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Norm of Difference of General Polynomial Weighted Differentiation Composition Operators from Cauchy Transform Spaces into Derivative Hardy Spaces

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Abstract. In this paper, we study the boundedness of the difference of general polynomial weighted differentiation composition operators from the Cauchy transform spaces into the function spaces $S = \{f : \ f' \in H^1\}$ and $S^2 = \{f : \ f' \in H^2\}$ with derivative in Hardy spaces. We also derive an exact formula for the norm of these operators. Furthermore, as a corollary, we will prove that there is no composition isometry from the Cauchy transform spaces into the spaces S and S^2.

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1 Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the space of analytic functions on $\mathbb{D}$. For $0 < p < \infty$, a function $f \in H(\mathbb{D})$ is said to belong to the Hardy space H^p if

$$\|f\|_{H^p} = \sup_{0<r<1} \Big(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \Big)^{\frac{1}{p}} < \infty.$$

When $1 \leq p < \infty$, H^p is a Banach space with the norm $\| \cdot \|_p$.

The space S^p consists of all functions $f \in H(\mathbb{D})$ such that $f' \in H^p$. This space is Banach space with the following norm

$$\|f\|_{S^p} = |f(0)| + \|f'\|_{H^p}.$$

The space of Cauchy transform functions can be viewed as a connection between analytic function theory and measure theory. If f is analytic in $\overline{\mathbb{D}}$, then using the Cauchy formula, we have

$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(\zeta)}{\zeta - z} d\zeta, \qquad z \in \mathbb{D},$$

where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. The above formula is a special case of the Cauchy transform: A function f is Cauchy transform or $f \in \mathcal{F}$ if it has a representation as

$$f(z) = \int_{\mathbb{T}} \frac{1}{1 - \overline{\zeta} z} d\mu(\zeta), \qquad z \in \mathbb{D},$$

where $\mu \in \mathcal{M}$, $\mathcal{M}$ is the space of all complex valued Borel measures on $\mathbb{T}$ with the total variation norm.

For $\varphi \in \mathcal{S}(\mathbb{D}) = \{\varphi \in H(\mathbb{D}) : \varphi(\mathbb{D}) \subset \mathbb{D}\}$ and $\psi \in H(\mathbb{D})$, the composition operator C_φ, the multiplication operator M_ψ, and the iterated differentiation operator D^n are defined respectively, as

$$C_\varphi f = f \circ \varphi, \quad M_\psi f = \psi f, \quad D^n f = f^{(n)}, \quad f \in H(\mathbb{D}),$$

where $f^{(0)} = f$. These operators and products of these operators and their differences have been studied extensively on spaces of analytic functions in the past four decades, (see, e.g., [1] -[24] and the related references therein).

Stević et. al. came to an idea of investigating the sums of two product-type operators in [19], [20] and [21]. After publishing [21] in 2015, Stević proposed to study the operator

$$T_{\vec{u},\varphi}f(z)=\sum_{k=0}^{n}u_k(z)f^{(k)}(\varphi)(z),\ f\in H(\mathbb{D}),$$

where $\vec{u}=(u_0,...,u_n)$ such that $\{u_k\}_{k=0}^{n}\subset H(\mathbb{D})$ and $\varphi\in\mathcal{S}(\mathbb{D})$. Investigations of these types of operators as well as some related ones attracted some attention during the last decade (see, for instance, [1, 2, 3, 4, 9, 15, 17, 22, 24] and the related references therein). Recently Stević [16] introduced general polynomial differentiation composition operator which includes the previous operators and is defined as

$$T^n_{\vec{u},\vec{\varphi}}f(z)=\sum_{j=0}^{n}u_j(z)f^{(j)}(\varphi_j(z))=\sum_{j=0}^{n}(D^j_{u_j,\varphi_j}f)(z),\quad n\in\mathbb{N}_0,$$

where $u_j\in H(\mathbb{D})$ and $\varphi_j\in S(\mathbb{D})$.

Let $i,n\in\mathbb{N}_0$, $u_i,v_i\in H(\mathbb{D})$ and $\varphi_i,\psi_i\in S(\mathbb{D})$. We set

$$L=\sum_{j=0}^{n}(D^j_{u_j,\varphi_j}-D^j_{v_j,\psi_j})=T^n_{\vec{u},\vec{\varphi}}-T^n_{\vec{v},\vec{\psi}}.$$

The purpose of this paper is to characterize differences of general polynomial differentiation composition operator L from the Cauchy transform spaces into the spaces S and S^2. Also we will obtain an exact formula for the norm of these operators. Furthermore, as a corollary, we will prove that there is no composition isometry from the Cauchy transform spaces into the spaces $S=\{f:\quad f'\in H\}$ and $S^2=\{f:\quad f'\in H^2\}$.

2 Main Results

In the following theorem, we consider the boundedness of the operator $L:\mathcal{F}\to S$ and we find the exact formula for the norm of this operator.

Theorem 2.1. *Let $0 \leq j \leq n$, $\varphi_j, \psi_j \in S(\mathbb{D})$ and $u_j, v_j \in H(\mathbb{D})$. Then the operator $L : \mathcal{F} \to S$ is bounded if and only if*

$$\sup_{\xi \in \mathbb{T}} \left(Q_1(\xi) + Q_2(\xi) \right) < \infty, \tag{1}$$

where

$$Q_1(\xi) = \left| \sum_{j=0}^{n} j! \bar{\xi}^j \left(\frac{u_j(0)}{(1-\bar{\xi}\varphi_j(0))^{j+1}} - \frac{-v_j(0)}{(1-\bar{\xi}\psi_j(0))^{j+1}} \right) \right|,$$

$$Q_2(\xi) = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \Big| \sum_{j=0}^{n} \sum_{t=0}^{1} (j+t)! \bar{\xi}^{j+t} \Big(\frac{E^t_{u_j,\varphi_j}(re^{i\theta})}{(1-\bar{\xi}\varphi_j(re^{i\theta}))^{j+t+1}} - \frac{E^t_{v_j,\varphi_j}(re^{i\theta})}{(1-\bar{\xi}\psi_j(re^{i\theta}))^{j+t+1}} \Big) \Big| d\theta,$$

$E^0_{u_j,\varphi_j}(re^{i\theta}) = u_j'(re^{i\theta})$ *and* $E^1_{u_j,\varphi_j}(re^{i\theta}) = u_j(re^{i\theta})\varphi'(re^{i\theta})$.
Moreover, in this case,

$$\|L\| = \sup_{\xi \in \mathbb{T}} \left(Q_1(\xi) + Q_2(\xi) \right).$$

Proof. For any $\xi \in \mathbb{T}$, the function

$$f_\xi(z) = \frac{1}{1-\bar{\xi}z}, \qquad z \in \mathbb{D}$$

belong to $\mathcal{F}$ with $\|f_\xi\|_{\mathcal{F}} = 1$ (see[23]) also for any $j \in \mathbb{N}_0$, $f_\xi^{(j)}(z) = \frac{j!\bar{\xi}^j}{(1-\bar{\xi}z)^{j+1}}$. Therefore, for any $\xi \in \mathbb{T}$, we have

$$\begin{aligned} |(Lf_\xi)(0)| &= \left| \sum_{j=0}^{n} (u_j(0) f_\xi^{(j)}(\varphi_j(0)) - v_j(0) f_\xi^{(j)}(\psi_j(0)) \right| \\ &= \left| \sum_{j=0}^{n} j! \bar{\xi}^j \left(\frac{u_j(0)}{(1-\bar{\xi}\varphi_j(0))^{j+1}} - \frac{v_j(0)}{(1-\bar{\xi}\psi_j(0))^{j+1}} \right) \right| \\ &= Q_1(\xi) \end{aligned} \tag{2}$$

and

$$\begin{aligned}
\|(Lf_\xi)'\|_{H^1} &= \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |(Lf_\xi)'(re^{i\theta})| d\theta \\
&= \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} \Big| \sum_{j=0}^{n}\sum_{t=0}^{1} E^t_{u_j,\varphi_j}(re^{i\theta}) f_\xi^{(j+t)}(\varphi_j(re^{i\theta})) \\
&\quad - E^t_{v_j,\psi_j}(re^{i\theta}) f_\xi^{(j+t)}(\psi_j(re^{i\theta}))\Big| d\theta \\
&= \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} \Big| \sum_{j=0}^{n}\sum_{t=0}^{1} (j+t)!\bar{\xi}^{j+t}\Big(\frac{E^t_{u_j,\varphi_j}(re^{i\theta})}{(1-\bar{\xi}\varphi_j(re^{i\theta}))^{j+t+1}} \\
&\quad - \frac{E^t_{v_j,\varphi_j}(re^{i\theta})}{(1-\bar{\xi}\psi_j(re^{i\theta}))^{j+t+1}}\Big)\Big| d\theta \\
&= Q_2(\xi).
\end{aligned}$$

Let $L : \mathcal{F} \to S$ be bounded. Then for any $\xi \in \mathbb{T}$, we get

$$\|L\| \geq \|Lf_\xi\|_{S^1} \geq |(Lf_\xi)(0)| + \|(Lf_\xi)'\|_{H^1} = Q_1(\xi) + Q_2(\xi).$$

Therefore, by taking supremum in the above inequality, we obtain

$$\sup_{\xi\in\mathbb{T}} (Q_1(\xi) + Q_2(\xi)) \leq \|L\|. \tag{3}$$

Now we assume that the condition of (1) holds. For any $f \in \mathcal{F}$ there exists $\mu \in \mathcal{M}$ such that

$$f(z) = \int_{\mathbb{T}} \frac{d\mu(\xi)}{1-\bar{\xi}z}, \qquad z \in \mathbb{D},$$

and $\|f\|_{\mathcal{F}} = \|\mu\|$. Therefore, for each $k \in \mathbb{N}_0$, we have

$$f^{(k)}(z) = k!\int_{\mathbb{T}} \frac{\bar{\xi}^k d\mu(\xi)}{(1-\bar{\xi}z)^{k+1}}, \qquad z \in \mathbb{D}.$$

Hence,

$$
\begin{aligned}
|(Lf)(0)| &= \left|\sum_{j=0}^{n} u_j(0)f^{(j)}(\varphi_j(0)) - v_j(0)f^{(j)}(\psi_j(0))\right| \qquad (4)\\
&= \left|\sum_{j=0}^{n} j! \int_{\mathbb{T}} \bar{\xi}^j \Big(\frac{u_j(0)}{(1-\bar{\xi}\varphi_j(0))^{j+1}} - \frac{v_j(0)}{(1-\bar{\xi}\psi_j(0))^{j+1}}\Big) d\mu(\xi)\right|\\
&\leq \int_{\mathbb{T}} \left|\sum_{j=0}^{n} j!\bar{\xi}^j \Big(\frac{u_j(0)}{(1-\bar{\xi}\varphi_j(0))^{j+1}} - \frac{v_j(0)}{(1-\bar{\xi}\psi_j(0))^{j+1}}\Big)\right| d|\mu|(\xi)\\
&= \int_{\mathbb{T}} Q_1(\xi) d|\mu|(\xi),
\end{aligned}
$$

and by using Fubini's Theorem, we get

$$
\begin{aligned}
\|(Lf)'\|_{H^1} &= \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \Big| \sum_{j=0}^{n} \sum_{t=0}^{1} (j+t)! \int_{\mathbb{T}} \bar{\xi}^{j+t} \Big(\frac{E^t_{u_j,\varphi_j}(re^{i\theta})}{(1-\bar{\xi}\varphi_j(re^{i\theta}))^{j+t+1}}\\
&\quad - \frac{E^t_{v_j,\psi_j}(re^{i\theta})}{(1-\bar{\xi}\psi_j(re^{i\theta}))^{j+t+1}}\Big) d\mu(\xi)\Big| d\theta\\
&\leq \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{T}} \Big| \sum_{j=0}^{n} \sum_{t=0}^{1} (j+t)! \bar{\xi}^{j+t} \Big(\frac{E^t_{u_j,\varphi_j}(re^{i\theta})}{(1-\bar{\xi}\varphi_j(re^{i\theta}))^{j+t+1}}\\
&\quad - \frac{E^t_{v_j,\psi_j}(re^{i\theta})}{(1-\bar{\xi}\psi_j(re^{i\theta}))^{j+t+1}}\Big)\Big| d|\mu|(\xi) d\theta\\
&\leq \int_{\mathbb{T}} \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \Big| \sum_{j=0}^{n} \sum_{t=0}^{1} (j+t)! \bar{\xi}^{j+t} \Big(\frac{E^t_{u_j,\varphi_j}(re^{i\theta})}{(1-\bar{\xi}\varphi_j(re^{i\theta}))^{j+t+1}}\\
&\quad - \frac{E^t_{v_j,\psi_j}(re^{i\theta})}{(1-\bar{\xi}\psi_j(re^{i\theta}))^{j+t+1}}\Big)\Big| d\theta d|\mu|(\xi)\\
&\leq \int_{\mathbb{T}} Q_2(\xi) d|\mu|(\xi).
\end{aligned}
$$

By applying the last inequalities, we obtain

$$\begin{aligned}|(Lf)(0)| + \|(Lf)'\|_{H^1} &\leq \int_{\mathbb{T}} (Q_1(\xi) + Q_2(\xi)) d|\mu|(\xi)\\ &\leq \sup_{\xi\in\mathbb{T}} \left(Q_1(\xi) + Q_2(\xi)\right) \int_{\mathbb{T}} d|\mu|(\xi)\\ &\leq \sup_{\xi\in\mathbb{T}} \left(Q_1(\xi) + Q_2(\xi)\right) \|\mu\|\\ &\leq \sup_{\xi\in\mathbb{T}} \left(Q_1(\xi) + Q_2(\xi)\right) \|f\|_{\mathcal{F}}.\end{aligned}$$

Therefore,

$$\|L\| \leq \sup_{\xi\in\mathbb{T}} \left(Q_1(\xi) + Q_2(\xi)\right).$$

By using (3) and the preceding inequality, we have

$$\|L\| = \sup_{\xi\in\mathbb{T}} \left(Q_1(\xi) + Q_2(\xi)\right).$$

The proof is completed. □

In the following theorem, we examine the boundedness of the operator $L : \mathcal{F} \to S^2$ and provide an approximation for the norm of this operator.

Theorem 2.2. *Let $0 \leq j \leq n$, $\varphi_j, \psi_j \in S(\mathbb{D})$ and $u_j, v_j \in H(\mathbb{D})$. Then the operator $L : \mathcal{F} \to S^2$ is bounded if and only if*

$$\sup_{\xi\in\mathbb{T}} \Big(Q_1(\xi) + Q_3(\xi)\Big) < \infty, \tag{5}$$

where

$$\begin{aligned}Q_3(\xi) = \sup_{0<r<1} \Big(\frac{1}{2\pi} \int_0^{2\pi} \Big| \sum_{j=0}^{n} \sum_{0}^{1} (j+t)! \bar{\xi}^{j+t} \Big(&\frac{E^t_{u_j,\varphi_j}(re^{i\theta})}{(1-\bar{\xi}\varphi_j(re^{i\theta}))^{j+t+1}}\\ &- \frac{E^t_{v_j,\psi_j}(re^{i\theta})}{(1-\bar{\xi}\psi_j(re^{i\theta}))^{j+t+1}}\Big)\Big|^2 d\theta\Big)^{\frac{1}{2}}\end{aligned}$$

and $Q_1(\xi)$, $E^t_{u_j,\varphi_j}$ are defined in Theorem 2.1. Moreover, in this case,

$$\sup_{\xi\in\mathbb{T}} \Big(Q_1(\xi) + Q_3(\xi)\Big) \leq \|L\| \leq \sup_{\xi\in\mathbb{T}} Q_1(\xi) + \sup_{\xi\in\mathbb{T}} Q_3(\xi).$$

Proof. Let the operator $L : \mathcal{F} \to S^2$ be bounded, therefore for any $\xi \in \mathbb{D}$, we have

$$\begin{aligned}
\|(Lf_\xi)'\|_{H^2}^2 &= \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left|(Lf_\xi)'(re^{i\theta})\right|^2 d\theta \\
&= \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \Big| \sum_{j=0}^{n} \sum_{t=0}^{1} (j+t)! \bar{\xi}^{j+t} \Big(\frac{E^t_{u_j,\varphi_j}(re^{i\theta})}{(1-\bar{\xi}\varphi_j(re^{i\theta}))^{j+t+1}} \\
&\quad - \frac{E^t_{v_j,\psi_j}(re^{i\theta})}{(1-\bar{\xi}\psi_j(re^{i\theta}))^{j+t+1}} \Big) \Big|^2 d\theta \\
&= Q_3(\xi)^2.
\end{aligned}$$

Applying (2) and preceding equation, for any $\xi \in \mathbb{T}$, we obtain

$$\begin{aligned}
Q_1(\xi) + Q_3(\xi) &= |(Lf_\xi)(0)| + \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left|(Lf_\xi)'(re^{i\theta})\right|^2 d\theta \right)^{\frac{1}{2}} \\
&\le \|Lf_\xi\|_{S^2} \\
&\le \|L\| \|f_\xi\|_{\mathcal{F}} = \|L\|.
\end{aligned}$$

So,

$$\sup_{\xi \in \mathbb{T}} (Q_1(\xi) + Q_3(\xi)) \le \|L\|.$$

Now, we suppose that (5) holds. Thus, for any $f \in \mathcal{F}$, there exists $\mu \in \mathcal{M}$ such that

$$f(z) = \int_{\mathbb{T}} \frac{d\mu(\xi)}{1-\bar{\xi}z}, \qquad z \in \mathbb{D},$$

and $\|f\|_{\mathcal{F}} = \|\mu\|$. Therefore, for any $f \in \mathcal{F}$, by applying a similar calculation as in equation (4), we get

$$|(Lf)(0)| \le \int_{\mathbb{T}} Q_1(\xi) d|\mu|(\xi) \le \|\mu\| \sup_{\xi \in \mathbb{T}} Q_1(\xi).$$

By using Jensen's inequality and Fubini's Theorem, we have

$$\begin{aligned}
\|(Lf)'\|_{H^2}^2 &= \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left|(Lf)'(re^{i\theta})\right|^2 d\theta \\
&= \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \Big| \sum_{j=0}^{n} \sum_{t=0}^{1} (j+t)! \int_{\mathbb{T}} \bar{\xi}^{j+t} \big(\frac{E^t_{u_j,\varphi_j}(re^{i\theta})}{(1-\bar{\xi}\varphi_j(re^{i\theta}))^{j+t+1}} \\
&\quad - \frac{E^t_{v_j,\psi_j}(re^{i\theta})}{(1-\bar{\xi}\psi_j(re^{i\theta}))^{j+t+1}} \big) d\mu(\xi) \Big|^2 d\theta \\
&= \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \|\mu\|^2 \Big| \sum_{j=0}^{n} \sum_{t=0}^{1} (j+t)! \int_{\mathbb{T}} \bar{\xi}^{j+t} \big(\frac{E^t_{u_j,\varphi_j}(re^{i\theta})}{(1-\bar{\xi}\varphi_j(re^{i\theta}))^{j+t+1}} \\
&\quad - \frac{E^t_{v_j,\psi_j}(re^{i\theta})}{(1-\bar{\xi}\psi_j(re^{i\theta}))^{j+t+1}} \big) \frac{d\mu(\xi)}{\|\mu\|} \Big|^2 d\theta \\
&\le \int_{\mathbb{T}} \|\mu\|^2 \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \Big| \sum_{j=0}^{n} \sum_{t=0}^{1} (j+t)! \bar{\xi}^{j+t} \big(\frac{E^t_{u_j,\varphi_j}(re^{i\theta})}{(1-\bar{\xi}\varphi_j(re^{i\theta}))^{j+t+1}} \\
&\quad - \frac{E^t_{v_j,\psi_j}(re^{i\theta})}{(1-\bar{\xi}\psi_j(re^{i\theta}))^{j+t+1}} \big) \Big|^2 d\theta \frac{d|\mu|(\xi)}{\|\mu\|} \\
&\le \int_{\mathbb{T}} \|\mu\| Q_3^2(\xi) d|\mu|(\xi) \\
&\le \|\mu\|^2 (\sup_{\xi\in\mathbb{T}} Q_3(\xi))^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|(Lf)(0)| + \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left|(Lf)'(re^{i\theta})\right|^2 d\theta \right)^{\frac{1}{2}} &\le \\
\left(\sup_{\xi\in\mathbb{T}} Q_1(\xi) + \sup_{\xi\in\mathbb{T}} Q_3(\xi) \right) \|\mu\| &\le \\
\left(\sup_{\xi\in\mathbb{T}} Q_1(\xi) + \sup_{\xi\in\mathbb{T}} Q_3(\xi) \right) \|f\|_{\mathcal{F}}&.
\end{aligned}$$

Hence,

$$\|L\| \le \sup_{\xi\in\mathbb{T}} Q_1(\xi) + \sup_{\xi\in\mathbb{T}} Q_3(\xi).$$

The proof is completed. $\square$

By setting appropriate parameters in Theorems 2.1 and 2.2, we derive the following two corollaries.

Corollary 2.3. *Let $\varphi \in S(\mathbb{D})$. Then*

$$\|C_\varphi\|_{\mathcal{F}\to S} = \sup_{\xi\in\mathbb{T}}\left(\frac{1}{|1-\bar{\xi}\varphi(0)|} + \sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}\frac{|\varphi'(re^{i\theta})|}{|1-\bar{\xi}\varphi(z)|^2}d\theta\right).$$

Corollary 2.4. *Ler $\varphi \in S(\mathbb{D})$. Then*

$$\sup_{\xi\in\mathbb{T}}\left(\frac{1}{|1-\bar{\xi}\varphi(0)|} + \sup_{0<r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{|\varphi'(re^{i\theta})|^2}{|1-\bar{\xi}\varphi(z)|^4}d\theta\right)^{\frac{1}{2}}\right) \leq \|C_\varphi\|_{\mathcal{F}\to S^2} \leq$$

$$\sup_{\xi\in\mathbb{T}}\frac{1}{|1-\bar{\xi}\varphi(0)|} + \sup_{\xi\in\mathbb{T}}\sup_{0<r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{|\varphi'(re^{i\theta})|^2}{|1-\bar{\xi}\varphi(z)|^4}d\theta\right)^{\frac{1}{2}}.$$

Corollary 2.5. *Let $\varphi \in S(\mathbb{D})$ such that $\varphi(0) \neq 0$. Then*

$$\max\{\|C_\varphi\|_{\mathcal{F}\to S}, \|C_\varphi\|_{\mathcal{F}\to S^2}\} > 1.$$

Proof. Since $\varphi(0) \neq 0$, so there exists $0 \leq \theta_0 < 2\pi$ such that $\varphi(0) = |\varphi(0)|e^{i\theta_0}$, so

$$\sup_{\xi\in\mathbb{T}}\frac{1}{|1-\bar{\xi}\varphi(0)|} \geq \frac{1}{|1-\overline{e^{i\theta_0}}|\varphi(0)|e^{i\theta_0}|} = \frac{1}{1-|\varphi(0)|} > 1.$$

Hence,

$$\begin{aligned}\|C_\varphi\|_{\mathcal{F}\to S} &= \sup_{\xi\in\mathbb{T}}\left(\frac{1}{|1-\bar{\xi}\varphi(0)|} + \sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}\frac{|\varphi'(re^{i\theta})|}{|1-\bar{\xi}\varphi(z)|^2}d\theta\right)\\ &\geq \frac{1}{|1-\overline{e^{i\theta_0}}\varphi(0)|} + \sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}\frac{|\varphi'(re^{i\theta})|}{|1-\overline{e^{i\theta_0}}\varphi(z)|^2}d\theta\\ &> 1 + \sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}\frac{|\varphi'(re^{i\theta})|}{|1-\overline{e^{i\theta_0}}\varphi(z)|^2}d\theta.\end{aligned}$$

With the same calculation, we have $\|C_\varphi\|_{\mathcal{F}\to S^2} > 1$. $\square$

Corollary 2.6. *Let $\varphi \in S(\mathbb{D})$ such that $\varphi \neq 0$. If $\varphi(0) = 0$ then*

$$\max\{\|C_\varphi\|_{\mathcal{F}\to S}, \|C_\varphi\|_{\mathcal{F}\to S^2}\} > 1.$$

Proof. It is clear that for any $\xi \in \mathbb{T}$, we have

$$|1-\overline{\xi}\varphi(z)|^2 \leq (1+|\overline{\xi}\varphi(z)|)^2 \leq 2^2 = 4.$$

Let $\varphi(0)=0$, so we have

$$\begin{aligned}\|C_\varphi\|_{\mathcal{F}\to S} &= \sup_{\xi\in\mathbb{T}}\left(\frac{1}{|1-\overline{\xi}\varphi(0)|} + \sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}\frac{|\varphi'(re^{i\theta})|}{|1-\overline{\xi}\varphi(z)|^2}d\theta\right)\\ &= 1+\sup_{\xi\in\mathbb{T}}\sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}\frac{|\varphi'(re^{i\theta})|}{|1-\overline{\xi}\varphi(z)|^2}d\theta\\ &\geq 1+\frac{1}{8\pi}\sup_{0<r<1}\int_0^{2\pi}|\varphi'(re^{i\theta})|d\theta.\end{aligned}$$

If $\|C_\varphi\|_{\mathcal{F}\to S}=1$ then $\sup_{0<r<1}\int_0^{2\pi}|\varphi'(re^{i\theta})|d\theta = 0$. This implies that φ must be constant, resulting in $\varphi=\varphi(0)=0$, which contradicts the initial assumption. Similarly, we get $\|C_\varphi\|_{\mathcal{F}\to S^2} > 1$. □

Recall that a linear operator T between the normed linear spaces X and Y with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ is called an isometry, if for any $x\in X$

$$\|Tx\|_Y = \|x\|_X.$$

Theorem 2.7. *Let $\varphi \in S(\mathbb{D})$. There is no composition isometry from the Cauchy transform spaces into the space S or S^2.*

Proof. If $\varphi\equiv 0$ then the operator $C_\varphi : \mathcal{F}\to S(C_\varphi : \mathcal{F}\to S^2)$ is not injective, so it cannot be an isometry. When $\varphi\neq 0$ the proof is clear by using corollaries 2.5 and 2.6. □

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