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Original Research Paper

Existence Results for Some Fractional Stochastic Integro-differential Equations via Measures of Non-compactness

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Abstract. Using fixed point theorems is one method used to prove the existence of solutions in many types of integral equations. This study focuses on applying a generalization of Petryshyn's fixed point theorem to solve a general form of fractional stochastic integro-differential equations in the Banach algebra $C([0,a])$. Besides stating and proving the relevant theorem, the reasons for the superiority of the new method compared some similar methods, were explained. In addition, to confirm the efficiency and check the validity of results, a part of the paper dedicated to solving some stochastic integral equations.

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1 Introduction

Integral equations ($\mathbb{IE}$s) are an important tool for scientific expression of many phenomena and modeling of a wide range of scientific processes and have wide applications in various scientific fields such as mathematical physics, economics, biology, scattering theory, mechanics and population dynamics, [13, 19, 39, 40, 41, 47]. The importance of the existence of the solution in such studies cannot be overstated, as many times no analytical solution can be found for such problems.

So far, many researchers have been study in this field and have reflected the results of their research [6, 17]. Employing fixed point theorems (FPTs) to check the existence of solutions in different types of $\mathbb{IE}$s is one of the most important methods used by scientists in this field, for more instance, consider [2, 22, 25, 26]. Some systems are such that to model them, familiarity with fractional integral equations ($\mathbb{FIE}$s) is necessary. Also, in the solution's existence, we can refer to the works done in [8, 32, 48, 50], and other classes of these equations in [4, 34, 49], which all are based on FPTs. Some phenomena include random parameters that leads to encountering stochastic $\mathbb{IE}$s [38, 46]. Such systems have more complexities and it is important to make sure they have solution. Methods based on FPTs are some studies that researchers have done to ensure the existence of solutions in different classes of such equations [12, 16, 27, 42]. There are equations that contain a combination of random parameters and derivative of fractional order. Such complex equations can be found in [18, 23].

The introduction and study on the existence of the solution of fractional functional $\mathbb{IE}$ in the Riemann-Liouville (RL) sense, using FPT in Banach algebra is given in [5, 9, 31]. In 2021, Samei *et al.* investigated the following singular fractional q-integro-differential equation involving Caputo fractional q-derivative, for $0 < s < 1$,

$$^{C}D_q^{\sigma} y(s) = g\left(s, y(s), y'(s), {}^{C}D_q^{\zeta} y(s), \int_0^s y(r)\beta(r)\,\mathrm{d}r\right), \tag{1}$$

under boundary conditions $y(0) = 0$ and $y(1) = {}^{C}D_q^{\eta} y(\tau)$, where $y \in C([0,1])$, $\sigma \in [1,2)$, $\zeta, \eta, \tau \in (0,1)$, $\beta \in L^1([0,1])$ is nonnegative with $\|\beta\|_1 = m$ and $g(s.y_1, y_2, y_3, y_4)$ is singular at some points of s [44].

Aydogan in [7] considered the following k-dimensional hybrid differential system

$$\begin{cases} {}^{C}D_q^{\sigma}\left(m_1(s)\left(\frac{y_1(s)}{l_1(s,y_1(s))}\right)' - \tilde{m}_1(s)\tilde{h}_1(y_1(s))\right) + q_1(s)y_1(s) \\ \quad = f_1(s)h_1(y_1(s)), \\ {}^{C}D_q^{\sigma}\left(m_2(s)\left(\frac{y_2(s)}{l_2(s,y_2(s))}\right)' - \tilde{m}_2(s)\tilde{h}_2(y_2(s))\right) + q_2(s)y_2(s) \\ \quad = f_2(s)h_2(y_2(s)), \\ \quad \dots \\ {}^{C}D_q^{\sigma}\left(m_k(s)\left(\frac{y_k(s)}{l_k(s,y_k(s))}\right)' - \tilde{m}_k(s)\tilde{h}_k(y_k(s))\right) + q_k(s)y_k(s) \\ \quad = f_k(s)h_k(y_k(s)), \end{cases}$$

under the sigma boundary value conditions

$$\left(\frac{y_i(s)}{l_i(s,y_i(s))}\right)'\Big|_{s=0} = \frac{\tilde{m}_i(s)}{m_i(s)}\,\tilde{h}_i(y_i(s))\Big|_{s=0}, \qquad 1 \le i \le k,$$

and $\sum_{i=1}^{k}\zeta_i\left(\frac{y_i(a_i)}{l_i(a_i,y_i(a_i))}\right) = \rho_i\sum_{i=1}^{k}\frac{y_i(z_i)}{l_i(z_i,y_i(z_i))}$. Bhupeshwar *et al.* first, focused on examining the existence and uniqueness of solutions Ψ-Hilfer $\mathbb{FDI}$

$$ {}^{H}D^{\sigma,\zeta;\Psi}y(s) = g\big(s,y(s)\big), \qquad s \in [\mathfrak{s}_1,\mathfrak{s}_2] \subset \mathbb{R}^{>0}, \tag{2}$$

under conditions $y(\mathfrak{s}_1^+) = y'(\mathfrak{s}_1^+)$ and $y(u) = KI^{\sigma;\Psi}y(s)$, with $\eta = \sigma + \zeta(3-\sigma)$, where $2 < \sigma < 3$, $K \in \mathbb{R}$, $I^{\sigma;\Psi}y(s)$ is the Ψ-RL $\mathbb{FI}$ of order σ, and in the second stage, provided two distinct existence results for Ψ-Hilfer $\mathbb{FDI}$ (2) via new conditions

$$\begin{cases} y_{\Psi}^{(j)}(\mathfrak{s}_1) = 0,\ j = 0,1,2,\dots(n-2), \quad y_{\Psi}^{(j)} = \left(\frac{\mathrm{d}}{\Psi'(s)\mathrm{d}s}\right)^j y(s), \\ y_{\Psi}^{n-1}(u) = y_u \in \mathbb{R}, \end{cases}$$

where $g(\cdot,y(\cdot)) \in C([0,1])$ [11].

In this study, we examine the existence of the solution of following fractional stochastic integro-differential equations ($\mathbb{FSIDE}$s),

$$\begin{aligned} {}^{C}D^{\sigma}(y(s)+g(s,y(s))) = f\big(s,y(\alpha(s))\big) \\ + F\bigg(s,y(\beta(s)),\int_0^s k_1(s,t,y(\theta(t)))\,\mathrm{d}s, \\ \int_0^s k_2(s,t,y(\mu(t)))\,\mathrm{d}W(t)\bigg), \end{aligned} \tag{3}$$

for $s \in I_a := [0,a]$, under the initial conditions

$$y^{(i)}(0) = x_i, \qquad i = 0,1,\dots,n-1, \tag{4}$$

where, $y \in C(I_a)$ as the analytical solution of (3) is unknown and all other functions are known stochastic processes defined on the some probability space $(\Omega, \mathscr{F}, P)$, and $W(s)$ is the Brownian motion. Also, $y^{(i)}(0)$ is the i-th order of derivative of continuous function x at point 0 and x_i's are constant. In addition $\alpha, \theta, \mu \in C(I_a)$, $f, g \in C(I_a \times \mathbb{R})$, and $k_1, k_2 \in C(I_a \times I_a \times \mathbb{R})$ are continuous functions. The development of the concept of measures of non-compactness (M.N.C) was first done by Kuratowski [30]. Later, other researchers used this concept in investigating the existence of different types of solutions for the $\mathbb{IE}$s [3, 10, 15, 33]. This research examines the existence of a solution to $\mathbb{FSIDE}$ (3) by applying the concept of M.N.C and in this way, the FPT of Petryshyn is used.

2 Auxiliary Facts and Notations

In this section we review some definitions and theorems, by stating some auxiliary facts and notations. First, we provide some preliminary concepts of fractional calculus. Then some basic introductions about stochastic calculations, and in the next subsection about FPT of Petryshyn, which depend on the concept of M.N.C, brief explanations will be provided.

2.1 Fractional calculus

Definition 2.1 ([28])**.** The RL $\mathbb{FI}$ of order $\sigma > 0$ of a function ξ, is defined as

$$I^\sigma \xi(\tau) = \int_0^\tau \tfrac{(\tau-\mu)^{\sigma-1}}{\Gamma(\sigma)} \xi(\mu)\, \mathrm{d}\mu, \qquad \tau > 0.$$

Of course, to learn about the properties of the RL derivative, you can see [28]. In this article, the definition of Caputo derivative is considered, which can better model the phenomenon and is compatible with the initial conditions of the problems.

Definition 2.2 ([28])**.** The Caputo derivative of fractional order $\sigma \geq 0$ for a function $\xi(\tau)$ is defined by

$$\left({}^C D^\sigma \xi\right)(\tau) = \int_0^\tau \tfrac{(\tau-\mu)^{n-\sigma-1}}{\Gamma(n-\sigma)} \tau^{(n)}(\mu)\, \mathrm{d}\mu,$$

for $n-1 < \sigma \leq n$, $n \in \mathbf{N}$.

Lemma 2.3 ([28])**.** *Let $\sigma > 0$ and $n = [\sigma] + 1$. For two fractional operators defined above, the following properties yield*

(i) $\big(I^\sigma\, {}^C D^\sigma \xi\big)(\tau) = \xi(\tau) - \displaystyle\sum_{i=0}^{n-1} \frac{\xi^{(i)}(0)}{i!} \tau^i$;

(ii) $\big({}^C D^\sigma I^\sigma \xi\big)(\tau) = \xi(\tau)$.

2.2 Stochastic calculus

Systems types have been described and evolved using $\mathbb{DE}$s and $\mathbb{IE}$s, since their inception, based on their applications (i.e. economic, mechanical and social systems). These equations applied to model phenomena that in part deal with movement. $\mathbb{SDE}$ is a new branch of mathematics that defines the characteristics of random motion based on very broad mathematical foundations. Mathematical models involving measuring uncertainty, are key to the solution and play an important part in the branch of science and industry, which is why scientists use $\mathbb{SDE}$s as needed in systems modeling. The $\mathbb{SE}$s are equations in which one, or more terms are random processes. Therefore, the solution of $\mathbb{SE}$s may also be of the type of stochastic processes that despite the similarity to the methods of solving ordinary $\mathbb{DE}$s, there are differences. We studied the basic concepts of this discussion using the concept of Brownian motion.

Definition 2.4. ([29]) Brownian motion $W(s)$ which is the following properties is a stochastic process.

a) For $0 \leq s_1 < s_2 < \cdots < s_n$, the increments

$$W(s_1),\, W(s_2) - W(s_1),\, \ldots,\, W(s_n) - W(s_{n-1}),$$

are independent of the path;

b) $W(s) - W(t)$ having mean and variance 0 and variance $s - t$, respectively, has a normal distribution, as a result $W(s)$ has normal distribution with mean and variance 0 and variance s;

c) The $W(s)$, for $s \geq 0$ is a continuous functions.

The definition in part (a), (b) and (c), assumes the start of movement from s. The condition $P(W(0) = 0) = 0$ standardizes Brownian motion where it start at 0.

Before explaining the next theorem which implies the existence of a $\mathrm{S}\mathbb{I}$, it is necessary to state the following definition.

Definition 2.5 ([29])**.** When for all s, $Y(s)$ be $\tilde{F}_s$-measurable, the process Y is called adapted to the filtration $\tilde{F} = (\tilde{F}_s)$.

Theorem 2.6 ([29])**.** *If Y be a process that satisfies the continuous adapted condition, then the $\int_0^T Y(s)\,\mathrm{d}W(s)$ exists.*

If Brownian motion was derivable everywhere, its integral would not be a problem, but considering that it is not derivable anywhere, therefore the $\mathrm{S}\mathbb{I}$ cannot be calculated by normal methods. The common method for calculating the $\mathrm{S}\mathbb{I}$ is to use the integration by parts method, which converts the $\mathrm{S}\mathbb{I}$ into a computable normal or simple integral. So that for the differentiable and bounded function ϕ, we have [36]:

$$\int_0^s \phi(t)\,\mathrm{d}W(t) = \phi(s)W(s) - \int_0^s W(t)\phi'(t)\,\mathrm{d}t, \qquad 0 \leq s \leq 1, \qquad (5)$$

which is an alternative method for calculating $\mathrm{S}\mathbb{I}$s.

2.3 Petryshyn's fixed-point theorem

Here, we employ the symbol E for *Real Banach space*, the symbol $\bar{B}_r$ for *Closed ball with center* 0 *and radius* r, the symbol $\partial\bar{B}_r$ for *Sphere in* E *around* 0 *with radius* $r > 0$, and finally the symbol $C(I_a)$ for *Space of all continuous and real-valued functions on* $I_a = [0, a]$. We recall some definitions and theorems that are required for the sequel.

Definition 2.7 ([30])**.** Let $Y \subset E$ be a bounded set, then $\alpha(Y) = \inf \{\rho > 0 : Y$ can be covered by a finite number of sets with diameter $\leq \rho\}$, is said to be the Kuratowski M.N.C.

Definition 2.8 ([21])**.** Let $Y \subset E$ be a bounded set, then the Hausdorff M.N.C is given by $\mu(Y) = \inf \{\rho > 0 : Y$ has a finite ρ-net in $E\}$.

Theorem 2.9 ([21])**.** *Let $Y \subset E$ be a bounded set, then the* M.N.C α *and μ fulfill $\mu(Y) \leq \alpha(Y) \leq 2\mu(Y)$.*

The space $C(I_a)$ is a Banach space under the norm $\|y\| = \sup \{|y(s)| : s \in I_a\}$, and we shall write the modulus of continuity of a function $y \in C(I_a)$ as

$$\omega(y, \rho) = \sup \Big\{|y(s) - y(t)| : |s - t| \leq \rho\Big\}.$$

Since y is uniformly continuous on I_a, we have $w(y, \rho) \to 0$, as $\rho \to 0$.

Theorem 2.10 ([21])**.** *In Hausdorff* M.N.C*, for all bounded sets $Y \subset C(I_a)$*

$$\mu(Y) = \lim_{\rho\to 0} \Big\{\sup \omega(y, \rho) : y \in Y\Big\}. \tag{6}$$

Definition 2.11. [35] Let $Q : E \to E$ be a continuous map. Q is said to be a k-set contraction if for all $Y \subset C(I_a)$ be bounded, $Q(Y)$ is bounded and $\alpha(QY) \leq k\alpha(Y)$, $0 < k < 1$. Moreover, Q is is said to be condensing (densifying) map if $\alpha(QY) < \alpha(Y)$.

Note that, a k-set contraction with $0 < k < 1$ yields condensing (densifying) but not vice versa.

Theorem 2.12 ([37, 45])**.** *Suppose that $Q : \bar{B}_r \to E$ is a densifying mapping that satisfies the boundary condition,*

(P) *if $Q(Y) = kY$, for some Y in ∂B_r then $k \leq 1$.*

Then the set of fixed points of Q in $\bar{B}_r$ is nonempty.

3 Main Results

In this section, we examine the solvability of the $\mathbb{FSIDE}$ (3). Because of the continuity of g and f, we apply the operator I^σ on sides of Eq. (3),

$$\begin{aligned} y(s) = &\sum_{i=0}^{n-1} \tfrac{y^{(i)}(0)+g^{(i)}(0,y_0)}{i!} s^i - g(s, y(s)) \\ &+ \tfrac{1}{\Gamma(\sigma)} \int_0^s \tfrac{f(t,y(\alpha(t)))}{(s-t)^{1-\sigma}} \, \mathrm{d}t \\ &+ \tfrac{1}{\Gamma(\sigma)} \int_0^s \tfrac{F(t,y(\beta(t)),(H_1y)(t),(H_2y)(t))}{(s-t)^{1-\sigma}} \, \mathrm{d}t, \end{aligned} \tag{7}$$

where

$$\begin{aligned} (H_1 y)(s) &= \int_0^s k_1\big(s, t, y(\theta(t))\big) \, \mathrm{d}t, \\ (H_2 y)(s) &= \int_0^s k_2\big(s, t, y(\mu(t))\big) \, \mathrm{d}W(t). \end{aligned}$$

The Eq. (3) is equivalent to the $\mathbb{FSIE}$ (7). This means every solution of Eq. (7) is also a solution of Eq. (3), and vice versa. Next, we consider the following conditions for Eq. (7):

H1) $g, f \in C(I_a \times \mathbb{R})$, $F \in C(I_a \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$, $k_1, k_2 \in C(I_a \times I_a \times \mathbb{R})$, and α, β, θ, $\mu : I_a \to I_a$ are continuous;

H2) There exist nonnegative constants k, c_1, c_2, c_3, c_4, and $k < 1$ such that $|g(s,u) - g(s,\bar{u})| \leq k|u - \bar{u}|$, and

$$|F(s,u,v,w) - F(s,\bar{u},\bar{v},\bar{w})| \leq c_1|u-\bar{u}| + c_2|v-\bar{v}| + c_3|w-\bar{w}|;$$

H3) (Bounded condition) There exists nonnegative r_0 such that

$$\sup \left\{ L + A + \tfrac{M_1 a^\sigma}{\Gamma(1+\sigma)} + \tfrac{M_2 a^\sigma}{\Gamma(1+\sigma)} \right\} \leq r_0, \tag{8}$$

where

$$
\begin{aligned}
L &= \sup\left\{\left|\sum_{i=0}^{n-1} \tfrac{y^{(i)}(0)+g^{(i)}(0,y_0)}{i!} s^i\right| \,:\, \forall\, s\in I_a\right\},\\
A &= \sup\left\{|g(s,u)| \,:\, \forall\, s\in I_a,\, u\in[-r_0,r_0]\right\},\\
M_1 &= \sup\left\{|f(s,u)| \,:\, \forall\, s\in I_a,\; u\in[-r_0,r_0]\right\},\\
M_2 &= \sup\Big\{|F(s,u,v,w)| \,:\, \forall\, s\in I_a,\, u\in[-r_0,r_0],\\
&\qquad |v|\le aA_1,\, |w|\le \lambda B\Big\},
\end{aligned}
$$

and

$$
\begin{aligned}
A_1 &= \sup\left\{|k_1(s,t,u)| \,:\, \forall\, s,t\in I_a,\, u\in[-r_0,r_0]\right\},\\
B &= \sup\left\{|k_2(s,t,u)| \,:\, \forall s,t\in I_a,\, u\in[-r_0,r_0]\right\},\\
\lambda &= \sup\left\{|W(s)| \,:\, \forall\, s\in I_a\right\}.
\end{aligned}
$$

Theorem 3.1. *By conditions* (H1)-(H3) *on* $E=C(I_a)$*,* $\mathbb{FSIDE}$*.* (3) *has at least one solution.*

Proof. We define the operator $Q: B_{r_0}\to C(I_a)$, as follows

$$
\begin{aligned}
(Qy)(s) = &\sum_{i=0}^{n-1} \tfrac{y^{(i)}(0)+g^{(i)}(0,y_0)}{i!} s^i - g(s,y(s))\\
&+ \tfrac{1}{\Gamma(\sigma)}\int_0^s \tfrac{f(t,y(\alpha(t)))}{(s-t)^{1-\sigma}}\,\mathrm{d}t\\
&+ \tfrac{1}{\Gamma(\sigma)}\int_0^s \tfrac{F(t,y(\beta(t)),(H_1y)(t),(H_2y)(t))}{(s-t)^{1-\sigma}}\,\mathrm{d}t.
\end{aligned}
$$

We will demonstrate that the operator Q is continuous on the ball B_{r_0}. Take arbitrary $x,y\in B_{r_0}$ and $\varepsilon>0$ such that $\|x-y\|\le\varepsilon$, then for

$s \in I_a$, we get

$$\begin{aligned}
|(Qy)(s) - (Qx)(s)| &\leq |g(s,x(s)) - g(s,y(s))| \\
&\quad + \tfrac{1}{\Gamma(\sigma)} \int_0^s \tfrac{|f(t,y(\alpha(t)) - f(t,x(\alpha(t))|}{(s-t)^{1-\sigma}}\,\mathrm{d}t \\
&\quad + \tfrac{1}{\Gamma(\sigma)} \int_0^s \tfrac{1}{(s-t)^{1-\sigma}} \Big[\Big| F\left(t, y(\beta(t)), (H_1 y)(t), (H_2 y)(t)\right) \\
&\quad - F\left(t, x(\beta(t)), (H_1 x)(t), (H_2 x)(t)\right) \Big|\Big]\,\mathrm{d}t \\
&\leq k \parallel y - x \parallel + \tfrac{s^\sigma}{\Gamma(1+\sigma)} \omega(f, \omega(\alpha,\varepsilon)) \\
&\quad + \tfrac{1}{\Gamma(\sigma)} \int_0^s \tfrac{1}{(s-t)^{1-\sigma}} \Big[\Big| F\left(t, y(\beta(t)), (H_1 y)(t), (H_2 y)(t)\right) \\
&\quad - F\left(t, x(\beta(t)), (H_1 y)(t), (H_2 y)(t)\right) \Big|\Big]\,\mathrm{d}t \\
&\quad + \tfrac{1}{\Gamma(\sigma)} \int_0^s \tfrac{1}{(s-t)^{1-\sigma}} \Big[\Big| F\left(t, x(\beta(t)), (H_1 y)(t), (H_2 y)(t)\right) \\
&\quad - F\left(t, x(\beta(t)), (H_1 x)(t), (H_2 y)(t)\right) \Big|\Big]\,\mathrm{d}t \\
&\quad + \tfrac{1}{\Gamma(\sigma)} \int_0^s \tfrac{1}{(s-t)^{1-\sigma}} \Big[\Big| F\left(t, x(\beta(t)), (H_1 x)(t), (H_2 y)(t)\right) \\
&\quad - F\left(t, x(\beta(t)), (H_1 x)(t), (H_2 x)(t)\right) \Big|\Big]\,\mathrm{d}t \\
&\leq k \parallel y - x \parallel + \tfrac{s^\sigma}{\Gamma(1+\sigma)} \omega(f, \omega(\alpha,\varepsilon)) + \tfrac{c_1 s^\sigma}{\Gamma(1+\sigma)} \parallel y - x \parallel \\
&\quad + \tfrac{c_2 a s^\sigma}{\Gamma(1+\sigma)} \omega(k_1, \varepsilon) + \tfrac{c_3 \lambda s^\sigma}{\Gamma(1+\sigma)} \omega(k_2, \varepsilon),
\end{aligned}$$

where for $\sigma > 0$, we define

$$\begin{aligned}
\omega(f,\varepsilon) &= \sup \Big\{ |f(t,y) - f(t,x)| \,:\, t \in I_a, \\
&\qquad y, x \in [-r_0, r_0],\ \|y - x\| \leq \varepsilon \Big\}, \\
\omega(k_i,\varepsilon) &= \sup \Big\{ |k_i(s,t,y) - k_i(s,t,x)| \,:\, s,t \in I_a, \\
&\qquad y, x \in [-r_0, r_0],\ \|y - x\| \leq \varepsilon \Big\}, \qquad i = 0, 1.
\end{aligned}$$

Since the functions $f = f(t,y)$ and $k = k(s,t,y)$ are uniformly continuous on $I_a \times \mathbb{R}$ and $I_a \times I_a \times \mathbb{R}$, we indicate that $\omega(f, \omega(\alpha, \varepsilon)) \to 0$,

$\omega(k_1,\varepsilon)\to 0$ and $\omega(k_2,\varepsilon)\to 0$ as $\varepsilon\to 0$. Consequently, the operator Q is continuous on B_{r_0}. In the following, we prove the operator Q fulfills densifying condition in view of μ. To do this, we take arbitrary $\varepsilon>0$ and assumed that $x\in Y\subset C(I_a)$ is a bounded set. Here for $s_1,s_2\in I_a$ such that $s_1\le s_2$ while $s_2-s_1\le\varepsilon$, gives

$$\begin{aligned}
|(Qy)(s_2)-(Qy)(s_1)| = \Bigg| & \sum_{i=0}^{n-1}\tfrac{y^{(i)}(0)+g^{(i)}(0,y_0)}{i!}s_2^i \\
& - g(s_2,y(s_2)) + \tfrac{1}{\Gamma(\sigma)}\int_0^{s_2}\tfrac{f(t,y(\alpha(t)))}{(s_2-t)^{1-\sigma}}\,\mathrm{d}t \\
& + \tfrac{1}{\Gamma(\sigma)}\int_0^{s_2}\tfrac{F(t,y(\beta(t)),(H_1y)(t),(H_2y)(t))}{(s_2-t)^{1-\sigma}}\,\mathrm{d}t \\
& - \sum_{i=0}^{n-1}\tfrac{y^{(i)}(0)+g^{(i)}(0,y_0)}{i!}s_1^i \\
& + g(s_1,y(s_1)) - \tfrac{1}{\Gamma(\sigma)}\int_0^{s_1}\tfrac{f(t,y(\alpha(t)))}{(s_1-t)^{1-\sigma}}\,\mathrm{d}t \\
& - \tfrac{1}{\Gamma(\sigma)}\int_0^{s_1}\tfrac{F(t,y(\beta(t)),(H_1y)(t),(H_2y)(t))}{(s_1-t)^{1-\sigma}}\,\mathrm{d}t\Bigg| \\
\le & \left|\sum_{i=0}^{n-1}\tfrac{y^{(i)}(0)+g^{(i)}(0,y_0)}{i!}\left(s_2^i-s_1^i\right)\right| + |g(s_1,y(s_1)) \\
& - g(s_2,y(s_2))| + \tfrac{1}{\Gamma(\sigma)}\Bigg|\int_0^{s_1}\tfrac{f(t,y(\alpha(t)))}{(s_2-t)^{1-\sigma}}\,\mathrm{d}t \\
& + \int_{s_1}^{s_2}\tfrac{f(t,y(\alpha(t)))}{(s_2-t)^{1-\sigma}}\,\mathrm{d}t + \int_0^{s_1}\tfrac{f(t,y(\alpha(t)))}{(s_1-t)^{1-\sigma}}\,\mathrm{d}t\Bigg| \\
& + \tfrac{1}{\Gamma(\sigma)}\Bigg|\int_0^{s_1}\tfrac{F(t,y(\beta(t)),(H_1y)(t),(H_2y)(t))}{(s_2-t)^{1-\sigma}}\,\mathrm{d}t \\
& + \int_{s_1}^{s_2}\tfrac{F(t,y(\beta(t)),(H_1y)(t),(H_2y)(t))}{(s_2-t)^{1-\sigma}}\,\mathrm{d}t \\
& + \int_0^{s_1}\tfrac{F(t,y(\beta(t)),(H_1y)(t),(H_2y)(t))}{(s_1-t)^{1-\sigma}}\,\mathrm{d}t| \\
\le & |g(s_1,y(s_1))-g(s_1,y(s_2))| + |g(s_1,y(s_2)) \\
& - g(s_2,y(s_2))| + \tfrac{1}{\Gamma(\sigma)}\int_0^{s_1}\left|\tfrac{f(t,y(\alpha(t)))}{(s_2-t)^{1-\sigma}}\right.
\end{aligned}$$

$$- \frac{f(t,y(\alpha(t)))}{(s_1-t)^{1-\sigma}}\Big| \,\mathrm{d}t + \frac{1}{\Gamma(\sigma)} \int_{s_1}^{s_2} \Big|\frac{f(t,y(\alpha(t)))}{(s_2-t)^{1-\sigma}}\Big| \,\mathrm{d}t$$
$$+ \frac{1}{\Gamma(\sigma)} \int_0^{s_1} \Big|\frac{F(t,y(\beta(t)),(H_1y)(t),(H_2y)(t))}{(s_2-t)^{1-\sigma}}$$
$$- \frac{F(t,y(\beta(t)),(H_1y)(t),(H_2y)(t))}{(s_1-t)^{1-\sigma}}\Big| \,\mathrm{d}t$$
$$+ \frac{1}{\Gamma(\sigma)} \int_{s_1}^{s_2} \Big|\frac{F(t,y(\beta(t)),(H_1y)(t),(H_2y)(t))}{(s_2-t)^{1-\sigma}}\Big| \,\mathrm{d}t.$$

For simplicity we use the following notation:

$$\omega_g(I_a,\varepsilon) = \sup \Big\{|g(s,y) - g(\bar{s},y)| \,:\, |s-\bar{s}| \leq \; y \in [-r_0, r_0]\Big\},$$

and using the above relation we get

$$\begin{aligned}|(Qy)(s) - (Qx)(s)| &\leq k\omega(y,\varepsilon) + \omega_g(I_a,\varepsilon)\\ &+ \tfrac{M_1}{\Gamma(1+\sigma)}\left\{s_1^\sigma - s_2^\sigma + (s_2-s_1)^\sigma\right\} + \tfrac{M_1}{\Gamma(1+\sigma)}(s_2-s_1)^\sigma\\ &+ \tfrac{M_2}{\Gamma(1+\sigma)}\left\{s_1^\sigma - s_2^\sigma + (s_2-s_1)^\sigma\right\} + \tfrac{M_2}{\Gamma(1+\sigma)}(s_2-s_1)^\sigma\\ &\leq k\omega(x,\varepsilon) + \omega_g(I_a,\varepsilon) + \tfrac{3\varepsilon^\sigma M_1}{\Gamma(1+\sigma)} + \tfrac{3\varepsilon^\sigma M_2}{\Gamma(1+\sigma)}.\end{aligned}$$

Taking limit as $\varepsilon \to 0$, we obtain $\omega(Qy,\varepsilon) \leq k\omega(y,\varepsilon)$, for $y \in Y$. Therefore, $\mu(QY) \leq k\mu(Y)$. Now, we get Q is a condensing mapping with constant $k < 1$. It remains to verify condition (P) of Theorem 2.12. Suppose $y \in \partial\bar{B}_{r_0}$. If $Qy = ky$ then we have $kr_0 = k\|y\| = \|Qy\|$ and by condition (H3), we concluded that

$$\begin{aligned}|Qy(s)| = \Big|&\sum_{i=0}^{n-1} \tfrac{y^{(i)}(0)+g^{(i)}(0,y_0)}{i!} s^i - g(s,y(s))\\ &+ \tfrac{1}{\Gamma(\sigma)} \int_0^s \tfrac{f(t,y(\alpha(t)))}{(s-t)^{1-\sigma}} \,\mathrm{d}t\\ &+ \tfrac{1}{\Gamma(\sigma)} \int_0^s \tfrac{F(t,y(\beta(t)),(H_1y)(t),(H_2y)(t))}{(s-t)^{1-\sigma}} \,\mathrm{d}t\Big| \leq r_0,\end{aligned}$$

hence $\|Qy\| \leq r_0$, which gives $k \leq 1$. □

The following corollary which is the main results of Dadsetadi *et al.* [14], would be obtained from Theorem 3.1.

Corollary 3.2 ([14]). *Suppose*

M1) $g \in C(I_a \times \mathcal{R})$, $f \in C(I_a \times \mathcal{R})$, $F \in C(I_a \times \mathcal{R}^2)$, $k \in C(I_a^2 \times \mathcal{R})$ *and*, $\mu : I_a \to I_a$ *are continuous;*

M2) *There exist non negative constants* k_1, k_2, c_1, c_2, *and* c_3, *so that* $k_1 < 1$,

$$|g(\vartheta, \omega_1) - g(\vartheta, \varpi_1)| \leq k_1 |\omega_1 - \varpi_1|,$$

and

$$|F(\vartheta, \omega_1, \omega_2) - F(\vartheta, \varpi_1, \varpi_2)| \leq c_1 |\omega_1 - \varpi_1| + c_2 |\omega_2 - \varpi_2|;$$

M3) $\exists \delta_0 \geq 0$ *such that*

$$\sup \left\{ L + A + \tfrac{M_1 a^\sigma}{\Gamma(1+\sigma)} + \tfrac{M_2 a^\sigma}{\Gamma(1+\sigma)} \right\} \leq \delta_0,$$

where,

$$\begin{aligned} L &= \sup \left\{ \left| \sum_{i=0}^{n-1} \tfrac{\chi^{(i)}(0) + g^{(i)}(0, \chi_0)}{i!} \vartheta^i \right| \,:\, \forall\, \vartheta \in I_a \right\}; \\ A &= \sup \left\{ |g(\vartheta, \omega_1)| \,:\, \forall \vartheta \in I_a,\, \omega_1 \in [-\delta_0, \delta_0] \right\}; \\ M_1 &= \sup \left\{ |f(\vartheta, \omega_1)| \,:\, \forall\, \vartheta \in I_a,\, \omega_1 \in [-\delta_0, \delta_0] \right\}; \\ M_2 &= \sup \left\{ |F(\vartheta, \omega_1, \omega_2)| \,:\, \forall \vartheta \in I_a,\, \omega_1 \in [-\delta_0, \delta_0], |\omega_2| \leq aB \right\}; \\ B &= \sup \left\{ |k(\vartheta, \ell, \omega_1)| \,:\, \forall \vartheta, \ell \in I_a,\, \omega_1 \in [-\delta_0, \delta_0] \right\}. \end{aligned}$$

Then

$$\begin{aligned} {}^C D^\sigma \big(y(\vartheta) + g(\vartheta, y(\vartheta)) \big) &= f(\vartheta, y(\vartheta)) \\ &\quad + F\bigg(\vartheta, y(\vartheta), \int_0^\vartheta k(\vartheta, \ell) H\big(y\big(\mu(\ell)\big)\big)\, \mathrm{d}\ell \bigg), \quad \vartheta \in I_a, \end{aligned} \tag{9}$$

with the initial conditions

$$y^{(i)}(0) = y_i, \qquad i = 0, 1, \ldots, n-1, \tag{10}$$

has at least a solution in I_a.

Proof. It is clear that Eq. (9) is a particular case of Eq. (3). Here $\alpha(\vartheta) = \beta(\vartheta) = \varsigma(\vartheta) = \vartheta$, $k(\vartheta, \ell, y(\mu(\ell))) = k(\vartheta, \ell)H(y(\mu(\ell)))$. By employing RL fractional integrating and Lemma 2.3, Eq. (9) changes into

$$y(s) = \sum_{i=0}^{n-1} \tfrac{y^{(i)}(0)+g^{(i)}(0,y_0)}{i!}\vartheta^i - g(\vartheta, y(\vartheta)) \\ + \tfrac{1}{\Gamma(\sigma)} \int_0^{\vartheta} \tfrac{f(\ell,y(\ell))}{(\vartheta-\ell)^{1-\sigma}} \,\mathrm{d}\ell + \tfrac{1}{\Gamma(\sigma)} \int_0^{\vartheta} \tfrac{F(\ell,y(\ell),(Hy)(\ell))}{(\vartheta-\ell)^{1-\sigma}} \,\mathrm{d}\ell.$$

The proof is connected to Theorem 3.1, so we can drop these parts. □

Remark 3.3. The above Corollary is the main result of [14], which was proved here using FPT of Petryshyn simpler and with fewer conditions, and this is the advantage of using Petryshyn's theorem.

Corollary 3.4. *If $g(s, y(s)) = k_1 \equiv 0$, $\alpha(s) = \mu(s) = s$ and $F(s, u, v, w) = w$, then Eq. (3) has the following form, which was studied in [24],*

$$^{C}D_s^{\sigma} y(s) = f(s, y(s)) + \int_0^s k_2(s, y(t)) \,\mathrm{d}W(t), \tag{11}$$

with $\sigma \in (0, 1)$ and the initial condition $y(0) = y_0$, where

$$^{C}D_s^{\sigma} \psi(z) = \tfrac{1}{\Gamma(1-\sigma)} \int_0^s \tfrac{\psi'(z)}{(s-z)^{\sigma}} \,\mathrm{d}z, \qquad s \in [0, a].$$

It is well-known that Eq. (11) is equivalent to the following $\mathbb{FSIE}$ with a weakly singular kernel of the form

$$y(s) = y_0 + \tfrac{1}{\Gamma(\sigma)} \int_0^s \tfrac{f(\zeta,y(\zeta))}{(s-\zeta)^{1-\sigma}} \,\mathrm{d}\zeta \\ + \tfrac{1}{\Gamma(\sigma)} \int_0^s \int_0^{\zeta} \tfrac{k_2(\zeta,y(t))}{(s-\zeta)^{1-\sigma}} \,\mathrm{d}W(t) \,\mathrm{d}\zeta.$$

4 Examples

Here, we provide two examples to confirm the efficiency and check the validity of the main results.

Example 4.1. Consider following $\mathbb{FSIDE}$ on $C(I_a)$, $a = 1$,

$$\begin{aligned}
{}^{C}D^{\sigma}\left(y(s) + \tfrac{\cos(s)}{\sqrt{64+s^2}}y(s)\right) &= \tfrac{e^{-4s}}{10+s^2}\sin\left(\tfrac{1}{1+s}\right) + \tfrac{y(1-s)\sqrt{s}}{7} \\
&+ \tfrac{3s^2y(s^2)}{5+5s^2} + \tfrac{e^{-s}}{6+5s}\int_0^s \tfrac{1}{1+s^3}\left[1 + \int_0^t \tfrac{\ln(1+|y(\xi)|)}{\sqrt{4+\xi+t}}\,\mathrm{d}\xi\right]\mathrm{d}t \\
&+ \tfrac{1}{5}\int_0^s \tfrac{e^{-3st}}{2+s^2+\ln(1+t)}\sin(y(t))\,\mathrm{d}W(t), \qquad (12)
\end{aligned}$$

under the initial conditions $y^{(i)}(0) = y_i$, $i = 0, 1$ for $\sigma = \frac{5}{4}, \frac{3}{2}, \frac{7}{4}$. Eq. (12) is a particular form of Eq. (3) such that $n = 2$,

$$\begin{aligned}
g(s, y(s)) &= \tfrac{\cos(s)}{\sqrt{64+s^2}}y(s), \\
f(s, y(\alpha(s))) &= \tfrac{e^{-4s}}{10+s^2}\sin\left(\tfrac{1}{1+s}\right) + \tfrac{y(1-s)\sqrt{s}}{7}, \\
F(s, u, v, w) &= \tfrac{3s^2u}{5+5s^2} + \tfrac{e^{-s}}{6+5s}v + \tfrac{1}{5}w,
\end{aligned}$$

and

$$\begin{aligned}
v &= \int_0^s \tfrac{1}{1+s^3}\left[1 + \int_0^t \tfrac{\ln(1+|y(\xi)|)}{\sqrt{4+\xi+t}}\,\mathrm{d}\xi\right]\mathrm{d}t, \\
w &= \int_0^s \tfrac{e^{-3st}}{2+s^2+\ln(1+t)}\sin(y(t))\,\mathrm{d}W(t).
\end{aligned}$$

It can be seen that $|g(s, u) - g(s, \bar{u})| \le \frac{1}{8}|u - \bar{u}|$ and

$$|F(s, u, v, w) - F(s, \bar{u}, \bar{v}, \bar{w})| \le \tfrac{3}{5}|u - \bar{u}| + \tfrac{1}{6}|v - \bar{v}| + \tfrac{1}{5}|w - \bar{w}|.$$

Here $k = \frac{1}{8} < 1$, $c_1 = \frac{3}{5}$, $c_2 = \frac{1}{6}$, $c_3 = \frac{1}{5}$. So, the conditions (H1) and (H2) hold. Moreover, for $\|y\| \le r_0$, $r_0 > 0$ and $y_0 = 0$, $y_1 = 1$, we have

$$\begin{aligned}
|y(s)| &= \Bigg| \sum_{i=0}^{1} \tfrac{y^{(i)}(0)+g^{(i)}(0,y_0)}{i!}s^i - g(s, y(s)) \\
&\quad + \tfrac{1}{\Gamma(\sigma)}\int_0^s \tfrac{f(t,y(\alpha(t)))}{(s-t)^{\sigma-1}}\,\mathrm{d}t \\
&\quad + \tfrac{1}{\Gamma(\sigma)}\int_0^s \tfrac{F(t,y(\beta(t)),y(\theta(t)),(Hy)(t))}{(s-t)^{\sigma-1}}\,\mathrm{d}t \Bigg| \\
&\le \tfrac{9}{8} + \tfrac{1}{8}r_0 + \tfrac{1}{\Gamma(\sigma+1)}\left(\tfrac{1}{10} + \tfrac{r_0}{7}\right) \\
&\quad + \tfrac{1}{\Gamma(\sigma+1)}\left[\tfrac{3r_0}{5} + \tfrac{1}{6}\left(\tfrac{r_0}{2} + 1\right) + \tfrac{\lambda}{10}\right], \qquad \forall\, s \in I_a.
\end{aligned}$$

Therefore (H3) holds if according to the Eq. (8),

$$\begin{aligned} L + A + \tfrac{M_1 a^\sigma}{\Gamma(1+\sigma)} + \tfrac{M_2 a^\sigma}{\Gamma(1+\sigma)} \\ \leq \tfrac{9}{8} + \tfrac{1}{8} r_0 + \tfrac{1}{\Gamma(\sigma+1)} \left(\tfrac{1}{10} + \tfrac{r_0}{7}\right) + \tfrac{1}{\Gamma(\sigma+1)} \Big[\tfrac{3r_0}{5} \\ + \tfrac{1}{6}\left(\tfrac{r_0}{2} + 1\right) + \tfrac{\lambda}{10}\Big] \simeq \left\{ \begin{array}{ll} 1.790, & \sigma = \frac{5}{4}, \\ 1.701, & \sigma = \frac{3}{2}, \\ 1.613, & \sigma = \frac{7}{4}, \end{array} \right\} \leq r_0. \end{aligned} \tag{13}$$

This shows that

$$r_0 \simeq \left\{ \begin{array}{ll} 1.215 + 0.088\lambda, & \sigma = \frac{5}{4}, \\ 1.072 + 0.075\lambda, & \sigma = \frac{3}{2}, \\ 0.929 + 0.062\lambda, & \sigma = \frac{7}{4}, \end{array} \right. \tag{14}$$

is a solution of the above inequality. Table 1 shows the numerical results of Eqs.(13) and (14). Furthermore, Figs. 1a and 1b show M_1 and M_2, respectively. One can see the 2D plot of suitable r_0 in Fig. 1c, well.

The result is followed from Theorem 3.1. Therefor, assumptions (H1)-(H3) be fulfilled and Theorem 3.1 indicates the solution of (12) in $C(I_a)$.

Example 4.2. Consider following $\mathbb{FSIDE}$,

$$\begin{aligned} {}^{C}D^\sigma \Big(y(s) + \tfrac{2+\ln(1+|y(s)|)}{(2s+3)^2} \Big) = \tfrac{1}{5} e^{-s} + \tfrac{3\cos(s) y(s^3)}{4+3s} \\ + \tfrac{1}{9} \sin\Big(\sqrt{\tfrac{\pi}{2}} y(\sqrt{s}) \Big) + \tfrac{s^2}{2(1+s^2)} \int_0^s \sqrt{\tfrac{s e^{-3t}}{1+s}} \Big[\tfrac{1}{5} \\ + \int_0^t \xi \Big(\tfrac{|y(\xi)|}{1+|y(\xi)|} + y(\xi) \Big) \, \mathrm{d}\xi \Big] \, \mathrm{d}t \\ + \tfrac{e^{-s}}{3+s^2} \int_0^s \tfrac{st \cos(sy(\sqrt{t}))}{2+t^2+5s} \, \mathrm{d}W(t), \end{aligned} \tag{15}$$

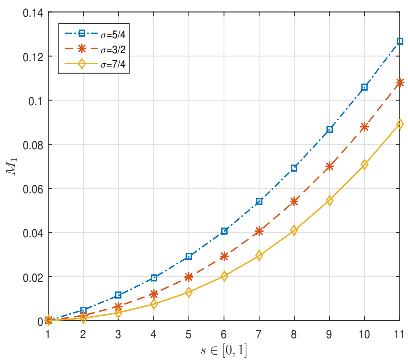
for $y \in C(I_a)$, $a = 1$ and for $\sigma = \frac{1}{5}, \frac{1}{3}, \frac{1}{2}$, via condition $y(0) = y_0 = 0$.

Table 1: Numerical results of A, M_1, M_2 and suitable r_0 of $\mathbb{FSIDE}$ (12) for three different values of σ in Example 4.1.

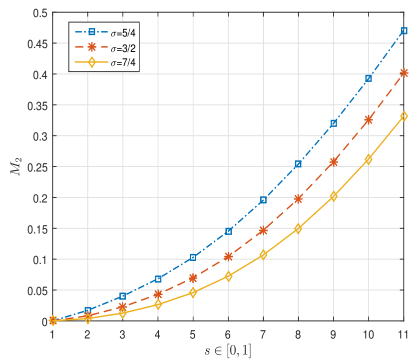
s	A	M_1	M_2	$r_0 \geq \dots$	r_0
			$\sigma = \frac{5}{4}$		
0.00	0.125	0.000	0.000	1.250	$1.215 + 0.088\lambda$
0.10	0.124	0.005	0.017	1.271	$1.215 + 0.088\lambda$
0.20	0.122	0.011	0.040	1.299	$1.215 + 0.088\lambda$
0.30	0.119	0.019	0.068	1.332	$1.215 + 0.088\lambda$
0.40	0.115	0.029	0.103	1.372	$1.215 + 0.088\lambda$
0.50	0.109	0.041	0.145	1.420	$1.215 + 0.088\lambda$
0.60	0.103	0.054	0.196	1.478	$1.215 + 0.088\lambda$
0.70	0.095	0.069	0.254	1.544	$1.215 + 0.088\lambda$
0.80	0.087	0.087	0.320	1.618	$1.215 + 0.088\lambda$
0.90	0.077	0.106	0.392	1.700	$1.215 + 0.088\lambda$
1.00	0.067	0.127	0.471	1.790	$1.215 + 0.088\lambda$
			$\sigma = \frac{3}{2}$		
0.00	0.125	0.000	0.000	1.250	$1.072 + 0.075\lambda$
0.10	0.124	0.002	0.008	1.260	$1.072 + 0.075\lambda$
0.20	0.122	0.007	0.023	1.277	$1.072 + 0.075\lambda$
0.30	0.119	0.012	0.043	1.300	$1.072 + 0.075\lambda$
0.40	0.115	0.020	0.070	1.329	$1.072 + 0.075\lambda$
0.50	0.109	0.029	0.104	1.368	$1.072 + 0.075\lambda$
0.60	0.103	0.040	0.147	1.415	$1.072 + 0.075\lambda$
0.70	0.095	0.054	0.198	1.472	$1.072 + 0.075\lambda$
0.80	0.087	0.070	0.258	1.539	$1.072 + 0.075\lambda$
0.90	0.077	0.088	0.326	1.616	$1.072 + 0.075\lambda$
1.00	0.067	0.108	0.401	1.701	$1.072 + 0.075\lambda$
			$\sigma = \frac{7}{4}$		
0.00	0.125	0.000	0.000	1.250	$0.929 + 0.062\lambda$
0.10	0.124	0.001	0.004	1.254	$0.929 + 0.062\lambda$
0.20	0.122	0.004	0.013	1.264	$0.929 + 0.062\lambda$
0.30	0.119	0.007	0.026	1.278	$0.929 + 0.062\lambda$
0.40	0.115	0.013	0.046	1.299	$0.929 + 0.062\lambda$
0.50	0.109	0.020	0.072	1.327	$0.929 + 0.062\lambda$
0.60	0.103	0.029	0.107	1.364	$0.929 + 0.062\lambda$
0.70	0.095	0.041	0.150	1.411	$0.929 + 0.062\lambda$
0.80	0.087	0.055	0.202	1.468	$0.929 + 0.062\lambda$
0.90	0.077	0.071	0.262	1.535	$0.929 + 0.062\lambda$
1.00	0.067	0.089	0.332	1.613	$0.929 + 0.062\lambda$

Here $n = 1$,

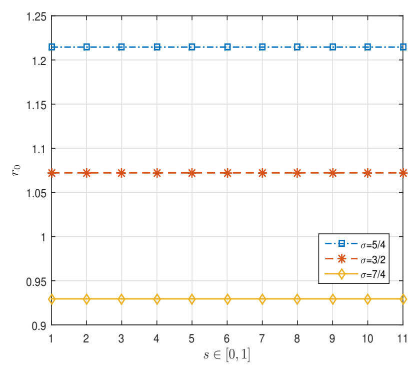
$$g(s, y(s)) = \frac{2+\ln(1+|y(s)|)}{(2s+3)^2},$$



(a) $M1$



(b) $M2$



(c) r_0

Figure 1: Graphical representation of M_1, M_2 and suitable r_0 of of $\mathbb{FSIDE}$ (12) for three different values of σ in Example 4.1.

$$f(s, y(\alpha(s))) = \tfrac{1}{5}e^{-s} + \tfrac{3\cos(s)y(s^3)}{4+3s},$$
$$F(s, u, v, w) = \tfrac{1}{9}\sin\left(\sqrt{\tfrac{\pi}{2}}u\right) + \tfrac{s^2}{2(1+s^2)}v + \tfrac{e^{-s}}{3+s^2}w,$$

and

$$v = \int_0^s \sqrt{\tfrac{se^{-3t}}{1+s}} \left(\tfrac{1}{5} + \int_0^t \xi \left[\tfrac{|y(\xi)|}{1+|y(\xi)|} + y(\xi)\right] \mathrm{d}\xi\right) \mathrm{d}t,$$

$$w = \int_0^s \tfrac{st\cos(sy(\sqrt{t}))}{2+t^2+5s}\,\mathrm{d}W(t).$$

It can be seen that we have $|g(s,u) - g(s,\bar{u})| \leq \frac{1}{9}|u-\bar{u}|$ and

$$|F(s,u,v,w) - F(s,\bar{u},\bar{v},\bar{w})| \leq \tfrac{\sqrt{\pi}}{18}|u-\bar{u}| + \tfrac{1}{2}|v-\bar{v}| + \tfrac{1}{3}|w-\bar{w}|.$$

So, we can choose $k = \frac{1}{9} < 1$, $c_1 = \frac{\sqrt{\pi}}{18}$, $c_2 = \frac{1}{2}$, $c_3 = \frac{1}{3}$. It follows that the conditions (H1) and (H2) hold. Moreover, for $\|y\| \leq r_0$, $r_0 > 0$, we have

$$\begin{aligned}|y(s)| &= \Big| y(0) + g(0,y_0) - g(s,y(s)) + \tfrac{1}{\Gamma(\sigma)}\int_0^s \tfrac{f(t,y(\alpha(t)))}{(s-t)^{1-\sigma}}\,\mathrm{d}t \\ &\quad + \tfrac{1}{\Gamma(\sigma)}\int_0^s \tfrac{F(t,y(\beta(t)),y(\theta(t)),(Hy)(t))}{(s-t)^{1-\sigma}}\,\mathrm{d}t\Big| \\ &\leq \tfrac{2}{9} + \tfrac{1}{\Gamma(\sigma+1)}\left(\tfrac{1}{5} + \tfrac{r_0}{4}\right) \\ &\quad + \tfrac{1}{\Gamma(\sigma+1)}\left[\tfrac{1}{9} + \tfrac{1}{4}\left(1 + r_0 + \tfrac{1}{5}\right) + \tfrac{1}{6}\lambda\right], \quad \forall\, s \in I_a.\end{aligned}$$

Therefore (H3) holds if according to the Eq. (8),

$$\begin{aligned} L + A + \tfrac{M_1 a^\sigma}{\Gamma(1+\sigma)} + \tfrac{M_2 a^\sigma}{\Gamma(1+\sigma)} & \\ \leq \tfrac{2}{9} + \tfrac{1}{\Gamma(\sigma+1)}\left(\tfrac{1}{5} + \tfrac{r_0}{4}\right) + \tfrac{1}{\Gamma(\sigma+1)}\left[\tfrac{1}{9} + \tfrac{1}{4}\left(1 + r_0 + \tfrac{1}{5}\right) + \tfrac{1}{6}\lambda\right] & \\ \simeq \left\{\begin{array}{ll} 1.128, & \sigma = \frac{1}{5}, \\ 1.151, & \sigma = \frac{1}{3}, \\ 1.158, & \sigma = \frac{1}{2}, \end{array}\right\} \leq r_0, & \end{aligned} \tag{16}$$

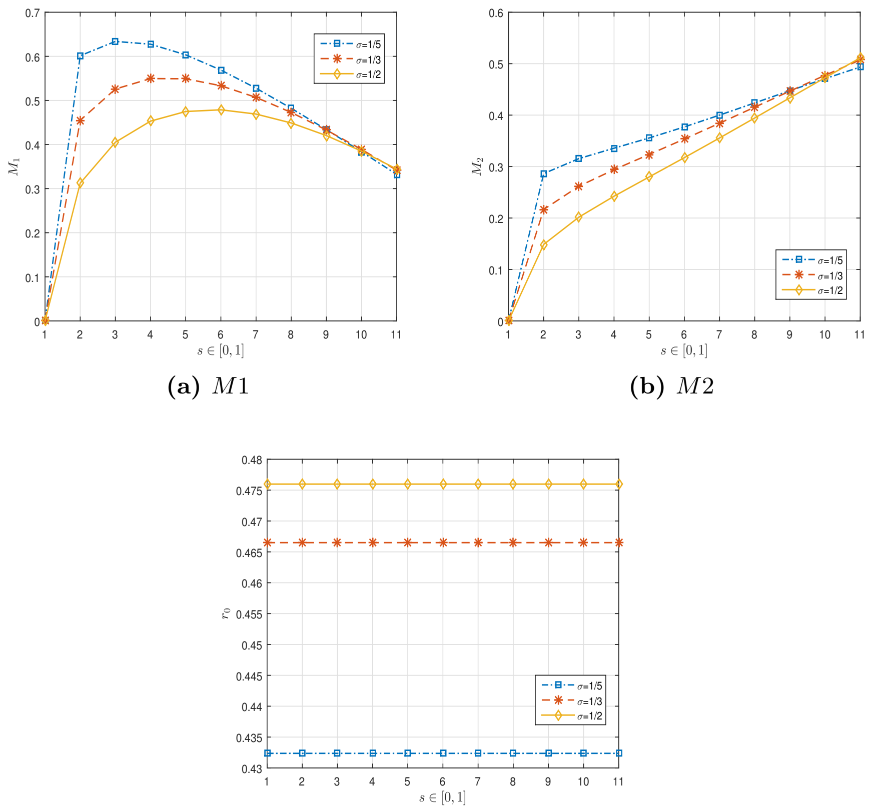
$|y(s)| \leq r_0$. This shows that

$$r_0 = \frac{2\sqrt{\pi} + 11 + 3\lambda}{9(\sqrt{\pi} - 1)} \simeq \begin{cases} 0.432 + 0.182\lambda, & \sigma = \frac{1}{5}, \\ 0.466 + 0.187\lambda, & \sigma = \frac{1}{3}, \\ 0.476 + 0.188\lambda, & \sigma = \frac{1}{2}. \end{cases} \tag{17}$$

Table 1 shows the numerical results of Eqs.(16) and (17). Furthermore, Figs. 2a and 2b show M_1 and M_2, respectively. One can see the 2D plot of suitable r_0 in Fig. 2c, well. The result is followed from Theorem 3.1.

Table 2: Numerical results of A, M_1, M_2 and suitable r_0 of $\mathbb{FSIDE}$ (15) for three different values of σ in Example 4.2.

s	A	M_1	M_2	$r_0 \geq \ldots$	r_0
			$\boldsymbol{\sigma = \frac{1}{5}}$		
0.00	0.222	0.000	0.000	0.444	$0.432 + 0.182\lambda$
0.10	0.195	0.601	0.286	1.305	$0.432 + 0.182\lambda$
0.20	0.173	0.634	0.315	1.345	$0.432 + 0.182\lambda$
0.30	0.154	0.628	0.336	1.340	$0.432 + 0.182\lambda$
0.40	0.139	0.603	0.356	1.320	$0.432 + 0.182\lambda$
0.50	0.125	0.569	0.377	1.293	$0.432 + 0.182\lambda$
0.60	0.113	0.528	0.400	1.263	$0.432 + 0.182\lambda$
0.70	0.103	0.482	0.424	1.231	$0.432 + 0.182\lambda$
0.80	0.095	0.434	0.448	1.198	$0.432 + 0.182\lambda$
0.90	0.087	0.384	0.471	1.163	$0.432 + 0.182\lambda$
1.00	0.080	0.332	0.493	1.128	$0.432 + 0.182\lambda$
			$\boldsymbol{\sigma = \frac{1}{3}}$		
0.00	0.222	0.000	0.000	0.444	$0.466 + 0.187\lambda$
0.10	0.195	0.455	0.217	1.089	$0.466 + 0.187\lambda$
0.20	0.173	0.526	0.262	1.183	$0.466 + 0.187\lambda$
0.30	0.154	0.550	0.294	1.220	$0.466 + 0.187\lambda$
0.40	0.139	0.549	0.324	1.233	$0.466 + 0.187\lambda$
0.50	0.125	0.533	0.354	1.234	$0.466 + 0.187\lambda$
0.60	0.113	0.507	0.384	1.227	$0.466 + 0.187\lambda$
0.70	0.103	0.473	0.415	1.214	$0.466 + 0.187\lambda$
0.80	0.095	0.433	0.447	1.196	$0.466 + 0.187\lambda$
0.90	0.087	0.389	0.477	1.175	$0.466 + 0.187\lambda$
1.00	0.080	0.342	0.507	1.151	$0.466 + 0.187\lambda$
			$\boldsymbol{\sigma = \frac{1}{2}}$		
0.00	0.222	0.000	0.000	0.444	$0.476 + 0.188\lambda$
0.10	0.195	0.312	0.149	0.878	$0.476 + 0.188\lambda$
0.20	0.173	0.405	0.202	1.002	$0.476 + 0.188\lambda$
0.30	0.154	0.453	0.242	1.072	$0.476 + 0.188\lambda$
0.40	0.139	0.475	0.280	1.116	$0.476 + 0.188\lambda$
0.50	0.125	0.479	0.317	1.143	$0.476 + 0.188\lambda$
0.60	0.113	0.469	0.356	1.160	$0.476 + 0.188\lambda$
0.70	0.103	0.449	0.394	1.169	$0.476 + 0.188\lambda$
0.80	0.095	0.420	0.434	1.171	$0.476 + 0.188\lambda$
0.90	0.087	0.385	0.473	1.167	$0.476 + 0.188\lambda$
1.00	0.080	0.344	0.511	1.158	$0.476 + 0.188\lambda$



(a) $M1$ **(b)** $M2$

(c) r_0

Figure 2: Graphical representation of M_1, M_2 and suitable r_0 of of $\mathbb{FSIDE}$ (15) for three different values of σ in Example 4.2.

5 Conclusion and Perspective

In this work, Theorem 2.12 and the M.N.C idea were used to analyze the of solutions some nonlinear functional $\mathbb{FSIDE}$ in the Banach algebra $C(I_a)$. The superiority of Theorem 2.12 compared to other similar FPTs, such as Darbo and Schauder, is that here the condition that involved operator maps a closed convex subset onto itself is not needed. Thus by applying weaker conditions, the method is extended and includes a larger range. In fact, Some valuable articles such as [1, 11, 20, 43], can be generalized or be used with the results of this research.

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