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Original Research Paper

Convexity of the Spectrum of a Multiplication Operator

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Abstract. Let F be a compact subset of the complex plane, m be the Lebesgue measure and $\nu = m|_F$. If A is the multiplication operator on $L^2(\nu)$ and $C^*(A)$ is the C^*-algebra generated by A, then F is convex if and only if the pure state space of $C^*(A)$ is convex.

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1 Introduction

Let H be a complex Hilbert space with inner product $\langle .,. \rangle$ and the norm $\|.\|$. Let $B(H)$ be the set of all bounded linear operators on H and $T \in B(H)$. The spectrum of T is the set of all $\lambda \in \mathbb{C}$ for which the operator $T - \lambda I$ does not have an inverse that is a bounded linear operator. The spectrum of T is denoted by $\sigma(T)$ and it is a non-empty

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compact subset of the complex plane. The numerical range $W(T)$ is defined by

$$W(T) = \{< Tx, x >: x \in H, \|x\| = 1\}.$$

The numerical range of an operator is a subset of the complex plane. One of the most important properties of the numerical range is its convexity. Another important property of $W(T)$ is that its closure contains the spectrum i.e. $\sigma(T) \subseteq \overline{W(T)}$. Numerical range is a connected set and for normal operator N,

$$\overline{W(N)} = co(\sigma(N)), \tag{1}$$

where the $co(\sigma(N))$ is the convex hull of $\sigma(N)$. A more complete discussion about the numerical range of operators on a Hilbert space is presented in sources [5] and [6] and recently in [14].

The attainment problem is one of the numerical range problems. The attainment problem asks: which subsets of the complex plane are the numerical ranges of a bounded linar operator in a Hilbert space? In Hilbert spaces with dimension $\geq 2^{\aleph}$, due to the crowdedness of the space, everything happens and it can be proved that every convex and compact subset of the complex plane is the numerical range of a diagonal, and hence normal, operator [8].

In this paper we aim to prove that each compact subset F of the complex plane is the spectrum of the multiplication operator A on $L^2(\nu)$, where ν is the Lebesgue meaure on $\mathbb{R}^2$ restricted on F. Let $C^*(A)$ be the C^*-algebra generated by A. We show that: F is convex if and only if the pure state space of $C^*(A)$, the space of all non-zero homomorphisms on $C^*(A)$ to $\mathbb{C}$ together with wk*-topology, is convex.

2 Main Results

Let $\mathcal{A}$ be a unital C*-algebra with dual $\mathcal{A}'$ and state space $\mathcal{S} = \{\varphi \in \mathcal{A}' : \varphi \geq 0, \varphi(1) = 1\}$. By the Krein-Milman theorem, the state space $\mathcal{S}$ has extreme points. The extreme points of the state space are termed pure states and is denoted by $\mathcal{M}(\mathcal{A})$. The C*-algebra numerical range of the element $a \in \mathcal{A}$, $V(a)$, is defined by

$$V(a) := \{\varphi(a) : \varphi \in \mathcal{S}\}.$$

Unlike the numerical range of bounded linear operators on infinite dimensional Hilbert spaces, which may not be closed, the numerical range of elements of C^*-algebras is always closed. Other properties, convexity, inclusion in the closed sphere to the center of the origin and radius $\|a\|$, and linearity it is similar to the numerical range of the operator (Refer to [2] and [9] for additional information).

The above definition is a generalization of the definition of the numerical range of bounded linear operators on Hilbert spaces, which means that, if $\mathcal{A} = B(H)$, for some complex Hilbert space H, and $T \in \mathcal{A}$, then $V(T)$ is the closure of $W(T)$.

If $a \in \mathcal{A}$, then a is normal if $aa^* = a^*a$. This definition stems from the definition of a normal linear operator. A bounded linear operator on a comlex Hilbert space is normal if commutes with its adjoint. For more details, see [4].
By an attainment problem, we have the following proposition:

Proposition 2.1. *Every non-empty compact subset of the complex plane is the spectrum of a normal operator.*

Remark 2.2. Any non-empty compact subset F of $\mathbb{C}$ is the spectrum of a normal operator acting on $\ell^2(\mathbb{N})$. One simply chooses a dense subset $\{d_n\}_n$ in F, and defines the diagonal (hence normal) operator $D_F = Diag(d_n)$ with respect to some orthonormal basis for $\ell^2(\mathbb{N})$. Then $\sigma(D_F) = \overline{\{d_n\}_n} = F$. Since the closure of the numerical range of a normal operator is the convex hull of its spectrum(see Problem 216 of [6]), thus, if F is non-empty, compact and convex, then with D_F as defined above,

$$W(D_F) = co(\sigma(D_F)) = co(F) = F.$$

The next theorem, a standard result (basic Gelfand theory) in C*-algebra theory (e.g. Theorem 2.1.13 of [11]), is the most important tool to prove the main result of this section.

Theorem 2.3. *If $\mathcal{A}$ is an abelian C*-algebra with pure state space $\mathcal{M}$ and $a \in \mathcal{A}$ such that $\mathcal{A} = C^*(a)$, then the map $\gamma : \mathcal{M} \to \sigma(a)$ defined by $\gamma(\phi) = \phi(a)$ is a homeomorphism.*

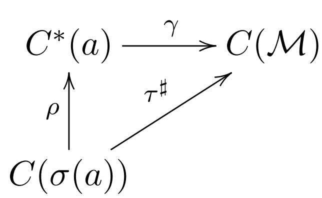
Proof. Since $\mathcal{A}$ is an abelian C*-algebra, then its pure state space, $\mathcal{M}$, is a wk* compact Hausdorff space. To prove this statement, suppose that $\{\phi_i\}$ is a net in $\mathcal{M}$ such that $\phi_i \to \phi$, in wk* topology, for some $\phi \in ball(\mathcal{A}^*)$. If $x, y \in \mathcal{A}$, then

$$\phi(xy) = \lim_i \phi_i(xy) = \lim_i \phi_i(x)\phi_i(y) = \phi(x)\phi(y).$$

So ϕ is a homomorphism. Since

$$\phi(1) = \lim_i \phi_i(1) = 1,$$

thus $\phi \neq 0$ and therefore $\phi \in \mathcal{M}$.
If $\phi \in \mathcal{M}$ and $\lambda = \phi(a)$, then $a - \lambda \in \ker\phi$. So $a - \lambda$ is not invertible and $\lambda \in \sigma(a)$; that is, $\mathcal{M}(a) \subseteq \sigma(a)$. Now assume that $\lambda \in \sigma(a)$; so $a - \lambda$ is not invertible and, hence, $(a - \lambda)\mathcal{A}$ is a proper ideal. Let Y be a maximal ideal in $\mathcal{A}$ such that $(a - \lambda)\mathcal{A} \subseteq Y$. If $\phi \in \mathcal{M}$ such that $Y = \ker\phi$, then $0 = \phi(a - \lambda) = \phi(a) - \lambda$; hence $\sigma(a) \subseteq \mathcal{M}(a)$. For continiuty of γ, let $\phi_i \to \phi$, in wk* topology in $\mathcal{M}$. Then $\gamma(\phi_i) = \phi_i(a) \to \phi(a) = \gamma(\phi)$. This implies γ is continuous. By Gelfand transform, $\mathcal{A}$ is $*$-isomorphism to $C(\mathcal{M})$, the commutative C^*-algebra of continuous complex functions on $\mathcal{M}$. If $\rho : C(\sigma(a)) \to C^*(a)$ is the functional calculus, $\tau^\sharp : C(\sigma(a)) \to C(\mathcal{M})$, $\tau^\sharp(f) = fo\tau$, where $\tau : \mathcal{M} \to \sigma(a)$, $\tau(\phi) = \phi(a)$, then the following diagram is comutative i.e, $\gamma o \rho = \tau^\sharp$,



And this completes the proof of the theorem. □

Remark 2.4. Suppose F is a compact subset of the complex plane. Then, by Proposition 2.1, there is a normal operator N on Hilbert space H such that $\sigma(N) = F$. Put $\mathcal{A} = C^*(N)$, the abelian C^*-algebra generated by N, which results from the normality of N and the continuity of involution and $\mathcal{S}$ be the state space of $\mathcal{A}$. Then by Theorem 2.3 the map $\gamma : \mathcal{S} \to F$ defined by $\gamma(\phi) = \phi(N)$ is a homeomorphism. Since

$\{\phi(N) : \phi \in \mathcal{S}\}$ is the C*-algebra numerical range of N, then, by convexity of numerical range, F is convex if and only if the pure state space of $\mathcal{A}$ is convex [3]. One simple counterexample is $F = \{0, 1\}$. It's the spectrum of the 2-by-2 diagonal matrix diag $(0, 1)$, but since it's disconnected, it cannot be homeomorphic to any compact convex subset. It turns out that the above main result (each compact subset is homeomorphic to a compact convex subset) is a special case of one proved by D. A. Herrero [7], saying that any nonempty bounded convex set in the complex plane is homeomorphic to the numerical range of some bounded linear operator on a Hilbert space. The states space $\mathcal{S}$ and the maximal ideals space $\mathcal{M}(\mathcal{A})$ are different spaces. There $\mathcal{M}(\mathcal{A})$ is a subspace of S which consists of multiplicative functionals and $\mathcal{S} = co\mathcal{M}(\mathcal{A})$. So, if $\mathcal{M}(\mathcal{A})$ is convex then, F is convex.

The question arises that there is such a space with these conditions. For the possibility of this condition or for the sake of non-obviousness the Remark 2.4, we provide an example.

Example 2.5. Let X be a compact Hausdorff topological space, γ be a self homeomorphism of X, $\Sigma = (X, \gamma)$ be the topological dynamical system respect to X and γ and $\ell^1(\Sigma)$ be a crossed product Banach $*$-algebra associated with these data. If X is a singleton set, then $\ell^1(\Sigma)$ is the group algebra of the integers. The commutant $C(X)_1^{'}$ of $C(X)$ in $\ell^1(\Sigma)$ is known to be a maximal abelian subalgebra which has non-zero intersection with each non-zero closed ideal. The maximal ideal space of $C(X)_1^{'}$ is described explicitly, and is seen to coincide with its pure state space and to be a topological quotient of $X \times \mathbb{T}$, where $\mathbb{T}$ is the unit circle in complex plane. For more details see [10].

The attainment problem for numerical range needs to be modified to keep it interesting. To make the prbolem interesting, we assume the condition of separability of Hilbert space as mandatory i.e., which nonempty bounded convex subset of $\mathbb{R}^2$ is the numerical range of some operator on a separable Hilbert space? For this, we must characterize the numerical range of bounded linear operators on separable Hilbert spaces. For example, the numerical range of any normal operator on a separable Hilbert space is a Borel set [12]. Also the numerical range of any operator on a separable Hilbert space is $G_{\delta\sigma}$ [1]. In a survey paper [15],

the authors considerd the problem of which nonempty bounded convex subset of the complex plane is the numerical range of some bounded linear operator on a complex separable Hilbert space. In [13], the authors asked, is every compact convex set the numerical range of some operator on separable Hilbert space. This question is still open. For further discussion, concepts will be needed. A closed subspace M of $\mathcal{H}$ is called reducing subspace for $T \in B(\mathcal{H})$ if $T(M) \subseteq M$ and $T(M^{\perp}) \subseteq M^{\perp}$. A vector x in $\mathcal{H}$ is a star-cyclic vector if $\mathcal{H}$ is the smallest reducing subspace for T that contains x. The operator T is star cyclic if it has a star-cyclic vector. A vector x is a star-cyclic vector for T if and only if $\mathcal{H} = \overline{\{Sx : S \in C^*(T)\}}$, where $C^*(T)$ is the C^*-algebra generated by T, i.e, $\{p(T,T^*)x : p = a\ polynomial\}$ is dense in $\mathcal{H}$. If T has a star-cyclic vector, then $\mathcal{H}$ is separable, because we can choose the polynomials with rational coefficient.
In the following example, we give the answer, although not completely, to the numerical range attainment problem.

Example 2.6. Let $F \subset \mathbb{R}^2$ be a compact convex set and ν is the Lebesgue meaure on $\mathbb{R}^2$ restricted on F. By the Lindelof Covering Theorem we can prove that ν is a regular Borel measure with support F, i.e, $\nu(F^c) = 0$. Define A on $L^2(\nu)$ by $Af(z) = zf(z)$ for each$f$ in $L^2(\nu)$. If $M = \sup\{|z| : z \in F\}$, then

$$\int_{\mathbb{R}^2} |z.f(z)|^2 d\nu = \int_F |z.f(z)|^2 d\nu \leq M^2 \int_{\mathbb{R}^2} |f(z)|^2 d\nu < \infty$$

So $z.f(z) \in L^2(\nu)$, and then A is well define. It is easy to check that A is normal. Suppose that λ is a point in support of ν. Let $U_n = B(\lambda, \frac{1}{n})$, then $U_n \cap F \neq \emptyset$ and so $\nu(U_n) > 0$. Put $f_n = \frac{1}{\sqrt{\nu(U_n)}} \chi_{U_n}$. Then

$$\begin{aligned} \|(z-\lambda)f_n\|_2^2 &= \int_{\mathbb{R}^2} |(z-\lambda).f_n(z)|^2 d\nu \\ &= \frac{1}{\nu(U_n)} \int_{U_n} |z-\lambda|^2 d\nu \\ &\leq \frac{1}{n^2} \to 0 \end{aligned}$$

that is $\lambda \in \sigma_{ap}(A) \subseteq \sigma(A)$. On the other hand if $\lambda \notin F$, then there is an open set U in $\mathbb{R}^2$ such that $\nu(U) = 0$ and $\lambda \in U$. Define $\psi(z) =$

$(\lambda - z)^{-1}\chi_{U^c} \in L^\infty(\nu)$ and $B : L^2(\nu) \to L^2(\nu)$ by $B(f(z)) = \psi(z)f(z)$. We have $(\lambda - A)B = B(\lambda - A) = I$. Thus, $\lambda \notin \sigma(A)$ and so $\sigma(A) = F$. Also $C^*(A) = \{M_h : h \in C(F)\}$, where M_h denote the multiplication operator by h on $L^2(\nu)$ and $C(F)$ is the continuous complex function on F. Then $\{T1 : T \in C^*(A)\} = \{M_h(1) : h \in C(F)\} = C(F)$, and $C(F)$ is dense in $L^2(\nu)$, so A is star cyclic operator and then $L^2(\nu)$ is separable. Now by using relation (1), we have $\overline{W(A)} = F$.

The lattice of the invariant subspaces of the multiplication operator has not yet been fully calculated and is an open problem. Therefore, we will consider this problem in the following remark.

Remark 2.7. If the closed linear span of the vectors comprising the orbit $x, Tx, T^2x, T^3x, ...$ of a vector $x \in \mathcal{H}$ under T is equal all of $\mathcal{H}$, then the vector x is said to be cyclic for T and T is cyclic operator. In other words, a vector x in $\mathcal{H}$ is a cyclic vector if $\{p(T)x : p = a\ polynomial\}$ is dense in $\mathcal{H}$. The operator T is cyclic if it has a cyclic vector. It is easy to see that any cyclic operator is star cyclic. If A is the multiplication operator defined in Example 2.6, then there is no known characterization of lattice of invariant subspace of A, i.e., $Lat(A)$. Since the cyclicity of operators and invariant subspaces are two related concepts and that the multiplicative operator is a star cyclic, so maybe this issue can be useful in solving the problem of finding invarinat subspaces of the multiplicative operator. For instance, if $x \in L^2(\nu)$ is a noncyclic vector for A then the closure of $\{p(A)x : p = a\ polynomial\}$ is a nontrivial element in $Lat(A)$.

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