual* $\{\Lambda_\omega (K^\dagger)^*\}_{\omega\in\omega}$ *is independent.*

Proof. Assume that $\{\Lambda_\omega\}_{\omega\in\Omega}$ is independent. By Lemma 2.7, the frame operator related to $\{\Lambda_\omega\}_{\omega\in\Omega}$ equals to KK^*. Since $K^\dagger$ is the Pseudo-inverse of the operator K, then

$$\int_\Omega {\Lambda_\omega}^* \Lambda_\omega f d\mu(\omega) = \int_\Omega {\Lambda_\omega}^* \Lambda_\omega (K^\dagger)^* K^* f d\mu(\omega), \qquad f \in H.$$

Therefore

$$\int_\Omega {\Lambda_\omega}^* (\Lambda_\omega - \Lambda_\omega (K^\dagger)^* K^*) f d\mu(\omega) = 0,$$

which implies that $\Lambda_\omega = \Lambda_\omega (K^\dagger)^* K^*$ and so ${\Lambda_\omega}^* = KK^\dagger {\Lambda_\omega}^*$, a.e. Now let $\int_\Omega (\Lambda_\omega (K^\dagger)^*)^* \Psi_\omega f d\mu(\omega) = 0$, for a c-g-Bessel sequence $\{\Psi_\omega\}_{\omega\in\Omega}$ and all $f \in H$. Then by Lemma 2.7 of [13], one can observe that

$$K \int_\Omega K^\dagger {\Lambda_\omega}^* \Psi_\omega f d\mu(\omega) = \int_\Omega KK^\dagger {\Lambda_\omega}^* \Psi_\omega f d\mu(\omega) = 0.$$

So, $\int_\Omega {\Lambda_\omega}^* \Psi_\omega f d\mu(\omega) = 0$ and by assumption, $\Psi_\omega = 0$, a.e.
Conversely, let $\{\Lambda_\omega (K^\dagger)^*\}_{\omega\in\Omega}$ be independent and for a c-g-Bessel sequence $\{\Psi_\omega\}_{\omega\in\Omega}$, we have $\int_\Omega {\Lambda_\omega}^* \Psi_\omega f d\mu(\omega) = 0$, for all $f \in H$. Then

$$\int_\Omega (\Lambda_\omega (K^\dagger)^*)^* \Psi_\omega f d\mu(\omega) = K^\dagger \int_\Omega {\Lambda_\omega}^* \Psi_\omega f d\mu(\omega) = 0.$$

By assumption, $\Psi_\omega = 0$, a.e. So $\{\Lambda_\omega\}_{\omega\in\Omega}$ is independent. □
Now, we intend to find the synthesis operator related to c-$K^\dagger K$-g-frame $\{\Lambda_\omega (K^\dagger)^*\}_{\omega\in\Omega}$. Recall from [1] or [4] that the space

$$\begin{aligned}
& (\bigoplus_{\omega\in\omega} H_\omega, \mu)_{L^2} \\
= & \{F \in \prod_{\omega\in\Omega} H_\omega : F \text{ is strongly measurable}, \int_\Omega \|F(\omega)\|^2 d\mu(\omega) < \infty\},
\end{aligned}$$

with inner product given by

$$\langle F, G\rangle = \int_\Omega \langle F(\omega), G(\omega)\rangle d\mu(\omega),$$

is a Hilbert space. We initiate with a theorem about the synthesis operator related to c-K-g-frames.

Theorem 3.5. (See [4]) *Let $K \in B(H)$ and (Ω, μ) be a σ-finite measure space. Suppose that $\{\Lambda_\omega \in B(H, H_\omega) : \omega \in \Omega\}$ is a family of operators such that $\{\Lambda_\omega f\}_{\omega\in\Omega}$ is strongly measurable, for each $f \in H$. Then $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ if and only if the operator*

$$T : (\bigoplus_{\omega\in\Omega} H_\omega, \mu)_{L^2} \longrightarrow H,$$

weakly defined by

$$\langle TF, g\rangle = \int_\Omega \langle \Lambda_\omega{}^* F(\omega), g\rangle d\mu(\omega), \qquad F \in (\bigoplus_{\omega\in\Omega} H_\omega, \mu)_{L^2},\ g \in H,$$

is bounded and $R(K) \subseteq R(T)$.

In this case, the adjoint T^* of T is defined by

$$(T^* g)(\omega) = \Lambda_\omega(g), \quad \omega \in \Omega,\ g \in H,$$

and $\|T\| \leq \sqrt{B}$, where B is the upper c-K-g-frame bound of $\{\Lambda_\omega\}_{\omega\in\Omega}$. The above operator T and its adjoint T^* are called the synthesis operator and analysis operator related to $\{\Lambda_\omega\}_{\omega\in\Omega}$, respectively.

Corollary 3.6. *Let $K \in B(H)$ such that K^* is surjective. Let μ be a σ-finite measure on Ω. If $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ and synthesis operator T, then the synthesis operator T_Γ related to $\Gamma = \{\Lambda_\omega(K^\dagger)^*\}_{\omega\in\Omega}$ is the bounded operator $T_\Gamma = K^\dagger T$. Also*

$$(T_\Gamma{}^* g)(\omega) = \Lambda_\omega (K^\dagger)^* g,$$

for all $\omega \in \Omega$ and $g \in H$.

Proof. By Theorems 3.1 and 3.2, Γ is a c-K^*-g-frame for H. Now, Theorem 3.5 implies that for all $F \in (\bigoplus_{\omega\in\Omega} H_\omega, \mu)_{L^2}$ and all $g \in H$:

$$\begin{aligned} \langle T_\Gamma F, g\rangle &= \int_\Omega \langle (\Lambda_\omega (K^\dagger)^*)^* F(\omega), g\rangle d\mu(\omega) \\ &= \int_\Omega \langle K^\dagger \Lambda_\omega{}^* F(\omega), g\rangle d\mu(\omega) \\ &= \langle K^\dagger T F, g\rangle, \end{aligned}$$

and then $T_\Gamma = K^\dagger T$. The second part of corollary, is also a direct consequence of Theorem 3.5. □

Finally, we find the frame operator S_Γ related to the c-g-frame $\Gamma = \{\Lambda_\omega(K^*)^\dagger\}_{\omega\in\Omega}$, as stated in Theorem 3.2. If S is the frame operator related to the Parseval c-K-g -frame $\{\Lambda_\omega\}_{\omega\in\Omega}$, then for any $f, g \in H$:

$$\begin{aligned} \langle S_\Gamma f, g\rangle &= \int_\Omega \langle \Lambda_\omega (K^*)^\dagger f, \Lambda_\omega (K^*)^\dagger g\rangle d\mu(\omega) \\ &= \int_\Omega \langle K^\dagger \Lambda_\omega{}^* \Lambda_\omega (K^*)^\dagger f, g\rangle d\mu(\omega) \\ &= \langle \int_\Omega K^\dagger \Lambda_\omega{}^* \Lambda_\omega (K^*)^\dagger f d\mu(\omega), g\rangle \\ &= \langle K^\dagger \int_\Omega \Lambda_\omega{}^* \Lambda_\omega (K^*)^\dagger f d\mu(\omega), g\rangle \\ &= \langle K^\dagger S (K^*)^\dagger f, g\rangle. \end{aligned}$$

Hence, $S_\Gamma = K^\dagger S (K^*)^\dagger$.

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