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Numerical Scheme for Solution of Coupled System of Initial Value Fractional Order Fredholm Integro-Differential Equations with Smooth Solutions

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Abstract. We study shifted Legendre polynomials and develop some operational matrices of integrations. We use these operational matrices and develop new sophisticated technique for numerical solutions to the following coupled system of fredholm integro differential equations

$$D^{\alpha}U(x) = f(x) + \lambda_{11}\int_0^1 K_{11}(x,t)U(t)dt + \lambda_{12}\int_0^1 K_{12}(x,t)V(t)dt,$$
$$D^{\alpha}V(x) = g(x) + \lambda_{21}\int_0^1 K_{21}(x,t)U(t)dt + \lambda_{22}\int_0^1 K_{22}(x,t)V(t)dt,$$
$$U(0) = C_1,\, V(0) = C_2,$$

where D^{α} is fractional derivative of order α with respect to x, $0 < \alpha \leqslant 1$, λ_{11}, λ_{12}, λ_{21}, λ_{22} are real constants, f, $g \in C([0,1])$ and K_{11}, K_{12}, K_{21}, $K_{22} \in C([0,1]\times[0,1])$. We develop simple procedure to reduce the coupled system of equations to a system of algebraic equations without discretizing the system. We provide examples and numerical simulations to show the applicability and simplicity of our results and to demonstrate that the results obtained using the new technique matches well with the exact solutions of the problem. We also provide error analysis.

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1. Introduction

Fredholm integro-differential equations arise in many applied problems such as floating structures and viscoelastic material dynamics [32], liquidity risk modeling [26], dynamics of fluid in porus media, water percolation [5, 24] etc. Fredholm integro-differential equations with integer order derivatives are well studied and numerous techniques such as differential transform method [3], Adomian decomposition [6, 4], Homotopy perturbation [7], Modified decomposition [1], Numerical scheme based on rationalized haar function and block pulses [20, 18], Galerkin method with hybrid functions [19], Tau method [16, 17] and Taylor series method [9, 31, 28, 21, 10] etc are available to approximate solutions of fredholm integro-differential equations analytically and numerically.
Recently, the study of Fredholm integro differential equations with fractional order derivatives has attracted some attentions, for example, A. Anguraj [2] developed some useful results for existence of solutions to fractional order integral equations via contraction mapping principle and the Krasnoselskii fixed point theorem. We also refer to [8, 33, 25] for the results on existence of solutions. Beside results on existence of solutions, another important task is to search for solutions of the problem. However, in most cases, exact analytic solutions of fractional order problems are not available. The non availability of the exact solutions of coupled system of fractional order Fredholm integro-differential equations and the wide range of their applications, motivated us to develop some numerical scheme for such system.
There do exist various numerical schemes, some cited above, for numerical solutions of differential equations including fractional order and partial differential equations. One of them is the scheme using operational matrices of integrations and differentiations. The techniques using operational matrices are simple and widely applicable for most problems in differential equations. Recently, we developed a scheme for the numerical solutions of coupled system of fredholm type integral equations [12], and coupled systems of PDEs and FDEs [13, 14, 11].
In this paper, we study the most simplest shifted Legendre polynomials and develop operational matrix of integrations. Based on the new operational matrix along with other matrices available in the literature, we develop a scheme for numerical solutions of the following fractional order coupled system of fredholm integro-differential equations of the form

$$
\begin{aligned}
&D^{\alpha}U(x) = f(x) + \lambda_{11}\int_0^1 K_{11}(x,t)U(t)dt + \lambda_{12}\int_0^1 K_{12}(x,t)V(t)dt,\\
&D^{\alpha}V(x) = g(x) + \lambda_{21}\int_0^1 K_{21}(x,t)U(t)dt + \lambda_{22}\int_0^1 K_{22}(x,t)V(t)dt,\\
&U(0) = C_1,\, V(0) = C_2.
\end{aligned} \tag{1}
$$

The technique convert the system (1) to a system of easily solvable algebraic equations without discretizing the system. We provide a simple numerical scheme which yields highly accurate results. It is worth mentioning that the scheme is computer oriented. We use matlab programming to carry out all the calculation.
The article is organized as follows: In sections 1 and 2, we provide introduction and preliminaries. In Section 3, operational matrices for the kernel function using shifted Legendre polynomials are developed and in section 4, we use the operational matrices for solutions of the coupled system of fredholm integro differential equations. In Section 5, the proposed method is applied to several examples. Finally in Section 6 a short conclusion and acknowledgment about the work is made.

2. Preliminaries

In this section, we recall some basic definitions and known results from fractional calculus, we refer to [27, 15] for more details.

Definition 2.1. *Given an interval $[a,b] \subset \mathbb{R}$, the Riemann-Liouville fractional order integral of order $\alpha \in \mathbb{R}_+$ of a function $\phi \in (L^1[a,b],\mathbb{R})$ is defined by*

$$\mathcal{I}^{\alpha}_{a+}\phi(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-s)^{\alpha-1}\phi(s)ds,$$

provided the integral on right hand side exists.

Definition 2.2. *For a given function $\phi(x) \in C^n[a,b]$, the Caputo fractional order derivative of order α is defined as*

$$D^{\alpha}\phi(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{\phi^{(n)}(t)}{(x-t)^{\alpha+1-n}}dt,\ \ n-1 \leqslant \alpha < n\ ,\ n \in N,$$

provided the right side is pointwise defined on (a,∞), where $n = [\alpha]+1$ in case α not an integer and $n = \alpha$ in case α is an integer.

Hence, it follows that $I^{\alpha}x^{k} = \frac{\Gamma(1+k)}{\Gamma(1+k+\alpha)}x^{k+\alpha}$ for $\alpha > 0$, $k \geqslant 0$, $D^{\alpha}C = 0$, for a constant C and

$$D^{\alpha}x^{k} = \frac{\Gamma(1+k)}{\Gamma(1+k-\alpha)}x^{k-\alpha}, \text{ for } k \geqslant [\alpha]. \tag{2}$$

2.1 The shifted legendre polynomials

The shifted Legendre polynomials are defined by

$$P_i(x) = \sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!}{(i-k)!}\frac{x^k}{(k!)^2}, i = 0, 1, 2, 3..., \tag{3}$$

where $P_i(0) = (-1)^i$, $P_i(1) = 1$ and the orthogonality condition is given by

$$\int_0^1 P_i(x)P_j(x)dx = \begin{cases} \frac{1}{2i+1}, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

Hence, it follows that any $f(x) \in C[0, 1]$ can be approximated as [29]

$$f(x) \approx \sum_{a=0}^{m} C_a P_a(x) = K_M^T \hat{P}_M(x), \tag{4}$$

where $C_a = (2a + 1)\int_0^1 f(x)P_a(x)dx$, $M = m + 1$, K_M is the coefficient vector and $\hat{P}_M(x)$ is M terms function vector. In two dimensional space, Legendre polynomials with two variables are defined as a product functions of two Legendre polynomials and the orthogonality condition is found to be

$$\int_0^1\int_0^1 P_i(x)P_j(t)P_a(x)P_b(t)dxdt = \begin{cases} \frac{1}{(2i+1)(2j+1)}, & \text{if } a = i, b = j; \\ 0, & \text{otherwise.} \end{cases}$$

Any function $f \in C([0, 1] \times [0, 1])$ can be approximated by two dimensional Legendre polynomial as

$$\begin{aligned} f(x,t) &\approx \sum_{i=0}^{m}\sum_{j=0}^{m} C_{ij}P_i(x)P_j(t), \\ C_{ij} &= (2i+1)(2j+1)\int_0^1\int_0^1 f(x,t)P_i(x)P_j(t)dxdt, \end{aligned} \tag{5}$$

which in vector notation, can be written as

$$f(x,t) \approx (\hat{P}_M(x))^T C_{(M\times M)}\hat{P}_M(t), \tag{6}$$

where $\hat{P}_M(x)$ and $\hat{P}_M(t)$ are column vectors of Legendre polynomials and $C_{M\times M}$ is the coefficient matrix. The following result guarantees the convergence of the two dimensional Legendre series.

2.2 Error estimates and convergence

It is known [22] that for a sufficiently smooth function $f(x,t)$ in some domain Ω, if

$$F_m(x,t)=\sum_{i=0}^{m}\sum_{j=0}^{m}C_{ij}P_i(x)P_j(t)=\sum_{i=0}^{m}\sum_{j=0}^{m}C_{ij}\psi_{ij}(x,t)=C^T_{M\times M}\Psi(x,t)$$

be the 2D shifted Legendre function expansion, where

$$C_{M\times M}=[C_{00},C_{01},...,C_{0m},...,C_{2m},....,C_{mm}]^T,$$

then there is a real number α such that $\|f(x,t)-F_m(x,t)\|_2\leqslant\frac{\alpha}{(m+1)!2^{2m+1}}$. Moreover, if $\bar{C}_{M\times M}=[\bar{C}_{00},\bar{C}_{01},...,\bar{C}_{0m},...,\bar{C}_{2m},....,\bar{C}_{mm}]^T$ be an approximation for the 2D shifted Legendre function coefficients vector $C_{M\times M}$ and $\bar{F}_m(x,t)=\bar{C}_{M\times M}\Psi(x,t)$, then there exists a real number β such that $\|f(x,t)-\bar{F}_m(x,t)\|_2\leqslant\frac{\alpha}{(m+1)!2^{2m+1}}+\beta\|F_m-\bar{F}_m)\|_2$. For the proof and more detail study on the theme, we refer to study [22].

3. Operational Matrices of Integrations

Now, we develop operational matrix of integration for the product function $\int_0^1 f(x,t)g(t)dt$ using shifted Legendre polynomials. We use this matrix and a matrix of fractional order integration to reduce the system of equations to a system of algebraic equations.

Lemma 3.1. *Let $f(x,t)\in C([0,1]\times[0,1])$ and $g(t)\in C([0,1])$, then*

$$\int_0^1 f(x,t)g(t)dt\approx K_M G_{M\times M}\hat{P}(x)$$

where K_M is the Legendre coefficient vector of the function $g(t)$ and the matrix $G_{M\times M}=[q_{ij}]$ where $q_{ij}=\frac{1}{2j+1}C_{ij}$, $i,j=1,2,...,M$.

Proof. In view of (4) and (5), we write

$$f(x,t)\approx\sum_{i=0}^{m}\sum_{j=0}^{m}C_{ij}P_i(x)P_j(t),$$

$$C_{ij}=(2j+1)(2i+1)\int_0^1\int_0^1 f(x,t)P_i(x)P_j(t)dxdt,$$

$$g(t) \approx \sum_{a=0}^{m} d_a P_a(t),\ d_a = (2a+1)\int_0^1 g(t)P_a(t)dt.$$

Hence,

$$\int_0^1 f(x,t)g(t)dt \approx \int_0^1 (\sum_{i=0}^{m}\sum_{j=0}^{m} C_{ij}P_i(x)P_j(t))(\sum_{a=0}^{m} d_a P_a(t))dt,$$

which implies that

$$\int_0^1 f(x,t)g(t)dt \approx \sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{a=0}^{m} d_a C_{ij}P_i(x)\int_0^1 P_j(t)P_a(t)dt.$$

Using the orthogonality relation, we obtain

$$\int_0^1 f(x,t)g(t)dt \approx \sum_{i=0}^{m}\sum_{j=0}^{m} d_j C_{ij}P_i(x)(\frac{1}{2j+1}) = \sum_{j=0}^{m}\sum_{i=0}^{m} d_j q_{ij}P_i(x), \tag{7}$$

where $q_{ij} = \frac{1}{2j+1}C_{ij}$. In matrix notation, (7) can be written as

$$\int_0^1 f(x,t)g(t)dt \approx K_M G_{M\times M}\hat{P}(x),$$

where $G_{M\times M} = [q_{ij}]_{M\times M}$ and the desire result follows. $\square$

The following result is known [30].

Lemma 3.2. *Let $\hat{P}(t)$ be the function vector as defined in (4) then the fractional integration of order α of $\hat{P}(t)$ is given by*

$$I^\alpha(\hat{P}(t)) \simeq P^\alpha \hat{P}(t)$$

where P^α is the operational matrix of integration of order α and is defined as

$$P^\alpha = \begin{pmatrix} \sum_{k=0}^{0}\Theta_{0,0,k} & \sum_{k=0}^{0}\Theta_{0,1,k} & \cdots & \sum_{k=0}^{0}\Theta_{0,j,k} & \cdots & \sum_{k=0}^{0}\Theta_{0,m,j} \\ \sum_{k=0}^{1}\Theta_{1,0,k} & \sum_{k=0}^{1}\Theta_{1,1,k} & \cdots & \sum_{k=0}^{1}\Theta_{1,j,k} & \cdots & \sum_{k=0}^{1}\Theta_{1,m,j} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{k=0}^{i}\Theta_{i,0,k} & \sum_{k=0}^{i}\Theta_{i,1,k} & \cdots & \sum_{k=0}^{i}\Theta_{i,j,k} & \cdots & \sum_{k=0}^{i}\Theta_{i,m,k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{k=0}^{m}\Theta_{m,0,k} & \sum_{k=0}^{m}\Theta_{m,1,k} & \cdots & \sum_{k=0}^{m}\Theta_{m,j,k} & \cdots & \sum_{k=0}^{m}\Theta_{m,m,k} \end{pmatrix},$$

where

$$\Theta_{i,j,k} = (2j+1)\sum_{l=0}^{j}\frac{(-1)^{i+j+k+l}(i+k)!(l+j)!}{(i-k)!k!\Gamma(k+\alpha+1)(j-l)!(l!)^2(k+l+\alpha+1)}.$$

4. Application of the Operational Matrices

Now, we develop a scheme for numerical solutions of the system (1). We seek the solutions $U(x)$ and $V(x)$ in terms of shifted Legendre polynomials such that the following hold

$$D^{\alpha}U(x) \approx H_M^T \hat{P}(x),\ D^{\alpha}V(x) \approx L_M^T \hat{P}(x), \tag{8}$$

where H_M and L_M are respectively are the unknown coefficient vector to be calculated. Applying I^{α} on both the side of (8), using (3) and the initial conditions, we obtain

$$U(x) - C_1 \approx H_M^T P_{M\times M}^{\alpha} \hat{P}(x),\ V(x) - C_2 \approx L_M^T P_{M\times M}^{\alpha} \hat{P}(x), \tag{9}$$

which implies that

$$\begin{aligned} U(x) &\approx (H_M^T P_{M\times M}^{\alpha} + \widetilde{C_1})\hat{P}(x) \\ V(x) &\approx (L_M^T P_{M\times M}^{\alpha} + \widetilde{C_2})\hat{P}(x) \end{aligned} \tag{10}$$

where $\widetilde{C_1}\hat{P}(x) \approx C_1$ and $\widetilde{C_2}\hat{P}(x) \approx C_2$. For simplicity, we use the notations

$$(H_M^T P_{M\times M}^{\alpha} + \widetilde{C_1}) = E_M,\ (L_M^T P_{M\times M}^{\alpha} + \widetilde{C_2}) = R_M. \tag{11}$$

Now, in view of Lemma (3.1) and (11), we obtain

$$\begin{aligned} &\int_0^1 K_{11}(x,t)U(t)dt \approx E_M G_{11}\hat{P}(x),\ \int_0^1 K_{12}(x,t)V(t)dt \approx R_M G_{12}\hat{P}(x), \\ &\int_0^1 K_{21}(x,t)U(t)dt \approx E_M G_{21}\hat{P}(x),\ \int_0^1 K_{22}(x,t)V(t)dt \approx R_M G_{22}\hat{P}(x), \end{aligned} \tag{12}$$

where G_{11}, G_{12}, G_{21}, G_{22} are $M \times M$ matrices corresponding to K_{11}, K_{12}, K_{21}, K_{22} respectively. Further, writing $f(x) \approx F_1\hat{P}(x)$, $g(x) \approx F_2\hat{P}(x)$, where F_1, F_2 are M terms coefficient vectors for f and g. Using (8) and (12) in (1), we obtain

$$\begin{aligned} H_M^T\hat{P}(x) &= \lambda_{11}E_M^T G_{11}\hat{P}(x) + \lambda_{12}R_M^T G_{12}\hat{P}(x) + F_1\hat{P}(x) \\ L_M^T\hat{P}(x) &= \lambda_{21}E_M^T G_{21}\hat{P}(x) + \lambda_{22}R_M^T G_{22}\hat{P}(x) + F_2\hat{P}(x), \end{aligned}$$

which in matrix notation can be rewritten as

$$\begin{pmatrix} H_M^T\hat{P}(x) \\ L_M^T\hat{P}(x) \end{pmatrix} = \begin{pmatrix} \lambda_{11}E_M^T G_{11}\hat{P}(x) \\ \lambda_{22}R_M^T G_{22}\hat{P}(x) \end{pmatrix} + \begin{pmatrix} \lambda_{12}R_M^T G_{12}\hat{P}(x) \\ \lambda_{21}E_M^T G_{21}\hat{P}(x) \end{pmatrix} + \begin{pmatrix} F_1\hat{P}(x) \\ F_2\hat{P}(x) \end{pmatrix},$$

Taking the transpose of the above matrix, we get

$$\begin{aligned} &\begin{pmatrix} H_M^T\hat{P}(x) & L_M^T\hat{P}(x) \end{pmatrix} = \begin{pmatrix} \lambda_{11}E_M^T G_{11}\hat{P}(x) & \lambda_{22}R_M^T G_{22}\hat{P}(x) \end{pmatrix} + \\ &\begin{pmatrix} \lambda_{12}R_M^T G_{12}\hat{P}(x) & \lambda_{21}E_M^T G_{21}\hat{P}(x) \end{pmatrix} + \begin{pmatrix} F_1\hat{P}(x) & F_2\hat{P}(x) \end{pmatrix}, \end{aligned}$$

and after simplification, we obtain

$$\begin{pmatrix} H_M^T & L_M^T \end{pmatrix} A = \begin{pmatrix} E_M^T & R_M^T \end{pmatrix} \begin{pmatrix} \lambda_{11}G_{11} & 0 \\ 0 & \lambda_{22}G_{22} \end{pmatrix} A +$$
$$\begin{pmatrix} E_M^T & R_M^T \end{pmatrix} \begin{pmatrix} 0 & \lambda_{21}G_{21} \\ \lambda_{12}G_{12} & 0 \end{pmatrix} A + \begin{pmatrix} F_1 & F_2 \end{pmatrix} A,$$

where

$$A = \begin{pmatrix} \hat{P}(x) & 0 \\ 0 & \hat{P}(x) \end{pmatrix}.$$

Hence, it follows that

$$\begin{pmatrix} H_M^T & L_M^T \end{pmatrix} - \begin{pmatrix} E_m^T & R_m^T \end{pmatrix} \begin{pmatrix} \lambda_{11}G_{11} & \lambda_{21}G_{21} \\ \lambda_{12}G_{12} & \lambda_{22}G_{22} \end{pmatrix} - \begin{pmatrix} F_1 & F_2 \end{pmatrix} = 0,$$

which in view of (11) yields

$$\begin{pmatrix} H_M^T & L_M^T \end{pmatrix} - \begin{pmatrix} H_M^T & L_M^T \end{pmatrix} \begin{pmatrix} \lambda_{11}P^\alpha G_{11} & \lambda_{21}P^\alpha G_{21} \\ \lambda_{12}P^\alpha G_{12} & \lambda_{22}P^\alpha G_{22} \end{pmatrix}$$
$$- \begin{pmatrix} \widetilde{C_1} & \widetilde{C_2} \end{pmatrix} \begin{pmatrix} \lambda_{11}G_{11} & \lambda_{21}G_{21} \\ \lambda_{12}G_{12} & \lambda_{22}G_{22} \end{pmatrix} - \begin{pmatrix} F_1 & F_2 \end{pmatrix} = 0,$$

a generalized Sylvestor type equation which can easily be solved for the unknown H_M and L_M by using any computational software. Using H_M and L_M in (10) we can get the approximate solution to the system.

5. Illustrative Examples

We apply the technique to some problems whose exact solutions are known.

Example 5.1. Consider the following system of equations

$$\begin{aligned} D^\alpha U(x) =& f(x) + 2\int_0^1 (x+t)U(t)dt - 2\int_0^1 (x-1)V(t)dt, \\ D^\alpha V(x) =& g(x) + 3\int_0^1 (x^2+t^2)U(t)dt - 3\int_0^1 (x^2-t^2)V(t)dt. \end{aligned}$$

subject to the conditions

$$U(0) = 10,\ V(0) = 14,$$

where $f(x) = -16x^3 + 9x^2 + \frac{83x}{10} - \frac{679}{30}$ and $g(x) = -28x^3 + 10x - \frac{231x^2}{20} - \frac{1539}{70}$. The exact solution of the problem is $U(x) = -4x^4 + 3x^2 + 2x^2 + 10$ and $V(x) = -7x^4 - 6x^2 + 5x^2 + 14$. We compare the approximate solution of the problem obtained with the method of the paper to the exact solution for different values of M. We see that the solution obtained via this new techniques matches well with the exact solution of the problem even for relatively small value of $M = 3$, which shows the rapid convergence. This phenomena is shown in Fig (1), where the blue dots and yellow dots respectively represent the exact solution $U(x)$, $V(x)$ while the red line and green line respectively represent the approximate solution $U(x)$, $V(x)$. Further, We provide numerical simulations of the the scheme for different values of α. It is observed that as $\alpha \to 1$, the various solutions approaches the exact solution of the problem at $\alpha = 1$. This phenomena is shown in Fig (2) and Fig (3). We see that the error decreases significantly as the value of M increases. This phenomena can easily be observed from the Fig (4) and Fig (5). Further, we note that if the exact solution is a polynomial of degree M. Then, at scale equal to M, this scheme will provide the exact solution as it also evident from the figure (Fig (4), Fig (5)).

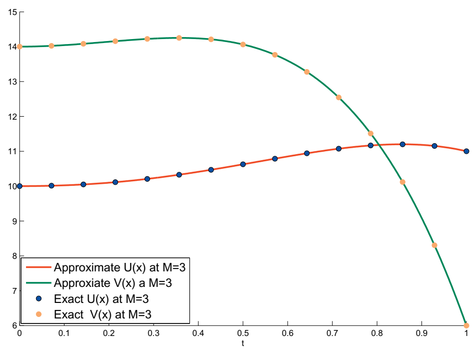

Figure 1: Comparison of the exact solution $U(x)$ and $V(x)$, with the solution obtained with the method of the paper at $M = 3$. Dots represents the exact solution while lines represents the approximate solution of the problem.

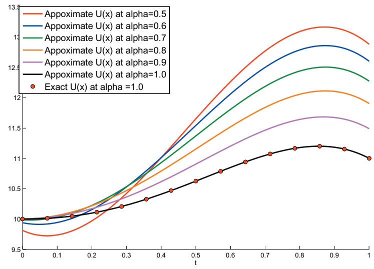

Figure 2: At $M = 4$, $U(x)$ is given for different values of α such as for $\alpha = 0.5$ (*red line*), $\alpha = 0.6$ (*blue line*), $\alpha = 0.7$ (*green line*), $\alpha = 0.8$ (*orange line*), $\alpha = 0.9$ (*pink line*), $\alpha = 1.0$ (*black line*) and the blue dots represents the exact $U(x)$.

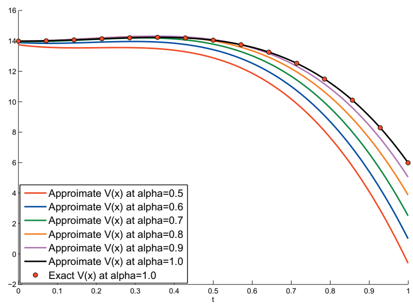

Figure 3: At $M = 4$, $V(x)$ is given for different values of α such as for $\alpha = 0.5$ (*red line*), $\alpha = 0.6$ (*blue line*), $\alpha = 0.7$ (*green line*), $\alpha = 0.8$ (*orange line*), $\alpha = 0.9$ (*pink line*), $\alpha = 1.0$ (*black line*) and the red dots represents the exact $V(x)$.

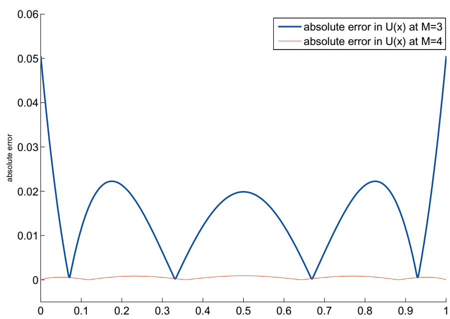

Figure 4: Absolute error in $U(x)$ obtained with new method at $M = 3$ (*blue line*) and $M = 4$ (*red line*).

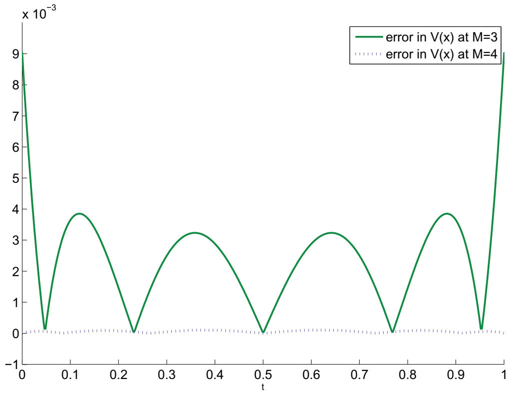

Figure 5: Absolute error in $V(x)$ obtained with new method at $M = 3$ (*green line*) and $M = 4$ (*purple line*).

Example 5.2. Consider the following system of equations

$$\begin{aligned} D^{\alpha}U(x) =& f(x) + \int_0^1 sin(x-t)U(t)dt + \int_0^1 cos(x+t)V(t)dt, \\ D^{\alpha}V(x) =& g(x) + \int_0^1 sin(x+t)U(t)dt + \int_0^1 sin(x-t)V(t)dt. \end{aligned}$$

subject to the conditions

$$U(0) = 0,\ V(0) = 1,$$

where $f(x) = \pi cos(\pi x) - \frac{sin(x+1)+sin(x)}{\pi^2-1} - \frac{\pi sin(x-1)+sin(x)}{\pi^2-1}$ and $g(x) = -(\frac{cos(x-1)+cos(x)}{\pi^2-1} + \pi sin(\pi x) + \frac{\pi sin(x+1)+\pi sin(x)}{\pi^2-1}$. The exact solution of the problem is $U(x) = sin(\pi x)$ and $V(x) = cos(\pi x)$. Comparison of the exact solution with that of the approximate solution obtained with the new method for different value of M is shown in Fig (6) which demonstrate that as the scale level M increases the approximate solutions approaches the exact solution of the problem. The solutions for different values of α are displayed in Fig (7), Fig (8) and the same conclusion follows as in Example (5.1). The absolute error is shown in Fig (9) and Fig (10). We note that the error decreases significantly as the scale level M increases. As evident from Fig (9) and Fig (10) that the absolute error is << 10^{-3}, which is much more acceptable number and guarantees the high accuracy of the method.We also investigate the behavior of solution at high scale level, that is $M = 10, 20, 25, 30$. We observe that the solution becomes more and more accurate. See for example Fig (11) and Fig (12). One can see that the error of approximation is also much more less than 10^{-15}. This guarantees the convergence of approximate solutions to the exact solutions.

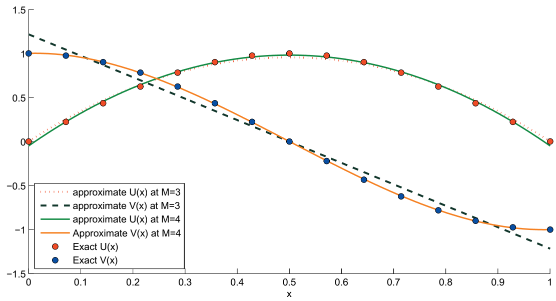

Figure 6: Comparing the exact solution with the solution obtained with the new method at $(M = 4, 5)$ red dots and blue dots represents the exact $U(x)$ and $V(x)$, while the dashed line represents the approximate solutions at $M = 3$ and solid lines represents approximate solution at $M = 4$.

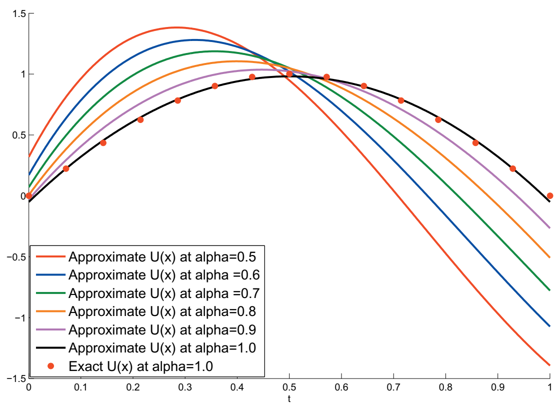

Figure 7: Approximate U(x) at $M = 4$ and for $\alpha = 0.5$ (*red line*), $\alpha = 0.6$ (*blue line*), $\alpha = 0.7$ (*green line*), $\alpha = 0.8$ (*orange line*), $\alpha = 0.9$ (*pink line*), $\alpha = 1.0$ (*black line*) and the red dots represents the exact U(x).

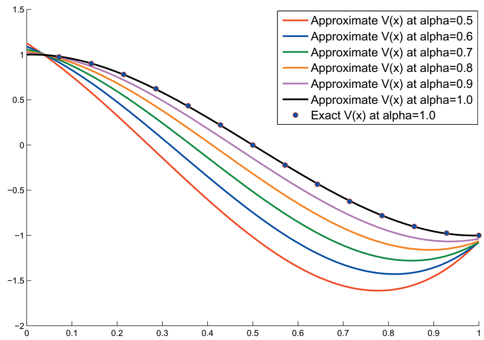

Figure 8: Approximate V(x) at $M = 4$ and for $\alpha = 0.5$ (*red line*), $\alpha = 0.6$ (*blue line*), $\alpha = 0.7$ (*green line*), $\alpha = 0.8$ (*orange line*), $\alpha = 0.9$ (*pink line*), $\alpha = 1.0$ (*black line*) and the blue dots represents the exact U(x).

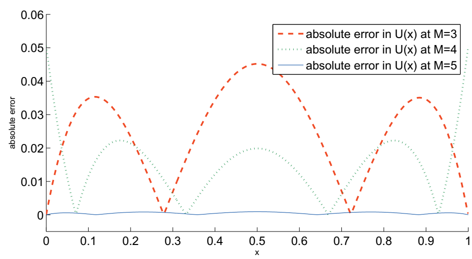

Figure 9: Amount of absolute error in U(x) at different value of M,i.e. red line (M=3), green lines (M=4) and blue lines for (M=5).

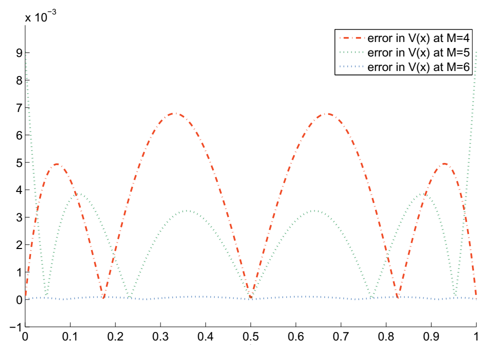

Figure 10: Amount of absolute error in V(x) at different value of M,i.e. red line (M=4), green lines (M=5) and blue lines for (M=6).

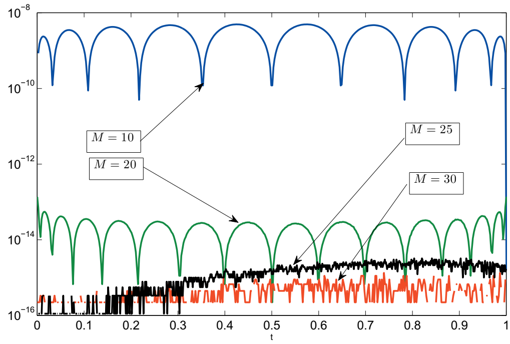

Figure 11: Amount of absolute error in U(x) at different value of M.

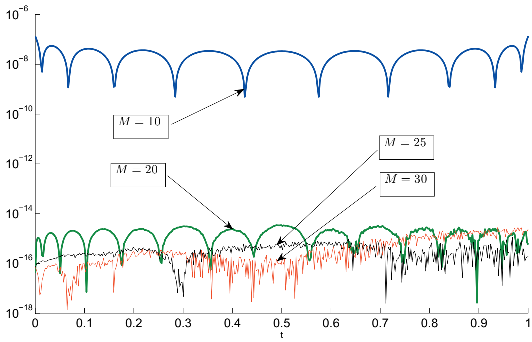

Figure 12: Amount of absolute error in V(x) at different value of M.

Example 5.3. From the above two examples we see that the the solution is too much accurate and also the convergence of the approximate solution is shown but only for integer order. In this example we will show that the solution also converges to the exact solution at fractional value of α. Consider the following coupled system

$$\begin{aligned} D^{8/10}U(x) =& f(x) + \int_0^1 (x+3+t)^2 U(t)dt + \int_0^1 (x+t)^2 V(t)dt, \\ D^{8/10}V(x) =& g(x) + \int_0^1 (x+4t+2)U(t)dt + \int_0^1 (x^2+t^3+xt)V(t)dt, \end{aligned}$$

We select the source terms

$f(x) = \frac{55666529}{60690000}x + \frac{7847968538941487}{36028797018963968}(\frac{15625}{924}x^{21/5} - \frac{1875}{22}x^{16/5} + \frac{1300}{11}x^{11/5} - \frac{503}{6}x^{6/5} + \frac{1251}{50}x^{1/5}) + \frac{8543}{30000}(x+3)^2 + \frac{3279}{5780}x^2 + \frac{643897}{606900}$, and $g(x) =$

$+\frac{17836292133957925}{874635076432369287168}$

$(180625\, x^{21/5}+151725\, x^{16/5}+149940\, x^{11/5}-200200\, x^{6/5}-52668x^{1/5})+$

$\frac{3279}{5780}x^2+\frac{12526163}{20230000}x+\frac{111591679}{91035000}.$

The exact solutions of this problem is known and is defined as $U(x) = (x-1)^5-(x-1/10)^4$ and $V(x) = ((x+1)^3-(x-2/17)^2)x(x-1)$. We approximate solutions of this problem at different scale level and observe that the approximation solutions become more and more accurate with the increase of scale level.This phenomena is shown in Fig (13). The absolute error of the solution $U(x)$ and $V(x)$ is shown in Fig (14) and Fig (15) respectively.One can see that the error is much more less than 10^{-4}. Which is acceptable number.

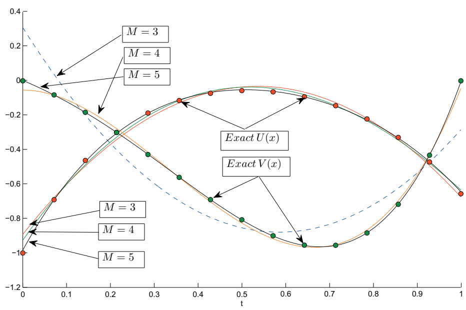

Figure 13: Comparison of exact solution of Example 5.3 with approximate solutions at different scale level. The solid dots represents the exact solution while the lines represents the approximate solutions.

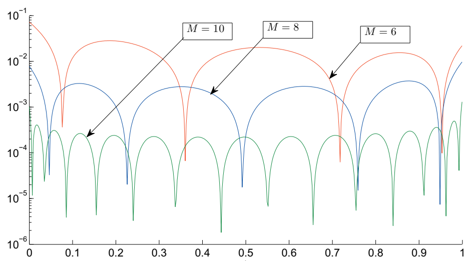

Figure 14: Amount of absolute error in U(x) of Example 5.3 at high scale level.

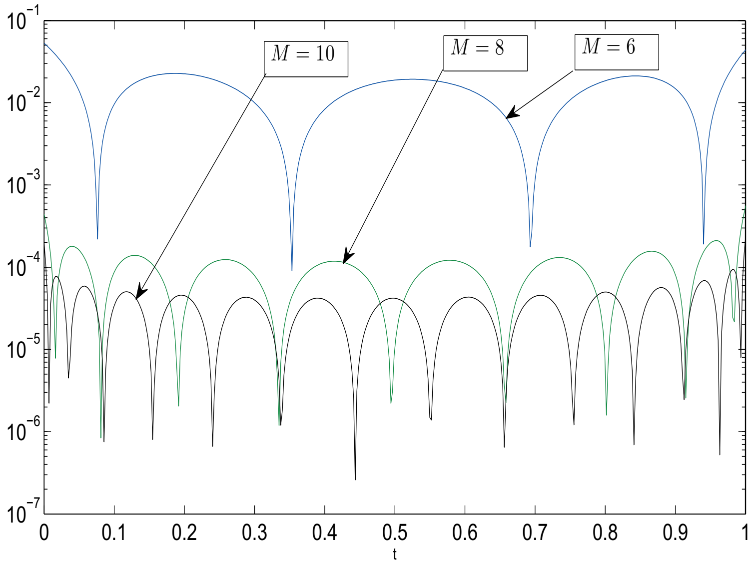

Figure 15: Amount of absolute error in V(x) of Example 5.3 at high scale level.

6. Conclusion and future work

From the above analysis and calculations we observe that the method provides high accurate estimator of the approximate solutions. The method works well for coupled system with initial conditions. The method approximate the solutions of such systems with smooth solutions. Our future work is related to solve such problems with boundary conditions. We will also extend the method to solve problems having non smooth solutions. The accuracy obtained with the current method is satisfactory. It is also expected that the accuracy is improved by using other orthogonal polynomials like Brenstein polynomials, Jacobi polynomials, Laguerre polynomials etc.

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