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Invertible Subspace-Hypercyclic Operators

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Abstract. A bounded linear operator on a Banach space X is called subspace-hypercyclic for a subspace M if $Orb(T, x) \cap M$ is dense in M for a vector $x \in M$. In this paper we give conditions under which an operator is M-hypercyclic. Then by this result, we answer in the affirmative a question that recently raised by Madore and Martnez-Avendano.

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1. Introduction

Let X be a Banach space over the field $\mathcal{C}$ of complex numbers. In this article, the symbol $\mathcal{N}$ denotes the set of all positive integers and the symbol T stands for a bounded linear operator acting on X. Consider any subset D of X, the symbol $Orb(T, D)$ denotes the orbit of D under the operator T, i.e. $Orb(T, D) = \{T^n x : x \in D, n = 0, 1, 2, ...\}$. If $D = \{x\}$ is a singleton set and the orbit $Orb(T, x)$ is dense in X, then the

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operator T is called hypercyclic and the vector x is a hypercyclic vector for T. If $D = \{\lambda x : \lambda \in \mathcal{C}\}$ and the set $Orb(T, D)$ is dense in X, then the operator T is called supercyclic and the vector x is a supercyclic vector for T. Observe that in case the operator T is hypercyclic the underlying Banach space X should be separable. Then it is well known that an operator T is hypercyclic if and only if T is topologically transitive, to be precise, for every pair of nonempty open sets U, V of X there exists a non-negative integer n such that $T^n(U) \cap V \neq \emptyset$. See the recent books [1], [2] and the articles [7] and [8] for some details on hypercyclicity and supercyclicity and related properties.
Recently, B. F. Madore and R. A. Martinez-Avendano introduced the concept of subspace-hypercyclicity, [5]. An operator T is subspace-hypercyclic (or M-hypercyclic) for a subspace M of X if there exists a vector $x \in X$ such that;

$$\overline{Orb(T, x) \cap M} = M.$$

Also authors introduced the notion of subspace-transitivity with respect to subspace M (or M-transitivity) and show that M-transitivity implies M-hypercyclicity, and C. M. Le construct an operator T such that it is M-hypercyclic but it is not M-transitive, [4]. In [5], authors raised five questions relating subspace-hypercyclicity and we are interested in the first one; "let T be an invertible operator. If T is an M-hypercyclic operator, is T^{-1} subspace-hypercyclic for some space? If so, for which space?". In this paper, we investigate a condition under which both operators T and T^{-1} are M-hypercyclic. Other sources of examples and some properties of notions relating subspace-hypercyclicity are [3] and [6].

2. Basic Notions and Main Results

Let M be a subspace of a Banach space X. In this section we give some basic definitions that we need to state our main results. Indeed, we give equivalent conditions for an operator being M-mixing. Also, we give an example satisfying the main theorem and then by an example we see that the subspace-hypercyclicity does not imply subspace-mixing. Finally our main result helps us, that under a condition to provide an affirmative answer to the question that raised by Madore and Martnez-Avendano.

Definition 2.1. *An operator $T \in B(X)$ is called topologically mixing or mixing if for any nonempty open subsets U, V there exists a non-negative integer N such that $T^{-n}(U) \cap V \neq \emptyset$ for all $n \geqslant N$.*

Definition 2.2. *Let $T \in B(X)$. We say that T is M-*transitive *if for any nonempty subsets U and V, both relatively open in M, there exists $n \geqslant 0$ such that $T^{-n}(U) \cap V$ is a relatively open nonempty subset of M.*

Definition 2.3. *Let $T \in B(X)$. We say that T is M-*mixing *if for any nonempty sets $U \subseteq M$ and $V \subseteq M$, both relatively open, there exists $N \geqslant 0$ such that for all $n \geqslant N$, $T^{-n}(U) \cap V$ is a relatively open nonempty subset of M.*

Definition 2.4. *Let T be an operator. For every $x \in M$ the set*

$$
\begin{aligned}
J^{mix}(T, M, x) = \{y \in M : & \ for\ every\ relatively\ open\ neighborhoods \\
& U, V\ of\ x, y\ in\ M\ respectively,\ and\ every\ positive\ integer \\
& n \geqslant N,\ T^n(U) \cap V \neq \emptyset\ and\ T^n(M) \subseteq M\ for\ some\ N \in \mathcal{N}\}
\end{aligned}
$$

denote the $M-$extended mixing limit set of x under T.

Theorem 2.5. *Let T be an operator on X. Then the following conditions are equivalent:*

(i) T is an M-mixing.

(ii) For every $x \in M$, $J^{mix}(T, M, x) = M$.

Proof. We first prove that *(i)* implies *(ii)*. Let U and V be nonempty relatively open subsets of M. Then there exists $N \geqslant 0$ such that for all $n \geqslant N$, $W_n = T^{-n}(U) \cap V$ is a relatively open nonempty subset of M. Now fix $x \in M$, $n \geqslant N$, and since $W_n \subseteq T^{-n}(U)$, it follows that $T^n(W_n) \subseteq M$. Take $x_n \in W_n$, then for $r_n > 0$ small enough, we have $x_n + r_n x \in W_n$, and hence $T^n(x_n + r_n x) \in M$. Since $r_n > 0$ and $T^n(x_n) \in M$, it follows that $T^n x \in M$ and $T^n(M) \subseteq M$. Therefore $J^{mix}(T, M, x) = M$, for every $x \in M$.

Now we prove that *(ii)* $\Rightarrow$ *(i)*. Let $U \subseteq M$, $V \subseteq M$, both nonempty and relatively open. Consider $x_0 \in U$, $y_0 \in V$, and since $J^{mix}(T, M, y_0) = M$, then there exists a non-negative integer N such that for all $n \geqslant N$;

$$T^n(M) \subseteq M \quad and \quad T^n(V) \cap U \neq \emptyset.$$

Fix $n \geqslant N$, hence $(T^n)_{|M} : M \longrightarrow M$ is continuous and consequently $T^{-n}(U) \cap M$ is relatively open in M. Therefore $T^{-n}(U) \cap V$ is a relatively open nonempty subset of M. $\square$

Example 2.6. Let $T = 2B$ where B is the backward shift on l^2, i. e. for every $x = (x_0, x_1, x_2, ...) \in l^2$;

$$T^n(x_0, x_1, x_2, ...) = 2^n(x_n, x_{n+1}, ...), \;\; n \in \mathcal{N}.$$

It is well known that $T \oplus I : l^2 \times l^2 \longrightarrow l^2 \times l^2$ is M-hypercyclic where $M = l^2 \times \{0\}$, but $T \oplus I$ is not hypercyclic, see [5]. Suppose that $U \subseteq l^2$, $V \subseteq l^2$, both relatively open, and $x = (x_0, x_1, x_2, ...) \in U$, $y = (y_0, y_1, y_2, ...) \in V$, Then $U \times \{0\}$ and $V \times \{0\}$ are two relatively open sets of M. Now for all $n \in \mathcal{N}$ set;

$$z_n = (x_0, x_1, ..., x_{n-1}, \frac{y_0}{2^n}, \frac{y_1}{2^n}, ..., \frac{y_{n-1}}{2^n}, \frac{y_0}{2^{2n}}, \frac{y_1}{2^{2n}}, ..., \frac{y_{n-1}}{2^{2n}}, ...),$$

then $z_n \longrightarrow x$ and $T^n z_n \longrightarrow y$. Hence there exists a non-negative integer N such that;

$$\forall n \geqslant N, \qquad (T \oplus I)^n(U \times \{0\}) \cap (V \times \{0\}) \neq \emptyset.$$

Since $(T \oplus I)(M) \subseteq M$, then using a similar argument as in Theorem 2.5 we conclude that that the operator $T \oplus I$ is M-mixing.

The next example will show that subspace-hypercyclicity does not imply subspace-mixing with respect to an subspace M.

Example 2.7. Let $\lambda \in \mathbf{C}$ be of modulus greater than 1 and let B be the backward shift on l^2. Let m be a positive integer and M be the subspace of l^2 consisting of all sequences with zero on the first m entries, i. e.

$$M = \{\{a_n\}_{n=0}^{\infty} \in l^2 : \; a_n = 0 \quad for \quad n \leqslant m\},$$

then $T = \lambda B$ is M-hypercyclic, see [5]. Now consider

$$V = \{\{a_n\}_{n=0}^{\infty} \in l^2 : \ a_n = 0 \ for \ n \leqslant m \ \ and \ \mid a_n \mid > 0 \ for \ n > m\},$$

so V is relatively open subset of M. If $N = m + 1$, then for every $n > N$, $T^n(V) \cap M = \emptyset$. Thus for every $x \in M$, $J^{mix}(T, M, x) \neq M$ and consequently the operator T is not an M-mixing operator.

In [5] authors raised five questions relating subspace-hypercyclicity and we are interested in the first one. The characterization of M-mixing operators in Theorem 2.5 helps us to answer in the affirmative question in an additional assumption: “let T be an invertible operator. If T is an M-hypercyclic operator, is T^{-1} subspace-hypercyclic for some space? If so, for which space?”.

Theorem 2.8. *Let T be an invertible and M-mixing operator. Then T^{-1} is M-hypercyclic.*

Proof. Let $x, y \in M$ and U, V are relatively open subsets of M such that contain x, y respectively. Then Theorem 2.5 and the invertibility of T imply that there exists $N \geqslant 0$ such that for all $n \geqslant N$;

$$T^n(U) \cap V \neq \emptyset \qquad and \qquad T^{-n}(M) \subseteq M.$$

Thus $x \in J^{mix}(T^{-1}, M, y)$ and consequently;

$$\forall y \in M, \ \ M = J^{mix}(T^{-1}, M, y).$$

Therefore Theorem 2.5 implies that T^{-1} is an M-mixing operator, and obviously every M-mixing operator is an M-hypercyclic operator. □

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