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Weak and Strong Convergence Results for Split Variational Inclusion Problem and Fixed Point Problem

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Abstract. In this paper, we introduce a new iterative algorithm for finding a common solution for variational inclusion problem and fixed point problem in real Hilbert spaces. This method can be implemented more easily without the prior knowledge of the Lipschitz constant of component operators. The algorithm uses variable step-sizes which are updated at each iteration by a simple computation. In additional, weak and strong convergence results of the proposed algorithm are obtained under some mild conditions. Then, we establish the linear rate of convergence for the proposed algorithm. Finally, an example is given to illustrate the convergence of the algorithm.

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1 Introduction

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. Many important real-world problems can be formulated as the finding of zero(s) of an operator. The variational inclusion problem seeks a point $a^* \in \mathcal{H}$ such that

$$0 \in (\mathcal{G}_1 + \mathcal{B})(\mathfrak{a}^*), \tag{1}$$

where $\mathcal{G}_1 : \mathcal{H} \to 2^{\mathcal{H}}$ is a multivalued operator and $\mathcal{B} : \mathcal{H} \to \mathcal{H}$ is a single valued operator. For $\mathcal{B} = 0$,(1) becomes the inclusion problem introduced by Rockafellar [20]. Indeed, Problem (1) includes many important optimization problems like variational inequalities, minimization problems, split feasibility problems, fixed point problems, Nash equilibrium problems in noncooperative games, and many more. Also, many problems in signal processing, image recovery, and machine learning can be formulated as (1); see, e.g, [2, 6, 7, 8, 10, 11, 18, 19, 22, 23, 25, 26] and the references therein. The problem of finding common elements of the set of solutions of variational inequality problems and the set of fixed points of nonlinear operators has become an interesting area of research for many researchers working in the area of nonlinear operator theory see, e.g, [1, 2, 12, 13, 19, 22, 23, 26].

In this paper, we consider the following split problem which aims to find $a^* \in H$ such that

$$\mathfrak{a}^* \in (\mathcal{G}_1 + \mathcal{B})^{-1}(0) \cap \mathrm{Fix}(\mathcal{S}), \qquad \mathcal{A}\mathfrak{a}^* \in {\mathcal{G}_2}^{-1}(0), \tag{2}$$

where $\mathcal{G}_1, \mathcal{G}_2 : \mathcal{H} \to 2^{\mathcal{H}}$ are multivalued operators, $\mathcal{B}, \mathcal{S} : \mathcal{H} \to \mathcal{H}$ are single valued operators and $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ is a non-zero bounded linear operator which $\mathcal{A}^*$ is the adjoint of $\mathcal{A}$. Also $\mathrm{Fix}(\mathcal{S})$ is the set of fixed point of $\mathcal{S}$ and $(\mathcal{G}_1 + \mathcal{B})^{-1}(0)$ denotes the solution set of the variational inclusion of finding a point $\mathfrak{a}^* \in \mathcal{H}$ such that

$$0 \in (\mathcal{G}_1 + \mathcal{B})(\mathfrak{a}^*), \tag{3}$$

and ${\mathcal{G}_2}^{-1}(0)$ is defined in a similar way. There are several ways to solve the split problems, see [3-24].

A valuable approach to solve variational inclusion Problem (3) is the well-known forward-backward algorithm [14] defined by

$$\mathfrak{a}_{j+1} = \mathcal{J}_{\lambda_j}^{\mathcal{G}_1}(\mathcal{I} - \lambda_j \mathcal{B})\mathfrak{a}_j, \qquad \mathfrak{a}_0 \in \mathcal{H},$$

where the operator $\mathcal{B}$ is generally (inverse) strongly monotone. To relax the strong monotonicity condition imposed on $\mathcal{B}$, Tseng [21] suggested a modified forward-backward method defined by

$$\begin{cases} \mathfrak{b}_j = \mathcal{J}_{\lambda_j}^{\mathcal{G}_1}(\mathcal{I} - \lambda_j \mathcal{B})(\mathfrak{c}_j), \\ \mathfrak{c}_{j+1} = \mathfrak{b}_j - \mu_j(\mathcal{B}(\mathfrak{b}_j) - \mathcal{B}(\mathfrak{c}_j)), \end{cases}$$

where $\mathcal{B}$ is a monotone continuous Lipschitz operator. Cholamjiak *et al.* [8] proposed the following Tseng-type method to solve Problem (3) without the prior knowledge of Lipschitz constant of the operator $\mathcal{B}$,

$$\begin{cases} \mathfrak{b}_j = \mathcal{J}_{\lambda_j}^{\mathcal{G}_1}(I - \lambda_j \mathcal{B})(\mathfrak{c}_j), \\ \mathfrak{c}_{j+1} = (1-\theta_j)\mathfrak{c}_j + \theta_j \mathfrak{b}_j + \theta_j \lambda_j (\mathcal{B}(\mathfrak{c}_j) - \mathcal{B}(\mathfrak{b}_j)), \\ \mu_{j+1} = \min\left\{\lambda, \frac{\mu\|w_j - \mathfrak{b}_j\|}{\|\mathcal{B}(w_j) - \mathcal{B}(\mathfrak{b}_j)\|}\right\}. \end{cases}$$

By use of the Krasnoselśkii-Mann theorem ([2]), Akram *et al.* presented the following iterative algorithm for solving the common solution of split variational inclusion problem and fixed point problem by using appropriate assumptions:

$$\begin{cases} \mathfrak{b}_j = \mathfrak{a}_j - \tau\left[\mathcal{J}_{\lambda_j}^{\mathcal{G}_1}\mathfrak{a}_j + \mathcal{A}^*\left(\mathcal{I} - \mathcal{J}_{\lambda_2}^{\mathcal{G}_2}\right)\mathcal{A}\mathfrak{a}_j\right], \\ \mathfrak{a}_{j+1} = \alpha_j \psi(\mathfrak{a}_j) + (1-\alpha_j)\mathcal{S}\mathfrak{b}_j, \end{cases}$$

where $\tau = \frac{1}{\|\mathcal{A}\|^2}$, $\mathcal{S}$ is a nonexpansive mapping and ψ is a contraction.

Motivated and inspired by the works of [2, 8], we construct an iterative algorithm for finding a solution of the split Problem (2). This method consists of forward-backward method, fixed point method, self-adaptive method. Weak and strong convergence of the sequence generated by proposed algorithm are discussed.

This work is organized as follows: In Section 2, we recall some lemmas, theorems, and definitions as preliminaries. In Section 3, we suggest an algorithm and investigate its convergence and rate convergence analysis. Then we give an example to support our results. Finally, Section 4 is devoted to the conclusion.

2 Preliminaries

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let $\mathcal{G} : \mathcal{H} \to 2^{\mathcal{H}}$ be a multivalued operator. The graph of $\mathcal{G}$ is displayed with $\mathbf{G}(\mathcal{G})$ and defined by

$$\mathbf{G}(\mathcal{G}) = \Big\{ (\mathfrak{s}, \mathfrak{t}) \ : \mathfrak{s} \in \mathcal{H}, \ \mathfrak{t} \in \mathcal{G}(\mathfrak{s}) \Big\}.$$

Recall that $\mathcal{G}$ is said to be

1. monotone if for all $\mathfrak{s}, \mathfrak{a} \in \mathcal{H}$, $\mathfrak{t} \in \mathcal{G}(\mathfrak{s})$ and $\mathfrak{b} \in \mathcal{H}(\mathfrak{a})$,

$$\langle \mathfrak{s} - \mathfrak{a}, \mathfrak{t} - \mathfrak{b} \rangle \geq 0;$$

2. maximal monotone if and only if for $(\mathfrak{s}, \mathfrak{t}) \in \mathcal{H} \times \mathcal{H}$,

$$\langle \mathfrak{s} - \mathfrak{a}, \mathfrak{t} - \mathfrak{b} \rangle \geq 0, \qquad \forall \, (\mathfrak{a}, \mathfrak{b}) \in \mathbf{G}(\mathcal{G}),$$

implies that $(\mathfrak{s}, \mathfrak{t}) \in \mathbf{G}(\mathcal{G})$;

3. ρ-strongly monotone if there exists a constant $\rho > 0$ such that

$$\langle \mathfrak{s} - \mathfrak{a}, \mathfrak{t} - \mathfrak{b} \rangle \geq \rho \| \mathfrak{s} - \mathfrak{a} \|^2, \qquad \forall \, \mathfrak{s}, \mathfrak{a} \in \mathcal{H}, \ \mathfrak{t} \in \mathcal{G}(\mathfrak{s}), \ \mathfrak{b} \in \mathcal{G}(\mathfrak{a}).$$

Let $\mathcal{B} : \mathcal{H} \to \mathcal{H}$ be an operator, $\mathcal{B}$ is called

1. Lipschitz continuous if there exists a constant $\mathcal{L} > 0$ such that

$$\| \mathcal{B}(\mathfrak{a}) - \mathcal{B}(\mathfrak{b}) \| \leq \mathcal{L} \| \mathfrak{a} - \mathfrak{b} \|, \qquad \forall \, \mathfrak{a}, \mathfrak{b} \in \mathcal{H};$$

2. monotone if the following relation holds:

$$\langle \mathcal{B}(\mathfrak{a}) - \mathcal{B}(\mathfrak{b}), \mathfrak{a} - \mathfrak{b} \rangle \geq 0, \qquad \forall \, \mathfrak{a}, \mathfrak{b} \in \mathcal{H};$$

3. firmly nonexpansive if the following relation holds:

$$\langle \mathcal{B}(\mathfrak{a}) - \mathcal{B}(\mathfrak{b}), \mathfrak{a} - \mathfrak{b} \rangle \geq \|\mathcal{B}(\mathfrak{a}) - \mathcal{B}(\mathfrak{b})\|^2, \qquad \forall\, \mathfrak{a}, \mathfrak{b} \in \mathcal{H};$$

4. demiclosed if $\mathrm{Fix}(\mathcal{B}) \neq \varnothing$ and $\{\mathfrak{a}_\mathfrak{j}\}$ is a sequence in $\mathcal{H}$ and weakly convergent to $\mathfrak{a} \in \mathcal{H}$ such that $(\mathcal{I} - \mathcal{B})\mathfrak{a}_\mathfrak{j} \to 0$ where $\mathcal{I}$ is identity mapping in $\mathcal{H}$, implies $\mathcal{B}(\mathfrak{a}) = \mathfrak{a}$.

The resolvent mapping $\mathcal{J}_\lambda^{\mathcal{G}} : \mathcal{H} \to \mathcal{H}$ associated with the maximal monotone operator $\mathcal{G}$ is defined by

$$\mathcal{J}_\lambda^{\mathcal{G}}(\mathfrak{a}) = (\mathcal{I} + \lambda\mathcal{G})^{-1}(x), \qquad \forall\, \mathfrak{a} \in \mathcal{H},$$

for some $\lambda > 0$, where $\mathcal{I}$ is identity mapping in $\mathcal{H}$. It is known that for all $\lambda > 0$, the resolvent mapping $\mathcal{J}_\lambda^{\mathcal{G}}$ is single valued, nonexpansive and firmly nonexpansive.

There exist two basic concepts of convergence rate of a sequence [17, Chapter 9]. Let $\{\mathfrak{a}_\mathfrak{j}\}$ be a sequence in $\mathcal{H}$ and $\mathfrak{a} \in \mathcal{H}$. The sequence $\{\mathfrak{a}_\mathfrak{j}\}$ is called

(i) convergent $\mathcal{R}$-linearly to $\mathfrak{a}$ if

$$\limsup_{\mathfrak{j}\to\infty} \|\mathfrak{a}_\mathfrak{j} - \mathfrak{a}\|^{1/\mathfrak{j}} < 1;$$

(ii) convergent $\mathcal{Q}$-linearly to $\mathfrak{a}$ if there exist constant numbers $\mathfrak{r} \in (0,1)$ and $\mathfrak{j}_0 \geq 1$ such rhat

$$\|\mathfrak{a}_{\mathfrak{j}+1} - \mathfrak{a}\| \leq \mathfrak{r}\|\mathfrak{a}_\mathfrak{j} - \mathfrak{a}\|, \qquad \forall\, \mathfrak{j} \geq \mathfrak{j}_0.$$

It is known that $\mathcal{Q}$-linear convergence implies $\mathcal{R}$-linear convergence [17, Section 9.3]. The inverse in general is not true.

Lemma 2.1 ([5])**.** *Let $\mathcal{H}$ be a real Hilbert space. Let $\mathcal{G} : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone operator and $\mathcal{B} : \mathcal{H} \to \mathcal{H}$ be a monotone and Lipschitz continuous operator. Then $\mathcal{G} + \mathcal{B}$ is a maximal monotone operator.*

Lemma 2.2 ([4])**.** *If $\mathcal{B} : \mathcal{H} \to \mathcal{H}$ is nonexpansive, then $\mathcal{I} - \mathcal{B}$ is demi-closed at zero. Moreover, if $\mathcal{B}$ is firmly nonexpansive, then $\mathcal{I} - \mathcal{B}$ is firmly nonexpansive.*

Lemma 2.3 ([4])**.** *If $\mathcal{G} : \mathcal{H} \to 2^{\mathcal{H}}$ is a maximal monotone operator, then $\mathcal{J}_\lambda^{\mathcal{G}}$ and $\mathcal{I} - \mathcal{J}_\lambda^{\mathcal{G}}$ are firmly nonexpansive.*

Theorem 2.4 ([9], Theorem 3.1)**.** *If $\mathcal{B} : \mathcal{H} \to \mathcal{H}$ is a nonexpansive mapping and $\mathfrak{b} \in \mathrm{Fix}(\mathcal{B})$. Then for all $x \in H$,*

$$\langle \mathfrak{a} - \mathcal{B}\mathfrak{a}, \mathfrak{b} - \mathcal{B}\mathfrak{a} \rangle \leq \tfrac{1}{2}\|\mathfrak{a} - \mathcal{B}\mathfrak{a}\|^2.$$

3 Main Results

In this section, we introduce the proposed algorithm. Then we establish the weak convergence of the sequence generated by the algorithm. By additional assumption of strong monotonicity of the operator, the strong convergence of the sequence is attained. Then, we discuss on the rate convergence of the generated sequence. The proposed algorithm consists of forward-backward method, fixed point method and self-adaptive method. In addition it uses a simple step-size rule without the prior knowledge of Lipschitz constant of the operator.

Algorithm 3.1.

Initialization: Choose arbitrary initial point $\mathfrak{a}_0 \in \mathcal{H}$ and $\lambda', \lambda_0 > 0$. Take $\alpha, \eta \in (0, 1]$, $\mu \in (0, 1)$, and $\theta \in (0, 1]$ such that

$$\tfrac{2}{\mu+3} < \theta < 1. \tag{4}$$

Iterative Steps: Calculate $\mathfrak{a}_{\mathrm{j}+1}$ and $\lambda_{\mathrm{j}+1}$ as follows

Step 1. Assume that $\mathfrak{a}_\mathrm{j} \in \mathcal{H}$, and λ_j are given. Put $\tau = \frac{1}{\|\mathcal{A}\|^2}$, compute

$$\mathfrak{u}_\mathrm{j} = \mathfrak{a}_\mathrm{j} - \tau\mathcal{A}^*\left(\mathcal{I} - \mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\mathfrak{a}_\mathrm{j},$$

$$\mathfrak{f}_\mathrm{j} = (1-\alpha)\mathfrak{a}_\mathrm{j} + \alpha\mathfrak{u}_\mathrm{j}.$$

Step 2. Compute

$$\mathfrak{b}_j = \mathcal{J}_{\lambda_j}^{\mathcal{G}_1}(\mathcal{I} - \lambda_j \mathcal{B})\mathfrak{f}_j,$$
$$\mathfrak{t}_j = (1-\theta)\mathfrak{f}_j + \theta \mathfrak{b}_j + \theta\lambda_j(\mathcal{B}(\mathfrak{f}_j) - \mathcal{B}(\mathfrak{b}_j)), \tag{5}$$

and

$$\mathfrak{a}_{j+1} = \eta \mathfrak{t}_j + (1-\eta)\mathcal{S}\mathfrak{t}_j.$$

Step 3. Update

$$\lambda_{j+1} = \min\left\{\lambda_j, \tfrac{\mu\|\mathfrak{f}_j - \mathfrak{b}_j\|}{\|\mathcal{B}(\mathfrak{f}_j) - \mathcal{B}(\mathfrak{b}_j)\|}\right\}. \tag{6}$$

Define the set

$$\Upsilon = \left\{\mathfrak{a} \in \mathcal{H} \,:\, \mathfrak{a} \in (\mathcal{G}_1 + \mathcal{B})^{-1}(0) \cap \operatorname{Fix}(\mathcal{S}),\ \mathcal{A}\mathfrak{a} \in \mathcal{G}_2^{-1}(0)\right\}.$$

Now, we prove if $\Upsilon \neq \varnothing$, then under some conditions on operators, the sequence generated by Algorithm 3.1 is weakly convergent to a point in Υ. To prove the main result, we need the following lemma.

Lemma 3.2. *If θ, μ are real numbers in $[0,1]$ satisfying*

$$\tfrac{2}{\mu+3} < \theta < \tfrac{1}{1-\mu}, \qquad \mu \neq 1,$$

then

$$\tfrac{\theta(1-\mu^2)}{2-\theta+\theta\mu} + \tfrac{1-\theta}{\theta} < 1. \tag{7}$$

Proof. The inequality (7) is equivalent to

$$\left(\mu^2 + 2\mu - 3\right)\theta^2 + (5-\mu)\theta - 2 > 0,$$

that is a quadratic equation in term of θ. Since the discriminant of this equation is equal to $\Delta = (3\mu+1)^2$. So this equation has two roots as $\theta_1 = \frac{2}{\mu+3}$ and $\theta_2 = \frac{1}{1-\mu}$. Considering $(\mu^2 + 2\mu - 3) < 0$, it suffices to assume that $\frac{2}{\mu+3} < \theta < \frac{1}{1-\mu}$. □

Theorem 3.3. *Let $\mathcal{H}$ be a real Hilbert space. Let $\mathcal{G}_1, \mathcal{G}_2 : \mathcal{H} \to 2^{\mathcal{H}}$ be maximal monotone operators and $\mathcal{B} : \mathcal{H} \to \mathcal{H}$ be a monotone and Lipschitz continuous operator. Also assume that $\mathcal{S} : \mathcal{H} \to \mathcal{H}$ is a non-expansive operator and $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator which $\mathcal{A}^*$ is the adjoint of $\mathcal{A}$. If Υ is not empty, then the sequence generated by Algorithm 3.1, converges weakly to a solution of the Problem (2).*

Proof. Since Υ is not empty, let $\xi \in \Upsilon$. By the definition of $\{\mathfrak{t}_j\}$ in Algorithm 3.1, we have

$$\begin{aligned}\|\mathfrak{t}_j - \xi\|^2 &= \|(1-\theta)(\mathfrak{f}_j - \xi) + \theta(\mathfrak{b}_j + \lambda_j(\mathcal{B}(\mathfrak{f}_j) - \mathcal{B}(\mathfrak{b}_j)) - \xi\|^2 \\ &= (1-\theta)\|\mathfrak{f}_j - \xi\|^2 + \theta\|\mathfrak{b}_j + \lambda_j(\mathcal{B}(\mathfrak{f}_j) - \mathcal{B}(\mathfrak{b}_j)) - \xi\|^2 \\ &\quad - \theta(1-\theta)\|\mathfrak{b}_j - \mathfrak{f}_j + \lambda_j(\mathcal{B}(\mathfrak{f}_j) - \mathcal{B}(\mathfrak{b}_j))\|^2. \end{aligned} \tag{8}$$

Also, we have

$$\begin{aligned}&\|\mathfrak{b}_j + \lambda_j(\mathcal{B}(\mathfrak{f}_j) - \mathcal{B}(\mathfrak{b}_j)) - \xi\|^2 \\ &\quad = \|\mathfrak{b}_j + \mathfrak{f}_j - \mathfrak{f}_j + \lambda_j(\mathcal{B}(\mathfrak{f}_j) - \mathcal{B}(\mathfrak{b}_j)) - \xi\|^2 \\ &\quad \le \|\mathfrak{b}_j - \mathfrak{f}_j\|^2 + \|\mathfrak{f}_j - \xi\|^2 + {\lambda_j}^2\|\mathcal{B}(\mathfrak{f}_j) - \mathcal{B}(\mathfrak{b}_j)\|^2 \\ &\qquad + 2\langle \mathfrak{b}_j - \mathfrak{f}_j, \mathfrak{f}_j - \xi\rangle + 2\lambda_j\langle \mathfrak{b}_j - \mathfrak{f}_j, \mathcal{B}(\mathfrak{f}_j) - \mathcal{B}(\mathfrak{b}_j)\rangle \\ &\qquad + 2\lambda_j\langle \mathfrak{f}_j - \xi, \mathcal{B}(\mathfrak{f}_j) - \mathcal{B}(\mathfrak{b}_j)\rangle \\ &\quad = \|\mathfrak{f}_j - \xi\|^2 + \|\mathfrak{b}_j - \mathfrak{f}_j\|^2 + 2\langle \mathfrak{b}_j - \mathfrak{f}_j, \mathfrak{f}_j - \mathfrak{b}_j\rangle \\ &\qquad + 2\langle \mathfrak{b}_j - \mathfrak{f}_j, \mathfrak{b}_j - \xi\rangle + {\lambda_j}^2\|\mathcal{B}(\mathfrak{f}_j) - \mathcal{B}(\mathfrak{b}_j)\|^2 \\ &\qquad + 2\lambda_j\langle \mathfrak{b}_j - \mathfrak{f}_j, \mathcal{B}(\mathfrak{f}_j) - \mathcal{B}(\mathfrak{b}_j)\rangle \\ &\qquad + 2\lambda_j\langle \mathfrak{f}_j - \xi, \mathcal{B}(\mathfrak{f}_j) - \mathcal{B}(\mathfrak{b}_j)\rangle \\ &\quad \le \|\mathfrak{f}_j - \xi\|^2 - \|\mathfrak{b}_j - \mathfrak{f}_j\|^2 + {\lambda_j}^2\|\mathcal{B}(\mathfrak{f}_j) - \mathcal{B}(\mathfrak{b}_j)\|^2 \\ &\qquad + 2\lambda_j\langle \mathfrak{b}_j - \xi, \mathcal{B}(\mathfrak{f}_j) - \mathcal{B}(\mathfrak{b}_j)\rangle + 2\langle \mathfrak{b}_j - \mathfrak{f}_j, \mathfrak{b}_j - \xi\rangle \\ &\quad \le \|\mathfrak{f}_j - \xi\|^2 - \|\mathfrak{b}_j - \mathfrak{f}_j\|^2 + {\lambda_j}^2\|\mathcal{B}(\mathfrak{f}_j) - \mathcal{B}(\mathfrak{b}_j)\|^2 \\ &\qquad - 2\langle \mathfrak{b}_j - \xi, \mathfrak{f}_j - \mathfrak{b}_j - \lambda_j(\mathcal{B}(\mathfrak{f}_j) - \mathcal{B}(\mathfrak{b}_j))\rangle. \end{aligned} \tag{9}$$

From $\mathfrak{b}_j = \mathcal{J}_{\lambda_j}^{\mathcal{G}_1}(\mathcal{I} - \lambda_j\mathcal{B})\mathfrak{f}_j$, we get

$$\mathfrak{f}_j - \mathfrak{b}_j - \lambda_j(\mathcal{B}(\mathfrak{f}_j) - \mathcal{B}(\mathfrak{b}_j)) \in \lambda_j(\mathcal{G}_1 + \mathcal{B})\mathfrak{b}_j. \tag{10}$$

By Lemma 2.1, $\mathcal{G}_1 + \mathcal{B}$ is monotone. This together with $0 \in \lambda_j\ (\mathcal{G}_1 + \mathcal{B})\xi$ imply that

$$\langle \mathfrak{f}_j - \mathfrak{b}_j - \lambda_j(\mathcal{B}(\mathfrak{f}_j) - \mathcal{B}(\mathfrak{b}_j)), \mathfrak{b}_j - \xi\rangle \ge 0. \tag{11}$$

Using (6), (9) and (11), we conclude that

$$\begin{aligned}
&\|\mathfrak{b}_j + \lambda_j(\mathcal{B}(\mathfrak{f}_j) - \mathcal{B}(\mathfrak{b}_j)) - \xi\|^2 \\
&\quad \leq \|\mathfrak{f}_j - \xi\|^2 - \|\mathfrak{b}_j - \mathfrak{f}_j\|^2 + {\lambda_j}^2 \|\mathcal{B}(\mathfrak{f}_j) - \mathcal{B}(\mathfrak{b}_j)\|^2 \\
&\quad \leq \|\mathfrak{f}_j - \xi\|^2 - \|\mathfrak{b}_j - \mathfrak{f}_j\|^2 + \tfrac{{\lambda_j}^2}{\lambda_{j+1}^2}\mu^2 \|\mathfrak{f}_j - \mathfrak{b}_j\|^2 \\
&\quad \leq \|\mathfrak{f}_j - \xi\|^2 - \left(1 - \tfrac{\lambda_j^2}{\lambda_{j+1}^2}\mu^2\right) \|\mathfrak{f}_j - \mathfrak{b}_j\|^2. \qquad (12)
\end{aligned}$$

By (5), (8) and (12), we obtain

$$\begin{aligned}
\|\mathfrak{t}_j - \xi\|^2 &= (1-\theta)\|\mathfrak{f}_j - \xi\|^2 + \theta\|\mathfrak{b}_j + \lambda_j(\mathcal{B}(\mathfrak{f}_j) - \mathcal{B}(\mathfrak{b}_j)) - \xi\|^2 \\
&\quad - \theta(1-\theta)\|\mathfrak{b}_j + \lambda_j(\mathcal{B}(\mathfrak{f}_j) - \mathcal{B}(\mathfrak{b}_j)) - \xi\|^2 \\
&\leq (1-\theta)\|\mathfrak{f}_j - \xi\|^2 + \theta(\|\mathfrak{f}_j - \xi\|^2 \\
&\quad - \left(1 - \tfrac{\lambda_j^2}{\lambda_{j+1}^2}\mu^2\right) \|\mathfrak{f}_j - \mathfrak{b}_j\|^2 \\
&\quad - \theta(1-\theta)\|\mathfrak{b}_j + \lambda_j(\mathcal{B}(\mathfrak{f}_j) - \mathcal{B}(\mathfrak{b}_j)) - \xi\|^2 \\
&= \|f_j - \xi\|^2 - \theta\left(1 - \tfrac{{\lambda_j}^2}{\lambda_{j+1}^2}\mu^2\right) \|\mathfrak{f}_j - \mathfrak{b}_j\|^2 \\
&\quad - \theta(1-\theta)\|\mathfrak{b}_j - \mathfrak{f}_j + \lambda_j(\mathcal{B}(\mathfrak{f}_j) - \mathcal{B}(\mathfrak{b}_j))\|^2 \\
&= \|\mathfrak{f}_j - \xi\|^2 - \theta\left(1 - \tfrac{{\lambda_j}^2}{\lambda_{j+1}^2}\mu^2\right) \|\mathfrak{f}_j - \mathfrak{b}_j\|^2 \\
&\quad - \tfrac{1-\theta}{\theta}\|\mathfrak{t}_j - \mathfrak{f}_j\|^2, \qquad (13)
\end{aligned}$$

it follows from (5) and (6) that

$$\begin{aligned}
\|\mathfrak{t}_j - \mathfrak{b}_j\| &= \|(1-\theta)(\mathfrak{f}_j - \mathfrak{b}_j) + \theta\lambda_j(\mathcal{B}(\mathfrak{f}_j) - \mathcal{B}(\mathfrak{b}_j))\| \\
&\leq (1-\theta)\|(\mathfrak{f}_j - \mathfrak{b}_j)\| + \theta\lambda_j\|(\mathcal{B}(\mathfrak{f}_j) - \mathcal{B}(\mathfrak{b}_j))\| \\
&\leq (1-\theta)\|(\mathfrak{f}_j - \mathfrak{b}_j)\| + \tfrac{\lambda_j}{\lambda_{j+1}}\mu\theta\|\mathfrak{f}_j - \mathfrak{b}_j\| \\
&= \left((1-\theta) + \tfrac{\lambda_j}{\lambda_{j+1}}\mu\theta\right)\|\mathfrak{f}_j - \mathfrak{b}_j\|, \qquad (14)
\end{aligned}$$

which gives

$$\begin{aligned}\|\mathfrak{t}_j - \mathfrak{f}_j\| &\le \|\mathfrak{t}_j - \mathfrak{b}_j\| + \|\mathfrak{b}_j - \mathfrak{f}_j\| \\ &\le \left(1 - \theta + \tfrac{\lambda_j}{\lambda_{j+1}}\mu\theta\right)\|\mathfrak{f}_j - \mathfrak{b}_j\| + \|\mathfrak{f}_j - \mathfrak{b}_j\| \\ &= \left(2 - \theta + \tfrac{\lambda_j}{\lambda_{j+1}}\mu\theta\right)\|\mathfrak{b}_j - \mathfrak{f}_j\|, \end{aligned} \tag{15}$$

it turns out that

$$\|\mathfrak{b}_j - \mathfrak{f}_j\| \ge \left[2 - \theta + \tfrac{\lambda_j}{\lambda_{j+1}}\mu\theta\right]^{-1}\|\mathfrak{t}_j - \mathfrak{f}_j\|. \tag{16}$$

It follows from (13), (14) and (16), that

$$\begin{aligned}\|\mathfrak{t}_j - \xi\|^2 \le \|\mathfrak{f}_j - \xi\|^2 - \Bigg\{&\theta\left(1 - \tfrac{\lambda_j^2}{\lambda_{j+1}^2}\mu^2\right) \\ &\cdot\left[2 - \theta + \tfrac{\lambda_j}{\lambda_{j+1}}\mu\theta\right]^{-2} + \tfrac{1-\theta}{\theta}\Bigg\}\|t_j - \mathfrak{f}_j\|^2. \end{aligned} \tag{17}$$

We put

$$\mathcal{H}_j = \theta\left(1 - \tfrac{\lambda_j^2}{\lambda_{j+1}^2}\mu^2\right)\left[2 - \theta + \frac{\lambda_j}{\lambda_{j+1}}\mu\theta\right]^{-2} + \tfrac{1-\theta}{\theta}. \tag{18}$$

Thus,

$$\|\mathfrak{t}_j - \xi\|^2 \le \|\mathfrak{f}_j - \xi\|^2 - H_j\|\mathfrak{t}_j - \mathfrak{f}_j\|^2. \tag{19}$$

Also, we have

$$\begin{aligned}\|\mathfrak{f}_j - \xi\|^2 &= \|(1-\alpha)\mathfrak{a}_j + \alpha\mathfrak{u}_j - \xi\|^2 \\ &= \|(1-\alpha)(\mathfrak{a}_j - \xi) + \alpha(\mathfrak{u}_j - \xi)\|^2 \\ &= (1-\alpha)\|\mathfrak{a}_j - \xi\|^2 + \alpha\|\mathfrak{u}_j - \xi\|^2 \\ &\quad - \alpha(1-\alpha)\|\mathfrak{a}_j - \mathfrak{u}_j\|^2. \end{aligned} \tag{20}$$

By Lemma 2.3, $\mathcal{I}-\mathcal{J}_{\lambda'}^{\mathcal{G}_2}$ is firmly nonexpansive. So, we get

$$\begin{aligned}
&\langle\mathcal{A}^*\left(\mathcal{I}-\mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\mathfrak{a}-\mathcal{A}^*\left(\mathcal{I}-\mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\xi, a-\xi\Big\rangle\\
&\quad=\Big\langle\left(\mathcal{I}-\mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\mathfrak{a}-\left(\mathcal{I}-\mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\xi, \mathcal{A}a-\mathcal{A}\xi\Big\rangle\\
&\quad\geq\left\|\left(\mathcal{I}-\mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\mathfrak{a}-\left(\mathcal{I}-\mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\xi\right\|^2\\
&\quad=\tfrac{1}{\|\mathcal{A}^*\|^2}\cdot\|\mathcal{A}^*\|^2\left\|\left(\mathcal{I}-\mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\mathfrak{a}-\left(\mathcal{I}-\mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\xi\right\|^2\\
&\quad\geq\tfrac{1}{\|\mathcal{A}^*\|^2}\|\mathcal{A}^*\left(\mathcal{I}-\mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\mathfrak{a}-\mathcal{A}^*\left(\mathcal{I}-\mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\xi\|^2\\
&\quad=\tau\left\|\mathcal{A}^*\left(\mathcal{I}-\mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\mathfrak{a}-\mathcal{A}^*\left(\mathcal{I}-\mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\xi\right\|^2.
\end{aligned}$$

Using the above relations, we have,

$$\begin{aligned}
\|\mathfrak{u}_\mathfrak{j}-\xi\|^2&=\left\|\mathfrak{a}_\mathfrak{j}-\tau\mathcal{A}^*\left(\mathcal{I}-\mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\mathfrak{a}_\mathfrak{j}-\xi\right\|^2\\
&=\|\mathfrak{a}_\mathfrak{j}-\xi\|^2+\tau^2\|\mathcal{A}^*\left(\mathcal{I}-\mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\mathfrak{a}_\mathfrak{j}\|^2\\
&\quad-2\tau\left\langle\mathfrak{a}_\mathfrak{j}-\xi,\mathcal{A}^*\left(\mathcal{I}-\mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\mathfrak{a}_\mathfrak{j}\right\rangle\\
&=\|\mathfrak{a}_\mathfrak{j}-\xi\|^2+\tau^2\Big\|\mathcal{A}^*\left(\mathcal{I}-\mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\mathfrak{a}_\mathfrak{j}\\
&\quad-\mathcal{A}^*\left(\mathcal{I}-\mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\xi\Big\|^2\\
&\quad-2\tau\Big\langle\mathfrak{a}_\mathfrak{j}-\xi,\mathcal{A}^*\left(\mathcal{I}-\mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\mathfrak{a}_\mathfrak{j}\\
&\quad-\mathcal{A}^*\left(\mathcal{I}-\mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\xi\Big\rangle\\
&\leq\|\mathfrak{a}_\mathfrak{j}-\xi\|^2-\tau^2\Big\|A^*\left(\mathcal{I}-\mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\mathfrak{a}_\mathfrak{j}\\
&\quad-\mathcal{A}^*\left(\mathcal{I}-\mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\xi\Big\|^2\\
&\leq\|\mathfrak{a}_\mathfrak{j}-\xi\|^2. \qquad (21)
\end{aligned}$$

By definition of $\{\mathfrak{f}_\mathfrak{j}\}$, it follows that

$$\begin{aligned}
\|\mathfrak{t}_\mathfrak{j}-\mathfrak{f}_\mathfrak{j}\|^2&=\|\mathfrak{t}_\mathfrak{j}-(1-\alpha)\mathfrak{a}_\mathfrak{j}-\alpha\mathfrak{u}_\mathfrak{j}\|^2\\
&=\|\mathfrak{t}_\mathfrak{j}-\mathfrak{a}_\mathfrak{j}-\alpha(\mathfrak{u}_\mathfrak{j}-\mathfrak{a}_\mathfrak{j})\|
\end{aligned}$$

$$\begin{aligned}
&= \|\mathfrak{t}_j - \mathfrak{a}_j\|^2 + \alpha^2\|\mathfrak{u}_j - \mathfrak{a}_j\|^2 - 2\alpha\langle \mathfrak{t}_j - \mathfrak{a}_j, \mathfrak{u}_j - \mathfrak{a}_j\rangle \\
&\geq \|\mathfrak{t}_j - \mathfrak{a}_j\|^2 + \alpha^2\|\mathfrak{u}_j - \mathfrak{a}_j\|^2 - \alpha\|\mathfrak{t}_j - \mathfrak{a}_j\|^2 - \alpha\|\mathfrak{u}_j - \mathfrak{a}_j\|^2 \\
&= (1-\alpha)\|\mathfrak{t}_j - \mathfrak{a}_j\|^2 - \alpha(1-\alpha)\|\mathfrak{u}_j - \mathfrak{a}_j\|^2. \qquad (22)
\end{aligned}$$

From (17), (20),(21) and (22), we get

$$\begin{aligned}
\|\mathfrak{t}_j - \xi\|^2 &\leq (1-\alpha)\|\mathfrak{a}_j - \xi\|^2 + \alpha\|u_j - \xi\|^2 \\
&\quad - \alpha(1-\alpha)\|\mathfrak{a}_j - \mathfrak{u}_j\|^2 - \mathcal{H}_j\Big((1-\alpha)\|\mathfrak{t}_j - \mathfrak{a}_j\|^2 \\
&\quad - \alpha(1-\alpha)\|\mathfrak{u}_j - \mathfrak{a}_j\|^2\Big) \\
&\leq (1-\alpha)\|\mathfrak{a}_j - \xi\|^2 + \alpha\|\mathfrak{a}_j - \xi\|^2 \\
&\quad + \alpha(1-\alpha)(\mathcal{H}_j - 1)\|\xi_j - \mathfrak{a}_j\|^2 - \mathcal{H}_j(1-\alpha)\|\mathfrak{t}_j - \mathfrak{a}_j\|^2 \\
&= \|\mathfrak{a}_j - \xi\|^2 + \alpha(1-\alpha)(\mathcal{H}_j - 1)\|\mathfrak{u}_j - \mathfrak{a}_j\|^2 \\
&\quad - \mathcal{H}_j(1-\alpha)\|\mathfrak{t}_j - \mathfrak{a}_j\|^2. \qquad (23)
\end{aligned}$$

Now, we show that $\lim_{j\to\infty}\lambda_j = \lambda$. If for some j, $\mathcal{B}(\mathfrak{b}_j) = \mathcal{B}(\mathfrak{f}_j)$, then $\lambda_{j+1} = \lambda_j$. Otherwise since $\mathcal{B}$ is Lipschitz continuous, then

$$\|\mathcal{B}(\mathfrak{b}_j) - \mathcal{B}(\mathfrak{f}_j)\| \leq L\|\mathfrak{b}_j - \mathfrak{f}_j\|.$$

Thus

$$\tfrac{\mu\|\mathfrak{b}_j - \mathfrak{f}_j\|}{\|\mathcal{B}(\mathfrak{b}_j) - \mathcal{B}(\mathfrak{f}_j)\|} \geq \tfrac{\mu}{L}\cdot\tfrac{\|\mathfrak{b}_j - \mathfrak{f}_j\|}{\|\mathfrak{b}_j - \mathfrak{f}_j\|} = \tfrac{\mu}{L}.$$

The above relation shows that the sequence $\{\lambda_j\}$ is bounded from below. Since $\{\lambda_j\}$ is nonincreasing, then it is convergent. Let $\lim_{j\to\infty}\lambda_j = \lambda$. Considering $\frac{2}{\mu+3} < \theta < 1$, for enough large j we have

$$\begin{aligned}
0 \leq \mathcal{H}_j &= \theta\left(1 - \tfrac{{\lambda_j}^2}{{\lambda_{j+1}}^2}\mu^2\right)\left[2 - \theta + \tfrac{\lambda_j}{\lambda_{j+1}}\mu\theta\right]^{-2} + \tfrac{1-\theta}{\theta} \\
&\leq \lim_{j\to\infty}\mathcal{H}_j = \tfrac{\theta(1-\mu^2)}{(2-\theta+\theta\mu)^2} + \tfrac{1-\theta}{\theta} \\
&\leq \tfrac{\theta(1-\mu^2)}{(2-\theta+\theta\mu)} + \tfrac{1-\theta}{\theta} < 1. \qquad (24)
\end{aligned}$$

Now, by definition of $\{\mathfrak{a}_{j+1}\}$ and from nonexpansivity of $\mathcal{S}$, we get

$$\begin{aligned}
\|\mathfrak{a}_{j+1} - \xi\| &= \|\eta t_j + (1-\eta)\mathcal{S}\mathfrak{t}_j - \xi\| \\
&= \|\eta(\mathfrak{t}_j - \xi) + (1-\eta)(\mathcal{S}\mathfrak{t}_j - \mathcal{S}\xi)\| \\
&\leq \eta\|\mathfrak{t}_j - \xi\| + (1-\eta)\|\mathfrak{t}_j - \xi\| = \|\mathfrak{t}_j - \xi\|. \qquad (25)
\end{aligned}$$

From Eqs. (23), (25) and considering (24), for enoughlarge j, we get

$$\|\mathfrak{a}_{j+1} - \xi\| \leq \|\mathfrak{a}_j - \xi\|.$$

So the sequence $\|\mathfrak{a}_j - \xi\|$ is noninceasing and bounded from below. We can assume that $\lim_{j\to\infty} \|\mathfrak{a}_j - \xi\| = \gamma$. Also, again by (23) and (25), we conclude that

$$\|\mathfrak{a}_{j+1} - \xi\|^2 \leq \|\mathfrak{a}_j - \xi\|^2 + \alpha(1-\alpha)(\mathcal{H}_j - 1)\|\mathfrak{u}_j - \mathfrak{a}_j\|^2. \tag{26}$$

By taking lim inf of both sides of inequality (26) and that $\mathcal{H} < 1$, we observe that

$$\gamma \leq \gamma + \alpha(1-\alpha)(\mathcal{H} - 1) \limsup_{j\to\infty} \|\mathfrak{u}_j - \mathfrak{a}_j\|^2,$$

and so

$$\limsup_{j\to\infty} \|\mathfrak{u}_j - \mathfrak{a}_j\| = 0.$$

By a similar way, $\limsup_{j\to\infty} \|\mathfrak{t}_j - \mathfrak{a}_j\| = 0$. Also

$$\|\mathfrak{f}_j - \mathfrak{a}_j\| = \|(1-\alpha)\mathfrak{a}_j + \alpha\mathfrak{u}_j - \mathfrak{a}_j\| = \alpha\|\mathfrak{u}_j - \mathfrak{a}_j\|.$$

Above inequality implies that $\lim_{j\to\infty} \|\mathfrak{f}_j - \mathfrak{a}_j\| = 0$ and so $\{\mathfrak{f}_j\}$ is bounded. In addition,

$$\|\mathfrak{f}_j - t_j\| = \|\mathfrak{f}_j - \mathfrak{a}_j\| + \|\mathfrak{a}_j - \mathfrak{t}_j\|.$$

Hence, the above inequality implies that $\lim_{j\to\infty} \|\mathfrak{f}_j - \mathfrak{t}_j\| = 0$ and so $\{\mathfrak{t}_j\}$ is bounded. Next, we show that $\{\mathfrak{b}_j\}$ is bounded.

$$\begin{aligned} \|\mathfrak{b}_j - \xi\|^2 &= \|\mathcal{J}_{\lambda_j}^{\mathcal{G}_1}(\mathcal{I} - \lambda_j\mathcal{B})\mathfrak{f}_j - \mathcal{J}_{\lambda_j}^{\mathcal{G}_1}(\mathcal{I} - \lambda_j\mathcal{B})\xi\|^2 \\ &\leq \|(\mathcal{I} - \lambda_j\mathcal{B})\mathfrak{f}_j - (\mathcal{I} - \lambda_j\mathcal{B})\xi\|^2 \\ &\leq \|\mathfrak{f}_j - \xi\|^2 + \lambda_j^2\|\mathcal{B}\mathfrak{f}_j - \mathcal{B}\xi\|^2 \\ &\leq \|\mathfrak{f}_j - \xi\|^2 + \frac{\lambda_j^2}{{\lambda_{j+1}}^2}\mu^2\|\mathfrak{f}_j - \xi\|^2. \end{aligned} \tag{27}$$

Above inequality shows that $\{\mathfrak{b}_j\}$ is bounded. From (14), we observe

$$\begin{aligned} \|\mathfrak{t}_j - \mathfrak{b}_j\| &\leq \left(1 - \theta + \frac{\lambda_j}{\lambda_{j+1}}\mu\theta\right) \|\mathfrak{f}_j - \mathfrak{b}_j\| \\ &\leq \left(1 - \theta + \frac{\lambda_j}{\lambda_{j+1}}\mu\theta\right) (\|\mathfrak{f}_j - \mathfrak{t}_j\| + \|\mathfrak{t}_j - \mathfrak{b}_j\|). \end{aligned} \tag{28}$$

Since $(1-\theta+\mu\theta)<1$ and $\{\mathfrak{b}_\mathfrak{j}\}$ is bounded, by taking $\limsup$ of both sides of above inequality, we see that $\lim_{\mathfrak{j}\to\infty}\|\mathfrak{t}_\mathfrak{j}-\mathfrak{b}_\mathfrak{j}\|=0$. Since $\{\mathfrak{a}_\mathfrak{j}\}$ is bounded, it has a weakly convergent subsequence, take $\{\mathfrak{a}_m\}$ is a subsequence that is weakly convergent to $\vartheta\in\mathcal{H}$. In the sequel, we show that $\vartheta\in\Upsilon$. For the convenience, we divide the rest of the proof into four steps.
Step 1. $\vartheta\in(\mathcal{G}_1+\mathcal{B})^{-1}(0)$. Since $\{\mathfrak{a}_m\}$ is weakly convergent to ϑ and $\|\mathfrak{a}_m-\mathfrak{b}_m\|\to 0$, so $\mathfrak{b}_m\rightharpoonup\vartheta$. Let $(\mathfrak{u},\mathfrak{v})\in\mathbf{G}(\mathcal{G}_1+\mathcal{B})$, that is, $\mathfrak{v}-\mathcal{B}(\mathfrak{u})\in\mathcal{G}_1(\mathfrak{u})$. On the other hand, the equality

$$\mathfrak{b}_m=\mathcal{J}^{\mathcal{G}_1}_{\lambda_\mathfrak{j}}(\mathcal{I}-\lambda_m\mathcal{B})\mathfrak{f}_m,$$

leads to

$$\tfrac{1}{\lambda_m}\left(\mathfrak{f}_m-\lambda_m\mathcal{B}\mathfrak{f}_m-\mathfrak{b}_m\right)\in\mathcal{G}_1\mathfrak{b}_m.$$

Maximal monotonicity of $\mathcal{G}_1$ leads to

$$\left\langle\mathfrak{v}-\mathcal{B}\mathfrak{u}-\tfrac{1}{\lambda_m}(\mathfrak{f}_m-\lambda_m\mathcal{B}\mathfrak{f}_m-\mathfrak{b}_m),\mathfrak{u}-\mathfrak{b}_m\right\rangle\geq 0,$$

thus

$$\begin{aligned}\langle\mathfrak{v},\mathfrak{u}-\mathfrak{b}_m\rangle&\geq\left\langle\mathcal{B}\mathfrak{u}+\tfrac{1}{\lambda_m}(\mathfrak{f}_m-\lambda_m\mathcal{B}\mathfrak{f}_m-\mathfrak{b}_m),\mathfrak{u}-\mathfrak{b}_m\right\rangle\\&=\tfrac{1}{\lambda_m}\langle\mathfrak{f}_m-\mathfrak{b}_m,\mathfrak{u}-\mathfrak{b}_m\rangle+\langle\mathcal{B}(\mathfrak{u})-\mathcal{B}(\mathfrak{b}_m),\mathfrak{u}-\mathfrak{b}_m\rangle\\&\quad+\langle\mathcal{B}(\mathfrak{b}_m)-\mathcal{B}(\mathfrak{f}_m),\mathfrak{u}-\mathfrak{b}_m\rangle\\&\geq-\tfrac{1}{\lambda_m}\|\mathfrak{f}_m-\mathfrak{b}_m\|\|\mathfrak{u}-\mathfrak{b}_m\|\\&\quad-\|\mathcal{B}(\mathfrak{b}_m)-\mathcal{B}(\mathfrak{f}_m)\|\|\mathfrak{u}-\mathfrak{b}_m\|.\end{aligned}\tag{29}$$

Since $\|\mathfrak{b}_m-\mathfrak{f}_m\|\to 0$, $\{\mathfrak{b}_m\}$ is bounded and $\mathcal{B}$ is Lipschitz, so $\langle\mathfrak{v},\mathfrak{u}-\vartheta\rangle\geq 0$. Maximality of $(\mathcal{G}_1+\mathcal{B})$ leads to $0\in(\mathcal{G}_1+\mathcal{B})\vartheta$, that is $0\in(\mathcal{G}_1+\mathcal{B})^{-1}(0)$.
Step 2. $\vartheta\in\mathrm{Fix}(S)$. Note that

$$\|\mathfrak{a}_\mathfrak{j}-\xi\|\leq\|\mathfrak{a}_\mathfrak{j}-\mathfrak{t}_\mathfrak{j}\|+\|\mathfrak{t}_\mathfrak{j}-\xi\|\leq 2\|\mathfrak{a}_\mathfrak{j}-\mathfrak{t}_\mathfrak{j}\|+\|\mathfrak{a}_\mathfrak{j}-\xi\|,$$

the relation $\lim_{\mathfrak{j}\to\infty}\|\mathfrak{a}_\mathfrak{j}-\xi\|=\gamma$ leads to $\lim_{\mathfrak{j}\to\infty}\|\mathfrak{t}_\mathfrak{j}-\gamma\|=\gamma$. Also,

$$\begin{aligned}\|\mathfrak{a}_{\mathfrak{j}+1}-\xi\|^2&\leq\eta\|t_\mathfrak{j}-\xi\|^2+(1-\eta)\|\mathcal{S}\mathfrak{t}_\mathfrak{j}-\mathcal{S}\xi\|^2\\&\quad-\eta(1-\eta)\|\mathfrak{t}_\mathfrak{j}-\mathcal{S}\mathfrak{t}_\mathfrak{j}\|^2\\&\leq\|\mathfrak{t}_\mathfrak{j}-\xi\|^2-\eta(1-\eta)\|\mathfrak{t}_\mathfrak{j}-\mathcal{S}\mathfrak{t}_\mathfrak{j}\|^2.\end{aligned}\tag{30}$$

Taking lim inf of the both sides of (30), we get

$$\gamma^2 \leq \gamma^2 - \eta(1-\eta)\limsup_{\mathfrak{j}\to\infty}\|\mathfrak{t}_\mathfrak{j} - \mathcal{S}\mathfrak{t}_\mathfrak{j}\|^2.$$

Then

$$\limsup_{\mathfrak{j}\to\infty}\|\mathfrak{t}_\mathfrak{j} - \mathcal{S}\mathfrak{t}_\mathfrak{j}\| = 0.$$

Since $\mathcal{S}$ is nonexpansive, it follows by Lemma 2.2 that $\mathcal{I}-\mathcal{S}$ is demiclosed, therefor $\vartheta = \mathcal{S}\vartheta$.
Step 3. $A\vartheta \in \mathcal{G}_2^{-1}(0)$. Note that

$$\lim_{\mathfrak{j}\to\infty}\|\mathfrak{u}_\mathfrak{j} - \mathfrak{a}_\mathfrak{j}\| = \tau\lim_{\mathfrak{j}\to\infty}\left\|A^*\left(\mathcal{I} - \mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\mathfrak{a}_\mathfrak{j}\right\| = 0.$$

By Theorem 2.4, we can deduce that

$$\begin{aligned}
&\left\langle \mathfrak{a}_m - \xi, \mathcal{A}^*\left(\mathcal{I} - \mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\mathfrak{a}_m\right\rangle \\
&\quad = \left\langle \mathcal{A}\mathfrak{a}_m - \mathcal{A}\xi, \left(\mathcal{I} - \mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\mathfrak{a}_m\right\rangle \\
&\quad = \left\langle \mathcal{A}\mathfrak{a}_m - \mathcal{A}\xi + \left(\mathcal{I} - \mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\mathfrak{a}_m \right. \\
&\qquad \left. - \left(\mathcal{I} - \mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\mathfrak{a}_m, \left(\mathcal{I} - \mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\mathfrak{a}_m\right\rangle \\
&\quad = \left\langle \mathcal{J}_{\lambda'}^{\mathcal{G}_2}\mathcal{A}\mathfrak{a}_m - \mathcal{A}\xi, \left(\mathcal{I} - \mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\mathfrak{a}_m\right\rangle \\
&\qquad + \left\|\left(\mathcal{I} - \mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\mathfrak{a}_m\right\|^2 \\
&\quad = -\left\langle \mathcal{A}\xi - \mathcal{J}_{\lambda'}^{\mathcal{G}_2}\mathcal{A}\mathfrak{a}_m, \left(\mathcal{I} - \mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\mathfrak{a}_m\right\rangle \\
&\qquad + \left\|\left(\mathcal{I} - \mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\mathfrak{a}_m\right\|^2 \\
&\quad \geq -\tfrac{1}{2}\left\|\left(\mathcal{I} - \mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\mathfrak{a}_m\right\|^2 + \left\|\left(\mathcal{I} - \mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\mathfrak{a}_m\right\|^2 \\
&\quad = \tfrac{1}{2}\left\|\left(\mathcal{I} - \mathcal{J}_{\lambda'}^{\mathcal{G}_2}\right)\mathcal{A}\mathfrak{a}_m\right\|^2.
\end{aligned} \tag{31}$$

Since $\{\mathfrak{a}_m\}$ is weakly convergent to ϑ, so $\mathcal{A}\mathfrak{a}_\mathfrak{m}$ is weakly convergent to $\mathcal{A}\vartheta$. By (31) and (31), we obtain

$$\|(\mathcal{I} - \mathcal{J}_{\lambda'}^{\mathcal{G}_2})\mathcal{A}\mathfrak{a}_m\| \to 0.$$

Since $(\mathcal{I}-\mathcal{J}_{\lambda'}^{\mathcal{G}_2})$ is nonexpansive, so is demiclosed, by Lemma 2.3, $\mathcal{J}_{\lambda'}^{\mathcal{G}_2}\mathcal{A}\vartheta = \mathcal{A}\vartheta$, that is $A\vartheta \in \mathcal{G}_2^{-1}(0)$. Note that we established $\vartheta \in \Upsilon$.
Step 4. Whole sequence $\{\mathfrak{a}_{\mathrm{j}}\}$ is weakly convergent to ϑ. Let $\{\mathfrak{a}_m\}$ and $\{\mathfrak{a}_k\}$ be two subsequences of $\{\mathfrak{a}_{\mathrm{j}}\}$ that weakly converge to ϑ_1 and ϑ_2 respectively. Note that

$$2\langle \mathfrak{a}_{\mathrm{j}}, \vartheta_1 - \vartheta_2 \rangle = \|\vartheta_1\|^2 - \|\vartheta_2\|^2 + \|\mathfrak{a}_{\mathrm{j}} - \vartheta_1\|^2 - \|\mathfrak{a}_{\mathrm{j}} - \vartheta_2\|^2.$$

Since $\lim_{\mathrm{j}\to\infty} \|\mathfrak{a}_{\mathrm{j}} - \vartheta_1\|$ and $\lim_{\mathrm{j}\to\infty} \|\mathfrak{a}_{\mathrm{j}} - \vartheta_2\|$ exist, so we can let

$$\lim_{\mathrm{j}\to\infty} \langle \mathfrak{a}_{\mathrm{j}}, \vartheta_1 - \vartheta_2 \rangle = \zeta.$$

Consequently

$$\langle \vartheta_1, \vartheta_1 - \vartheta_2 \rangle = \langle \vartheta_2, \vartheta_1 - \vartheta_2 \rangle.$$

Thus $\vartheta_1 = \vartheta_2$, that is, whole sequence $\{\mathfrak{a}_{\mathrm{j}}\}$ has weakly unique limit. So $\{\mathfrak{a}_{\mathrm{j}}\}$ is weakly convergent to ϑ. This completes the proof. □

In the sequel, by adding the assumption of strongly monotonicity on $\mathcal{B}$, we consider the rate convergence of Algorithm 3.1. Now, we have the following result.

Theorem 3.4. *Under the hypothesis of Theorem 3.3, in addition, let $\mathcal{B}$ is strongly monotone. Then the sequence generated by Algorithm 3.1 converges at least R-linearly to the unique solution of Problem (2).*

Proof. We have

$$\begin{aligned}
&\|\mathfrak{b}_{\mathrm{j}} - \mathfrak{f}_{\mathrm{j}} + \lambda_{\mathrm{j}}(\mathcal{B}(w_{\mathrm{j}} - \mathcal{B}(\mathfrak{b}_{\mathrm{j}}))\| \\
&\qquad \geq \|\mathfrak{b}_{\mathrm{j}} - \mathfrak{f}_{\mathrm{j}}\| - \lambda_{\mathrm{j}}\|\mathcal{B}(w_{\mathrm{j}}) - \mathcal{B}(\mathfrak{b}_{\mathrm{j}})\| \\
&\qquad \geq \|\mathfrak{b}_{\mathrm{j}} - \mathfrak{f}_{\mathrm{j}}\| - \frac{\lambda_{\mathrm{j}}}{\lambda_{\mathrm{j}+1}}\mu\|\mathfrak{f}_{\mathrm{j}} - \mathfrak{b}_{\mathrm{j}}\| \\
&\qquad = \left(1 - \tfrac{\lambda_{\mathrm{j}}}{\lambda_{\mathrm{j}+1}}\mu\right)\|\mathfrak{f}_{\mathrm{j}} - \mathfrak{b}_{\mathrm{j}}\|. \qquad (32)
\end{aligned}$$

Note that for enough large j, $(1 - \frac{\lambda_{\mathrm{j}}}{\lambda_{\mathrm{j}+1}}\mu) \geq 0$. In addition, arguing similarly to the proof of Theorem 3.3, we get

$$\begin{aligned}\|\mathfrak{t}_j-\xi\|^2 &\leq \|\mathfrak{f}_j-\xi\|^2-\theta\left(1-\frac{\lambda_j{}^2}{\lambda_{j+1}{}^2}\mu^2\right)\|\mathfrak{f}_j-\mathfrak{b}_j\|^2\\ &\quad -\theta(1-\theta)\|\mathfrak{b}_j-\mathfrak{f}_j+\lambda_j(\mathcal{B}(\mathfrak{f}_j)-\mathcal{B}(\mathfrak{b}_j))\|^2\\ &\leq \|\mathfrak{f}_j-\xi\|^2-\theta\Big[\left(1-\frac{\lambda_j{}^2}{\lambda_{j+1}{}^2}\mu^2\right)\\ &\quad +(1-\theta)\left(1-\frac{\lambda_j}{\lambda_{j+1}}\mu\right)^2\Big]\|\mathfrak{f}_j-\mathfrak{b}_j\|^2. \end{aligned} \tag{33}$$

Now, because

$$\mathfrak{b}_j-\lambda_j(\mathcal{B}(\mathfrak{f}_j)-\mathcal{B}(\mathfrak{b}_j))\in\lambda_j(\mathcal{G}_1+\mathcal{B})\mathfrak{b}_j,$$

$0\in\lambda_j(\mathcal{G}_1+\mathcal{B}))\xi$ and $\lambda_j(\mathcal{G}_1+\mathcal{B}))$ is $\nu\lambda_j-$ strongly monotone, we conclude that

$$\langle\mathfrak{f}_j-\mathfrak{b}_j-\lambda_j(\mathcal{B}(\mathfrak{f}_j)-\mathcal{B}(\mathfrak{b}_j)),\mathfrak{b}_j-\xi\rangle\geq\nu\lambda_j\|\mathfrak{b}_j-\xi\|^2. \tag{34}$$

So

$$\begin{aligned}\nu\lambda_j\|\mathfrak{b}_j-\xi\|^2 &\leq \langle\mathfrak{f}_j-\mathfrak{b}_j,\mathfrak{b}_j-\xi\rangle-\lambda_j\langle(\mathcal{B}(\mathfrak{f}_j)-\mathcal{B}(\mathfrak{b}_j)),\mathfrak{b}_j-\xi\rangle\\ &\leq \|\mathfrak{f}_j-\mathfrak{b}_j\|\|\mathfrak{b}_j-\xi\|+\frac{\lambda_j}{\lambda_{j+1}}\mu\|\mathfrak{f}_j-\mathfrak{b}_j\|\|\mathfrak{b}_j-\xi\|\\ &= \left(1+\frac{\lambda_j}{\lambda_{j+1}}\mu\right)\|\mathfrak{f}_j-\mathfrak{b}_j\|\|\mathfrak{b}_j-\xi\|, \end{aligned} \tag{35}$$

which one deduce that

$$\begin{aligned}\|\mathfrak{f}_j-\xi\| &\leq \|\mathfrak{f}_j-\mathfrak{b}_j\|+\|\mathfrak{b}_j-\xi\|\\ &\leq \left[1+\frac{1}{\lambda_j\nu}\left(1+\frac{\lambda_j}{\lambda_{j+1}}\mu\right)\right]\|\mathfrak{f}_j-\mathfrak{b}_j\|, \end{aligned} \tag{36}$$

it turns into

$$\|\mathfrak{f}_j-\mathfrak{b}_j\|\geq\left[1+\frac{1}{\lambda_j\nu}\left(1+\frac{\lambda_j}{\lambda_{j+1}}\mu\right)\right]^{-1}\|\mathfrak{f}_j-\xi\|. \tag{37}$$

It follows from (33) and (37) that

$$\begin{aligned}\|\mathfrak{t}_j-\xi\|^2 &\leq \Bigg\{1-\left[\theta\left(1-\frac{\lambda_j{}^2}{\lambda_{j+1}{}^2}\mu^2\right)+(1-\theta)\left(1-\frac{\lambda_j}{\lambda_{j+1}}\mu\right)^2\right]\\ &\quad\cdot\left[\left(1+\frac{1}{\lambda_j\nu}\left(1+\frac{\lambda_j}{\lambda_{j+1}}\mu\right)\right)^2\right]^{-1}\Bigg\}\|\mathfrak{f}_j-\xi\|^2, \end{aligned} \tag{38}$$

by (21) and (22), we observe that $\|\mathfrak{f}_j - \xi\|^2 \leq \|\mathfrak{a}_j - \xi\|^2$. From (30) and that $\|\mathfrak{a}_{j+1} - \xi\|^2 \leq \|\mathfrak{t}_j - \xi\|^2$ and (38), it is deduced that

$$\|\mathfrak{a}_{j+1} - \xi\|^2 \leq \rho_j \|\mathfrak{a}_j - \xi\|^2,$$

where

$$\rho_j = 1 - \left[\theta\left(1 - \frac{\lambda_j{}^2}{\lambda_{j+1}{}^2}\mu^2\right) + (1-\theta)\left(1 - \frac{\lambda_j}{\lambda_{j+1}}\mu\right)^2\right] \cdot \left[1 + \frac{1}{\lambda_j \nu}\left(1 + \frac{\lambda_j}{\lambda_{j+1}}\mu\right)\right]^2.$$

It is clear that, $\lim_{j\to\infty} \rho_j = 1 - k$, where

$$k = \left(\theta(1-\mu^2) + (1-\theta)(1-\mu)^2\right)\left[\left(1 + \frac{1+\mu}{\lambda\nu}\right)^2\right].$$

Since $\frac{2}{\mu+3} < \theta < 1$, $\mu \neq 1$, so $1 - 2\theta < 0$. Therefor

$$\theta(1-\mu^2) + (1-\theta)(1-\mu)^2 = (1-2\theta)\mu^2 + 2(\theta-1)\mu + 1 < 1.$$

Thus, $0 < k < 1$ that means $0 < \lim_{j\to\infty} \rho_j < 1$. So the sequence $\{\|\mathfrak{a}_j - \xi\|\}$ is $\mathcal{Q}-$linearly convergent. □

The following corollary is obtained by Theorem 3.3 with $\theta = 1$ and $\mathcal{S} = \mathcal{I}$.

Corollary 3.5. *Suppose that the hypothesis of Theorem 3.3 are satisfied. Take $\lambda', \lambda_0 > 0$ and three numbers $\mu, \alpha \in (0,1)$ and $\eta \in (0,1]$. Let $\{\mathfrak{a}_j\}$ be the sequence defined by the following iterative algorithm. Choose $\mathfrak{a}_0 \in H$ and compute*

$$\begin{aligned}
\mathfrak{u}_j &= \mathfrak{a}_j - \tau\mathcal{A}^*(\mathcal{I} - \mathcal{J}_{\lambda'}^{\mathcal{G}_2})\mathcal{A}\mathfrak{a}_j,\\
\mathfrak{f}_j &= (1-\alpha)\mathfrak{a}_j + \alpha\mathfrak{u}_j,\\
\mathfrak{b}_j &= \mathcal{J}_{\lambda_j}^{\mathcal{G}_1}(\mathcal{I} - \lambda_j\mathcal{B})\mathfrak{f}_j,\\
\mathfrak{t}_j &= \mathfrak{b}_j + \lambda_j(\mathcal{B}(\mathfrak{f}_j) - \mathcal{B}(\mathfrak{b}_j)),
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{a}_{j+1} &= \eta\mathfrak{t}_j + (1-\eta)\mathfrak{t}_j,\\
\lambda_{j+1} &= \min\left\{\lambda_j, \frac{\mu\|\mathfrak{f}_j - \mathfrak{b}_j\|}{\|\mathcal{B}(\mathfrak{f}_j) - \mathcal{B}(\mathfrak{b}_j)\|}\right\}.
\end{aligned}$$

Then the sequence $\{\mathfrak{a}_\mathrm{j}\}$ *converges weakly to a point in*

$$\left\{\mathfrak{a} \,:\, \mathfrak{a} \in (\mathcal{G}_1 + \mathcal{B})^{-1}(0),\, \mathcal{A}\mathfrak{a} \in \mathcal{G}_2^{-1}(0)\right\}.$$

The following corollary is obtained by Theorem 3.3 with $\theta = \eta = 1$.

Corollary 3.6. *Suppose that the hypothesis of Theorem 3.3 are satisfied. Take* $\lambda', \lambda_0 > 0$ *and two numbers* $\mu, \alpha \in (0,1)$. *Let* $\{\mathfrak{a}_\mathrm{j}\}$ *be the sequence defined by the following iterative algorithm. Choose* $\mathfrak{a}_0 \in H$ *and compute*

$$\begin{aligned}
\mathfrak{u}_\mathrm{j} &= \mathfrak{a}_\mathrm{j} - \tau\mathcal{A}^*\left(\mathcal{I} - \mathcal{J}_{\lambda'}^{\mathcal{A}}\mathfrak{a}_\mathrm{j}\right),\\
\mathfrak{f}_\mathrm{j} &= (1-\alpha)\,\mathfrak{a}_\mathrm{j} + \alpha\mathfrak{u}_\mathrm{j},\\
\mathfrak{b}_\mathrm{j} &= \mathcal{J}_{\lambda_\mathrm{j}}^{\mathcal{G}_1}\left(\mathcal{I} - \lambda_\mathrm{j}\mathcal{B}\right)\mathfrak{f}_\mathrm{j},
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{a}_{\mathrm{j}+1} &= \mathfrak{b}_\mathrm{j} + \lambda_\mathrm{j}\left(\mathcal{B}\left(\mathfrak{f}_\mathrm{j}\right) - \mathcal{B}\left(\mathfrak{b}_\mathrm{j}\right)\right),\\
\lambda_{\mathrm{j}+1} &= \min\left\{\lambda_\mathrm{j}, \frac{\mu\|\mathfrak{f}_\mathrm{j} - \mathfrak{b}_\mathrm{j}\|}{\|\mathcal{B}(\mathfrak{f}_\mathrm{j}) - \mathcal{B}(\mathfrak{b}_\mathrm{j})\|}\right\}.
\end{aligned}$$

Then the sequence $\{\mathfrak{a}_\mathrm{j}\}$ *converges weakly to a point in*

$$\left\{\mathfrak{a} \,:\, \mathfrak{a} \in (\mathcal{G}_1 + \mathcal{B})^{-1}(0),\, \mathcal{A}\mathfrak{a} \in \mathcal{G}_2^{-1}(0)\right\}.$$

Example 3.7. Let $H = \mathbb{R}^3$ with the inner product defined by

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + x_3y_3,$$

and

$$\|x\|^2 = {x_1}^2 + {x_2}^2 + {x_3}^2, \quad \forall\, x = (x_1, x_2, x_3),\, y = (y_1, y_2, y_3) \in \mathbb{R}^3.$$

We define the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ by

$$\mathcal{G}_1 = \begin{pmatrix} 1/3 & 0 & 0\\ 0 & 1/2 & 0\\ 0 & 0 & 1\end{pmatrix}, \qquad \mathcal{G}_2 = \begin{pmatrix} 1/4 & 0 & 0\\ 0 & 1/3 & 0\\ 0 & 0 & 1/2\end{pmatrix}.$$

Clearly $\mathcal{G}_1$ and $\mathcal{G}_2$ are maximal monotone operators and their resolvents are defined by

$$\mathcal{J}^{\mathcal{G}_1}_{\lambda_j} = (\mathcal{I} + \lambda_j \mathcal{G}_1)^{-1} = \begin{pmatrix} \frac{3}{3+\lambda_j} & 0 & 0 \\ 0 & \frac{2}{2+\lambda_j} & 0 \\ 0 & 0 & \frac{1}{1+\lambda_j} \end{pmatrix},$$

$$\mathcal{J}^{\mathcal{G}_2}_{\lambda'} = (\mathcal{I} + \lambda' \mathcal{G}_2)^{-1} = \begin{pmatrix} \frac{4}{4+\lambda'} & 0 & 0 \\ 0 & \frac{3}{3+\lambda'} & 0 \\ 0 & 0 & \frac{2}{2+\lambda'} \end{pmatrix}.$$

Now, consider a bounded linear operator $\mathcal{A}$ and its adjoint operator $\mathcal{A}^*$ as

$$\mathcal{A} = \begin{pmatrix} 1 & 3 & 2 \\ 4 & 2 & 1 \\ 1 & 3 & 5 \end{pmatrix}, \qquad \mathcal{A}^* = \begin{pmatrix} 1 & 4 & 1 \\ 3 & 2 & 3 \\ 2 & 1 & 5 \end{pmatrix}.$$

Not that $\|\mathcal{A}\|_2 = 7.56$ and so $\tau = 0.0175$. We define the mappings $\mathcal{B}$ and $\mathcal{S}$ by

$$\mathcal{B} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \qquad \mathcal{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Clearly $\mathcal{B}$ is monotone and Lipschitz continuous and $\mathcal{S}$ is nonexpansive. Choose the scalars in Algorithm 3.1 as $\lambda' = \frac{1}{2}$, $\lambda_0 = 1$, $\eta = 0.1$, $\alpha = \frac{1}{4}$, $\mu = 0.9$, $\theta = 0.6$. So all conditions of Theorem 3.3 are satisfied. Now, we consider arbitrary initial points

$$a_0 = (1, 1, 1)^T, \quad (2, 2, 3)^T, \quad (3, 4, 3)^T,$$

in Algorithm 3.1. Then, the sequence generated by the proposed algorithm converges to a solution $(0, 0, 0)^T$. The convergence graph of $\|a_j\|$ and $\|a_{j+1} - a_j\|$ is shown in Figs. 1 and 2. Moreover, Table 1 shows that the numerical results for the three initial points. All of the codes are written in MATLAB r2015a.

4 Conclusion

In this paper, we investigated an iterative algorithm for solving split variational inclusion problem and fixed point problem in real Hilbert

Table 1: Table of Figs. 1 and 2.

a_0		$\|a_j\|$	$\|a_{j+1} - a_j\|$
$(1,1,1)^T$		1.91×10^{-6}	1.17×10^{-6}
	No. Iter.	58	52
	CPU Time(s)	0.00007	0.00002
$(2,2,3)^T$		1.22×10^{-6}	1.15×10^{-6}
	No. Iter.	61	55
	CPU Time(s)	0.0002	0.00002
$(3,4,3)^T$		1.11×10^{-6}	1.19×10^{-6}
	No. Iter.	62	56
	CPU Time(s)	0.00037	0.00002

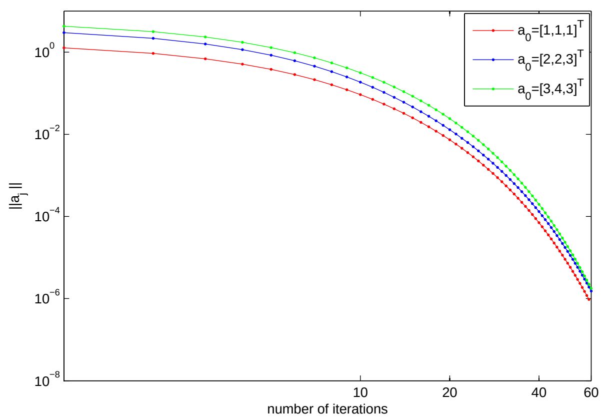

Figure 1: Convergence graph of $\|a_j\|$.

spaces. We proposed an iterative method which consist of fixed point method, Tseng-type splitting method and self-adaptive method for finding a solution of the considered problem. In addition the algorithm used a simple step-size rule without the prior knowledge of Lipschitz constant of the operator and without any linesearch procedure. The step-sizes were updated at each iteration. Also, weak and strong convergence results of the proposed algorithm are obtained under some mild conditions. In addition, we established the convergence rate of the proposed algorithm. Finally, we provided a numerical example to illustrate

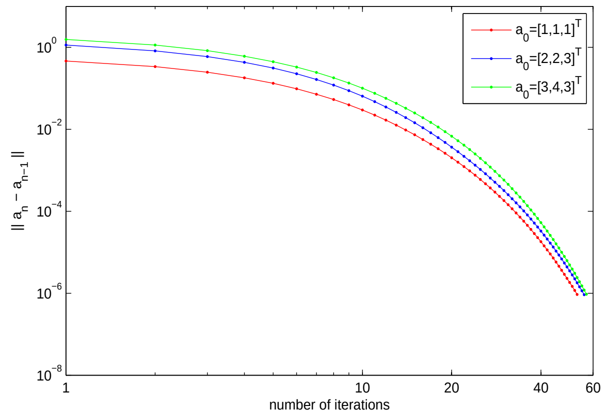

Figure 2: Convergence graph of $\|a_{j+1} - a_j\|$.

the computational performance of the new algorithm.

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