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Characterization of Bi-gyrosemigroups: An Eemphasis on Order 2

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Abstract. The practical implementation and recognition of gyrogroups have been significantly advanced through their association with relativistically admissible velocities within special relativity theory. This specific context provides a tangible representation where gyrogroup properties can be observed and studied effectively. By employing the Einstein velocity addition law to define the binary operation within this space, researchers have successfully extended traditional group theory concepts to encompass both gyrogroups and bi-gyrogroups. A bi-gyrogroup is a group-Like Structure which satisfies the group condition, in addition to that, for any pair (a, b) in this structure, there are two automorphisms with this property that fulfill left and right associativity properties. The study in this article is motivated by generalizing the bi-gyrogroups and semigroups led to introducing bi-gyrosemigroups. Accordingly, in this paper, some classes of this structure are presented. Also, all bi-gyrosemigroups of order 2 are characterized.

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1 Introduction

The concept of gyrogroups, which finds its roots in the algebra of Möbius transformations of the complex open unit disc, serves as a natural extension of groups and vector spaces. Gyrogroups can be classified into two categories: gyrocommutative and non-gyrocommutative gyrogroups. Interestingly, certain gyrocommutative gyrogroups allow for scalar multiplication, leading to the emergence of gyrovector spaces. These novel notions of gyrogroups and gyrovector spaces retain the essence of their classical counterparts while establishing a valuable connection between non-associative algebra and hyperbolic geometry. The exploration of this evolution from Möbius to gyrogroups originated from the realization that Einstein's velocity addition law encapsulates a profound structure, characterized as a gyrocommutative gyrogroup and a gyrovector space, [15]. Remarkably, this concept continues to be actively investigated even after more than 150 years since Möbius' initial findings.

Ungar's significant contribution to the field of mathematics lies in his work on the generalization of gyrogroups to bi-gyrogroups, [12] and [14]. In particular, he focused on generalized Lorentz transformation groups denoted as $\Gamma = SO(m, n)$, where m and n are natural numbers. These transformation groups possess a distinctive bi-decomposition structure denoted as $\Gamma = H_L B H_R$, where B represents a subset of Γ, while H_L and H_R denote subgroups within Γ. This bi-decomposition structure induces a group-like organization for B, which is referred to as a bi-gyrogroup.

A semigroup is an essential mathematical construct comprising a set along with an associative binary operation. Serving as an extension of groups, semigroups find applications in diverse fields such as automata theory, dynamical systems, algebraic geometry, functional analysis, and probability theory. Semigroups offer valuable tools for modeling complex systems and comprehending their behavior over time, [3, 4, 8].

This study is motivated by introducing a structure that is a generalization of semigroup and bi-gyrogroup call it bi-gyrosemigroups. Several examples and properties of this structure are investigated. We characterize bi-gyrosemigroups of order two up to bi-gyroisomorphism.

A groupoid G with a binary operation $\oplus$ is called a gyrogroup if its binary operation satisfies the following axioms:

Definition 1.1. (Gyrogroup) Let G be a non-empty set.

(1) Left gyroassociative law holds, it means for two element u and v in G there exists gyroautomorphism $gyr[u, v]$ such that for all $u \in G$:

$$u \oplus (v \oplus w) = (u \oplus v) \oplus gyr[u, v](w),$$

(2) For every $u, v \in G$, $gyr[u \oplus v, v] = gyr[u, v]$.

(3) Left identity law holds, that mean there is an element $0 \in G$, such that $0 \oplus u = u$, for all $u \in G$.

(4) For each $u \in G$, there exists $v \in G$ such that $v \oplus u = 0$.

Definition 1.2. (Bi-gyrogroup). A groupoid $(B, \oplus)$ is a bi-gyrogroup if its binary operation satisfies the following axioms.

(1) There is an element $0 \in B$ such that $0 \oplus a = a \oplus 0 = a$ for all $a \in B$.

(2) For each $a \in B$, there is an element $b \in B$ such that $b \oplus a = 0$.

(3) Each pair of a and b in B corresponds to a left automorphism $lgyr[a, b]$ and a right automorphism $rgyr[a, b]$ in $Aut(B, \oplus)$ such that for all $c \in B$,

$$(a \oplus b) \oplus lgyr[a, b]c = rgyr[b, c]a \oplus (b \oplus c)$$

(4) For all $a, b \in B$,

(a) $rgyr[a, b] = rgyr[a \oplus b, lgyr[a, b]b]$

(b) $lgyr[a, b] = lgyr[a \oplus b, rgyr[a, b]b]$

(5) For all $a \in B$, $lgyr[a, 0]$ and $rgyr[a, 0]$ are the identity automorphism of B.

2 Bi-Gyrosemigroup

In this section, we explore and analyze the notion of bi-gyrosemigroup. A groupoid which satisfies the third and forth axioms of Definition 1.2 is called a *bi-gyrosemigroup*.

We call a bi-gyrogroup $(G, \oplus)$ is trivial if $lgyr[a, b]$ and $rgyr[a, b]$ are identity functions, i.e. $lgyr[a, b]x = rgyr[a, b]x = x$, for every $x \in G$. Now, the question that arises is whether there is a non-trivial bi-gyrogroup of any order. The answer in (bi)-gyrogroups is negative. In fact Ashrafi (in [10]) demonstrates that there are no nondegenerate gyrogroups of order less than 8 and we in this papaer shows that there are no nondegenerate bi-gyrogroups of order 2(See Theorem 3.18). We show that the answer is positive in bi-gyrosemigroups. In this section, non-trivial bi-gyrosemigroups with non-identity left(right) gyrator are introduced. Moreover, we present a class of bi-gyrosemigroup of any order with non-trivial left and right gyrators.

Lemma 2.1. *Let S be a non-empty set. We define $\oplus : S \times S \to S$ by $a \oplus b = a$, for all S. Let $lgyr[a, b]$ be an automorphism of S and $rgyr[a, b]$ the identity automorphism for every $a, b \in S$. Then $rgyr[a, b] = rgyr[a \oplus b, lgyr[a, b]b]$ and $lgyr[a, b] = lgyr[a \oplus b, rgyr[a, b]b]$.*

Theorem 2.2. *If $a \oplus b = a$ for every $a, b \in S = \{1, \ldots, n\}$ and $rgyr[a, b] =$ identity, then for every $lgyr[a, b] \in Aut(S, \oplus)$,*

$$rgyr[b, c]a \oplus (b \oplus c) = (a \oplus b) \oplus lgyr[a, b]c.$$

Next, the bi-gyrosemigroups of order n with non-trivial left and right gyrators.

Theorem 2.3. *If $(S = \{1, \ldots, n\}, \oplus)$ be a groupoid with the following Cayley's table and $rgyr[a, b] = F$ for every $a, b \in S$, where $F : S \to S$ is a permutation of order n, then $(S, \oplus)$ is bi-gyrosemigroup for every*

$lgyr[a,b] \in Aut(S,\oplus)$.

$\cdot$	1	2	$\cdots$	n
1	$F(1)$	$F(1)$	$\cdots$	$F(1)$
2	$F(2)$	$F(2)$	$\cdots$	$F(2)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
n	$F(n)$	$F(n)$	$\cdots$	$F(n)$

Table 1: Bi-gyrosemigroup of order n with non-trivial left and right gyrators

Proof. For every $a, b, c \in S$, we have

$$rgyr[b,c]a \oplus (b \oplus c) = F(F(a)) = F(a \oplus b) = (a \oplus b) \oplus lgyr[a,b]c$$

and the result follows. □ Now by the Theorem 2.3 (or directly by Lemma 2.1 and Theorem 2.2), we obtain the next theorem:

Corollary 2.4. *If $(S,\oplus)$ be a groupoid with the following Cayley's table and $rgyr[a,b] = identity$ for every $a, b \in S$, then $(S,\oplus)$ is bi-gyrosemigroup for every $lgyr[a,b] \in Aut(S,\oplus)$.*

$\cdot$	1	2	$\cdots$	n
1	1	1	$\cdots$	1
2	2	2	$\cdots$	2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
n	n	n	$\cdots$	n

Table 2: Non-trivial bi-gyrosemigroup of order n

In the following, two spacial cases of Theorem 2.3 are presented.

Example 2.5. Let S be a non-empty set. We define $\oplus : S \times S \to S$ by $a \oplus b = a$, for all S. Let $lgyr[a,b]$ be a automorphism of S and $rgyr[a,b]$ the identity automorphism for every $a, b \in S$. Then $rgyr[a,b] = rgyr[a \oplus b, lgyr[a,b]b]$, $lgyr[a,b] = lgyr[a \oplus b, rgyr[a,b]b]$ and $(a \oplus b) \oplus lgyr[a,b]c = rgyr[b,c]a \oplus (b \oplus c)$, for every $a, b, c \in S$.

Example 2.6. Let $S = \{0, 1\}$ and $a \oplus b = a'$ for every $a, b \in S$, where $a' = 1$ for $a = 0$ and $a' = 0$ for $a = 1$. Then for every $aut \in Aut(S, \oplus)$,

$$rgyr[b, c]a \oplus (b \oplus c) = (a \oplus b) \oplus lgyr[a, b]c,$$

where for every $a, b, c \in S$ $lgyr[a, b] = aut$ and $rgyr[a, b] = T$, such that $T(c) = c'$ ($c' = 1$ for $c = 0$ and $c' = 0$ for $c = 1$). Also in this case $rgyr[a, b] = rgyr[a \oplus b, lgyr[a, b]b]$ and $lgyr[a, b] = lgyr[a \oplus b, rgyr[a, b]b]$.

Next, a class of groupoids are introduced which can not be a bi-gyrosemigroup with any non-trivial left gyrator.

Theorem 2.7. *Suppose that $(S, \oplus)$ is a groupoid with the following Cayley's table.*

$\cdot$	1	2	$\cdots$	n
1	1	2	$\cdots$	n
2	1	2	$\cdots$	n
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
n	1	2	$\cdots$	n

Table 3: Bi-gyrosemigroup of order n

Then $(S, \oplus)$ is bi-gyrosemigroup if and only if for every $a, b \in S$, $lgyr[a, b]$ is an identity function.

Proof. First suppose that $(S, \oplus)$ is bi-gyrosemigroup. Then for every $a, b, c \in S$, $rgyr[b, c]a \oplus (b \oplus c) = c$ and $(a \oplus b) \oplus lgyr[a, b]c = lgyr[a, b]c$. So $lgyr[a, b]c = c$ and $lgyr[a, b]$ is an identity function. Conversely, we have

$$rgyr[b, c]a \oplus (b \oplus c) = c = (a \oplus b) \oplus c = (a \oplus b) \oplus lgyr[a, b]c$$

and we are done. □

In the following, bi-gyrosemigroups whose structure is derived from Abelian groups are introduced.

Theorem 2.8. *If $(G, \cdot)$ is an Abelian group, then $(G, \oplus)$ is a bi-gyrosemigroup, when for every $a, b \in G$, $a \oplus b = a \cdot b^{-1}$, $lgyr[a, b]c = c^{-1}$ and $rgyr[a, b]c = c$.*

Example 2.9. Consider $(\mathbb{Z}_n, +)$ be a cyclic group of order n. We define $a \oplus b = a + (-b)$. $(\mathbb{Z}_n, \oplus)$ is not semigroup, but it is a bi-gyrosemigroup when for every $a, b \in G$, $rgyr[a, b]c = c$ and $lgyr[a, b]c = -c$.

Theorem 2.10. *If $(G, \cdot)$ is an Abelian group, then $(G, \oplus)$ is a bi-gyrosemigroup, when for every $a, b \in G$, $a \oplus b = a^{-1} \cdot b$, $lgyr[a, b]c = c$ and $rgyr[a, b]c = c^{-1}$.*

Example 2.11. Let $G = (\mathbb{Z}_n, +)$ be the cyclic group of order n. Define operation $\oplus : \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}_n$ by $a \oplus b = b - a$. Then, we have $rgyr[b, c]a \oplus (b \oplus c) = c - b + a = (a \oplus b) \oplus lgyr[a, b]c$ where for every $a, b, c \in \mathbb{Z}_n$, $rgyr[a, b]c = -c$ and $lgyr[a, b]c = c$.

At the end of this section, we will answer the question of whether there is a groupoid that forms a bi-gyrosemigroup for every non-identity gyrator.

Theorem 2.12. *Let $(S, \oplus)$ is a groupoid with non-identity automorphism, then there exists a left and right gyrators, $lgyr$ and $rgyr$, such that $(S, \oplus, lgyr, rgyr)$ is not a bi-gyrosemigroup.*

Proof. To the contrary assume that $(S, \oplus)$ is a groupoid with non-identity automorphism which forms a bi-gyrosemigroup for every left and right gyrators. Let $A_1 \neq A_2$ be two automorphisms of G and $lgyr_i = rgyr_i = A_i$, for $i \in \{1, 2\}$. Since $(S, \oplus)$ is bi-gyrosemigroup with $(lgyr_i, rgyr_j)$, for every $i, j \in \{1, 2\}$, we have $rgyr_1[b, c]a \oplus (b \oplus c) = (a \oplus b) \oplus lgyr_1[a, b]c = rgyr_2[b, c]a \oplus (b \oplus c) = (a \oplus b) \oplus lgyr_2[a, b]c$, for every $a, b, c \in S$. This shows that $a \oplus b$ is independent of a and b, i.e. $a \oplus b = a' \oplus b' = s$ for every $a, b, a', b' \in S$. So $Aut(S, \oplus) = \{identity\}$, a contradiction and the proof is complete. □

A gyrosemigroup is a bi-gyrosemigroup where $lgyr[a, b]$ is an identity. Gyrosemigroups introduced by Mirvakili and et. al. [11] and e characterize gyrosemigroups of order two up to gyroisomorphism.

Theorem 2.13. *If $(S, \oplus, gyr)$ is a gyrosemigroup then it is a bi-gyrosemigroup.*

Proof. Set $lgyr[a, b] = id$ and $rgyr[a, b] = gyr[a, b]$ for every $a, b \in S$. It is not difficult to see that $(S, \oplus, lgyr, rgyr)$ is a bi-gyrosemigroup. □

Example 2.14. The converse of Theorem 2.13 is not true. Let $S = \{0, 1\}$ by the following Cayley's Table:

$\cdot$	0	1
0	1	1
1	0	0

$lgyr$	0	1
0	A	T
1	T	A

$rgyr$	0	1
0	T	T
1	T	T

where A is identity automorphism and T is permutation (01). This means that $T(0) = 1$ and $T(1) = 0$.

We have $(S, \oplus, lgyr, rgyr)$ is a bi-gyrosemigroup but $(S, \oplus, rgyr)$ is not a gyrosemigroup.

3 Bi-gyrosemigroups of order 2

In this section all bi-gyrosemigroups of order 2 are introduced and characterized.

Let $G = \{0, 1\}$. If $(G, \oplus)$ is a groupoid, then $Aut(G) = \{A\}$ or $Aut(G) = \{A, T\}$, where A is identity automorphism and T is permutation (01). This means that $T(0) = 1$ and $T(1) = 0$.

Theorem 3.1. *There exist 5 non-isomorphic semigroups of order 2 among of 8 ones, as following Cayley's Tables:*

S_1

$\cdot$	0	1
0	0	0
1	0	0

S_2

$\cdot$	0	1
0	0	0
1	0	1

S_3

$\cdot$	0	1
0	0	0
1	1	1

S_4

$\cdot$	0	1
0	0	1
1	0	1

S_5

$\cdot$	0	1
0	0	1
1	1	0

Table 4: All semigroups of order 2 up to isomorphism

Theorem 3.2. *Let $(G, \oplus)$ be a semigroup and for every $a, b \in G$, $rgyr[a, b] = A$ and $lgyr[a, b] = A$. Then $(G, \oplus)$ is a bi-gyrosemigroup.*

Proof. For every $a, b, c \in G$, we have $rgyr[b, c]a \oplus (b \oplus c) = a \oplus (b \oplus c) = (a \oplus b) \oplus c = (a \oplus b) \oplus A(c) = (a \oplus b) \oplus lgyr[a, b]c$ and the proof is complete. □

Theorem 3.3. *Let $G = \{0, 1\}$. If $0 \oplus 0 = 1 \oplus 1$, then $Aut(G) = A$ and so $(G, \oplus)$ is a bi-gyrosemigroup if and only if $(G, \oplus)$ is a semigroup and for every $a, b \in G$, $rgyr[a, b] = A$ and $lgyr[a, b] = A$.*

Proof. To the contrary, let $T \in Aut(G)$. Assume that $0 \oplus 0 = 1 \oplus 1 = 0$ then

$$1 = T(0) = T(0 \oplus 0) = T(0) \oplus T(0) = 1 \oplus 1 = 0,$$

a contradiction. By a similar way, $0 \oplus 0 = 1 \oplus 1 = 1$ makes a contradiction. Therefore $Aut(G) = A$. By similar argument, we conclude that $rgyr[a, b] = A$.

Thus a groupoid of order 2 with $0 \oplus 0 = 1 \oplus 1$ can not be a bi-gyrosemigroup with any non trivial gyrator. Moreover, when the gyrator is trivial the concepts of semigroup and bi-gyrosemigroup coincide. The inverse statement is clear by Theorem 3.2 and we are done. □

Corollary 3.4. *Among 2048 bi-gyrogroupoids of order 2 with* $0 \oplus 0 = 1 \oplus 1$*, there are 2044 non-bi-gyrosemigroups and 4 bi-gyrosemigroups.*

Theorem 3.5. *Let* $G = \{0, 1\}$*. If* $(G, \oplus)$ *is a commutative groupoid, then* $(G, \oplus)$ *is a bi-gyrosemigroup if and only if* $(G, \oplus)$ *is a semigroup and for every* $a, b \in G$*,* $rgyr[a, b] = lgyr[a, b] = A$*.*

Proof. Let for some $a, b \in G$, $lgyr[a, b] \neq A$ so $lgyr[a, b](0) = 1$ and $lgyr[a, b](1) = 0$. Let $0 \oplus 1 = 1 \oplus 0 = 0$ then

$$1 = lgyr[a, b](0) = lgyr[a, b](0 \oplus 1) = lgyr[a, b](0) \oplus lgyr[a, b](1) = 1 \oplus 0 = 0.$$

Then it is a contradiction. Similarly, $0 \oplus 1 = 1 \oplus 0 = 1$ makes a contradiction. Therefore for every $a, b \in G$, $lgyr[a, b] = A$. In a similar way it is concluded that $rgyr[a, b] = A$. So, a commutative groupoid of order 2 can not be a bi-gyrosemigroup with any non trivial gyrator. The inverse is clear by Theorem 3.2 and the proof is complete. □

Corollary 3.6. *There are 1022 commutative non-bi-gyrosemigroup groupoids of order 2 and 2 commutative bi-gyrosemigroups such that* $0 \oplus 0 \neq 1 \oplus 1$*.*

Theorem 3.7. *Let* $a \oplus a = a \oplus b = a$*. Then,* $(G, \oplus)$ *is a bi-gyrosemigroup, if and only if* $rgyr[a, b] = A$*, for every* $a, b \in G = \{0, 1\}$*.*

Proof. First, let $rgyr[a, b] = A$. For every $a, b, c \in G$ and $lgyr \in Aut(G)$, we have $rgyr[b, c]a \oplus (b \oplus c) = rgyr[b, c]a = a = a \oplus b = (a \oplus b) \oplus lgyr[a, b]c$. Conversely, and $(a \oplus b) \oplus lgyr[a, b]c = a$ and $rgyr[b, c]a \oplus (b \oplus c) = rgyr[b, c]a$, so $rgyr[a, b] = A$ and the proof is complete. □

Corollary 3.8. *There are 16 bi-gyrosemigroups and and 240 non bi-gyrosemigroups of order 2 with $a \oplus a = a \oplus b = a$.*

Theorem 3.9. *If $(G, \oplus)$ is a bi-gyrosemigroup and $a \oplus b = b'$, where $b' = 0$ for $b = 1$ and $b' = 1$ for $b = 0$, then $(G, \oplus)$ is a bi-gyrosemigroup if and only if for every $a, b, c, d \in G$, $lgyr[a, b] = T$ and $rgyr[a, b] = rgyr[c, d]$.*

Proof. For every $a, b, c \in G$,

$$rgyr[b, c]a \oplus (b \oplus c) = c \text{ and } (a \oplus b) \oplus lgyr[a, b]c = (lgyr[a, b]c)' \Leftrightarrow (lgyr[a, b]c)' = c \Leftrightarrow lgyr[a, b] = T.$$

For the last part of theorem assume that $rgyr[0, 0] = A$. Then $rgyr[1, 1] = rgyr[1 \oplus 1, lgyr[1, 1]1] = rgyr[0, 0] = A$ and so $rgyr[1, 1] = A$. Moreover

$$rgyr[0, 0] = rgyr[0 \oplus 0, lgyr[0, 0]0] = rgyr[1, 1]$$

and

$$rgyr[1, 1] = rgyr[1 \oplus 1, lgyr[1, 1]1] = rgyr[0, 0].$$

Thus in this case for every $a, b \in G$, $rgyr[a, b] = A$. For $rgyr[0, 0] = T$, by similar argument, it is concluded that for every $a, b \in G$, $rgyr[a, b] = T$ and we are done. □

It is noteworthy that the groupoids in the last theorem do not satisfy conditions of Theorems 3.3, 3.5 and 3.7.

Corollary 3.10. *There are 254 non-bi-gyrosemigroup groupoids of order 2 with $0 \oplus 0 = 1 \oplus 0 = 1$, $1 \oplus 1 = 0 \oplus 1 = 0$ and 2 bi-gyrosemigroup with such property.*

Theorem 3.11. *If $a \oplus b = b$ for every $a, b \in G = \{0, 1\}$, then $(G, \oplus)$ is a bi-gyrosemigroup if and only if for every $a, b \in G$, $lgyr[a, b] = A$ and $rgyr[a, b] = rgyr[b, b]$.*

Proof. First assume that $lgyr[a, b] = A$ and $rgyr[a, b] = rgyr[b, b]$, for every $a, b \in G$. So

$$rgyr[b, c]a \oplus (b \oplus c) = c = lgyr[a, b]c = (a \oplus b) \oplus lgyr[a, b]c.$$

Also, $rgyr[a, b] = rgyr[b, b] = rgyr[a \oplus b, lgyr[a, b]b]$ and $lgyr[a, b] = lgyr[b, c] = lgyr[a \oplus b, rgyr[a, b]b]$. Thus $(G, \oplus)$ is a bi-gyrosemigroup.

conversely, suppose that $(G, \oplus)$ is a bi-gyrosemigroup. Then for every $a, b, c \in G$, $c = rgyr[b, c]a \oplus (b \oplus c) = (a \oplus b) \oplus lgyr[a, b]c = lgyr[a, b]c$ and so $lgyr[a, b] = A$. Also, $rgyr[a, b] = rgyr[a \oplus b, lgyr[a, b]b] = rgyr[b, b]$. □

Corollary 3.12. *There are 252 non-bi-gyrosemigroup groupoids of order 2 with $a \oplus b = b$ and 4 bi-gyrosemigroup with such property.*

Theorem 3.13. *Let $a \oplus b = a'$ for every $a, b \in G = \{0, 1\}$, where $a' = 0$ for $a = 1$ and $a' = 1$ for $a = 0$. Then $(G, \oplus)$ is a bi-gyrosemigroup if and only if for every $a, b \in G$, $lgyr[a, b] = lgyr[b, a]$, $lgyr[a, a] = lgyr[b, b]$ and $rgyr[a, b] = T$.*

Proof. First assume that $lgyr[a, b] = lgyr[b, a]$, $lgyr[a, a] = lgyr[b, b]$ and $rgyr[a, b] = T$, for every $a, b \in G$. So

$$rgyr[b, c]a \oplus (b \oplus c) = (rgyr[b, c]a)' = (T(a))' = a = (a')' = (a \oplus b) \oplus lgyr[a, b]c.$$

Also, $rgyr[a, b] = T = rgyr[a \oplus b, lgyr[a, b]b]$. Also, for $a \neq b$, $lgyr[a, b] = lgyr[b, a] = lgyr[a \oplus b, rgyr[a, b]b]$ and $lgyr[a, a] = lgyr[b, b] = lgyr[a \oplus a, rgyr[a, a]a]$. Thus $(G, \oplus)$ is a bi-gyrosemigroup. conversely, suppose that $(G, \oplus)$ is a bi-gyrosemigroup. Then for every $a, b, c \in G$, $(rgyr[b, c]a)' = rgyr[b, c]a \oplus (b \oplus c) = (a \oplus b) \oplus lgyr[a, b]c = (a')'$ and so $rgyr[b, c] = T$. Also, for $a \neq b$, $lgyr[a, b] = lgyr[a \oplus b, rgyr[a, b]b] = lgyr[b, a]$ and $lgyr[a, a] = lgyr[a \oplus a, rgyr[a, a]a] = lgyr[b, b]$ and the proof is complete. □

Corollary 3.14. *There are 252 non-bi-gyrosemigroup groupoids of order 2 with $a \oplus b = a'$, where $a' = 0$ for $a = 1$ and $a' = 1$ for $a = 0$. Also, There are 4 bi-gyrosemigroups with such property.*

Now, by summing up the results stated above, we can classify all bi-gyrosemigroups of order 2. The final result of this classification is summarized in the next theorem.

Theorem 3.15. *There are exactly 32 bi-gyrosemigroups of order 2 called $BGS_1, BGS_2, \ldots, BGS_{32}$, shown in table* 5.

Now, the question arises whether some of these bi-gyrosemigroups can be isomorphic. To answer the question, we first need to introduce the concept of isomorphism in bi-gyrosemigroups.

Definition 3.16. Let $(G, \oplus)$ and $(G', \oplus')$ be two bi-gyrosemigroups. If $f : G \to G'$ is a groupoid homomorphism such that $lgyr'[f(a), f(b)]f(c) = f(lgyr[a, b]c)$ and $rgyr'[f(a), f(b)]f(c) = f(rgyr[a, b]c)$, then f is called bi-gyrohomomorphism.

If bi-gyrohomomorphism f is onto and one to one, then we say that f is a bi-gyroisomorphism. In this case, G and G' are bi-gyroisomorph and denoted by $G \cong G'$.

By applying the last definition on the bi-gyrosemigroups of Corollary 3.15, we conclude the following result.

Theorem 3.17. *There are exactly 22 non-bi-gyroisomorphic bi-gyrosemigroups of order 2.*

Proof. By Definition 3.16, one can see that $GS_1 \simeq GS_{32}$, $GS_2 \simeq GS_{24}$, $GS_4 \simeq GS_{11}$, $GS_5 \simeq GS_7$, $GS_6 \simeq GS_{15}$, $GS_8 \simeq GS_{13}$, $GS_{10} \simeq GS_{17}$, $GS_{14} \simeq GS_{16}$, $GS_{20} \simeq GS_{21}$ and $GS_{23} \simeq GS_{25}$. $\square$

$\cdot_1$	0	1
0	0	0
1	0	0

$lgyr_1$	0	1
0	A	A
1	A	A

$rgyr_1$	0	1
0	A	A
1	A	A

$\cdot_2$	0	1
0	0	0
1	0	1

$lgyr_2$	0	1
0	A	A
1	A	A

$rgyr_2$	0	1
0	A	A
1	A	A

$\cdot_3$	0	1
0	0	0
1	1	1

$lgyr_3$	0	1
0	A	A
1	A	A

$rgyr_3$	0	1
0	A	A
1	A	A

$\cdot_4$	0	1
0	0	0
1	1	1

$lgyr_4$	0	1
0	A	A
1	A	T

$rgyr_4$	0	1
0	A	A
1	A	A

$\cdot_5$	0	1
0	0	0
1	1	1

$lgyr_5$	0	1
0	A	A
1	T	A

$rgyr_5$	0	1
0	A	A
1	A	A

$\cdot_6$	0	1
0	0	0
1	1	1

$lgyr_6$	0	1
0	A	A
1	T	T

$rgyr_6$	0	1
0	A	A
1	A	A

$\cdot_7$	0	1
0	0	0
1	1	1

$lgyr_7$	0	1
0	A	T
1	A	A

$rgyr_7$	0	1
0	A	A
1	A	A

$\cdot_8$	0	1
0	0	0
1	1	1

$lgyr_8$	0	1
0	A	T
1	A	T

$rgyr_8$	0	1
0	A	A
1	A	A

$\cdot_9$	0	1
0	0	0
1	1	1

$lgyr_9$	0	1
0	A	T
1	T	A

$rgyr_9$	0	1
0	A	A
1	A	A

$\cdot_{10}$	0	1
0	0	0
1	1	1

$lgyr_{10}$	0	1
0	A	T
1	T	T

$rgyr_{10}$	0	1
0	A	A
1	A	A

$\cdot_{11}$	0	1
0	0	0
1	1	1

$lgyr_{11}$	0	1
0	T	A
1	A	A

$rgyr_{11}$	0	1
0	A	A
1	A	A

$\cdot_{12}$	0	1
0	0	0
1	1	1

$lgyr_{12}$	0	1
0	T	A
1	A	T

$rgyr_{12}$	0	1
0	A	A
1	A	A

$\cdot_{13}$	0	1		$lgyr_{13}$	0	1		$rgyr_{13}$	0	1
0	0	0		0	T	A		0	A	A
1	1	1		1	T	A		1	A	A

$\cdot_{14}$	0	1		$lgyr_{14}$	0	1		$rgyr_{14}$	0	1
0	0	0		0	T	A		0	A	A
1	1	1		1	T	T		1	A	A

$\cdot_{15}$	0	1		$lgyr_{15}$	0	1		$rgyr_{15}$	0	1
0	0	0		0	T	T		0	A	A
1	1	1		1	A	A		1	A	A

$\cdot_{16}$	0	1		$lgyr_{16}$	0	1		$rgyr_{16}$	0	1
0	0	0		0	T	T		0	A	A
1	1	1		1	A	T		1	A	A

$\cdot_{17}$	0	1		$lgyr_{17}$	0	1		$rgyr_{17}$	0	1
0	0	0		0	T	T		0	A	A
1	1	1		1	T	A		1	A	A

$\cdot_{18}$	0	1		$lgyr_{18}$	0	1		$rgyr_{18}$	0	1
0	0	0		0	T	T		0	A	A
1	1	1		1	T	T		1	A	A

$\cdot_{19}$	0	1		$lgyr_{19}$	0	1		$rgyr_{19}$	0	1
0	0	1		0	A	A		0	A	A
1	0	1		1	A	A		1	A	A

$\cdot_{20}$	0	1		$lgyr_{20}$	0	1		$rgyr_{20}$	0	1
0	0	1		0	A	A		0	A	T
1	0	1		1	A	A		1	A	T

$\cdot_{21}$	0	1		$lgyr_{21}$	0	1		$rgyr_{21}$	0	1
0	0	1		0	A	A		0	T	A
1	0	1		1	A	A		1	T	A

$\cdot_{22}$	0	1		$lgyr_{22}$	0	1		$rgyr_{22}$	0	1
0	0	1		0	A	A		0	T	T
1	0	1		1	A	A		1	T	T

$\cdot_{23}$	0	1		$lgyr_{23}$	0	1		$rgyr_{23}$	0	1
0	0	1		0	A	A		0	A	A
1	1	0		1	A	A		1	A	A

$\cdot_{24}$	0	1		$lgyr_{24}$	0	1		$rgyr_{24}$	0	1
0	0	1		0	A	A		0	A	A
1	1	1		1	A	A		1	A	A

$\cdot_{25}$	0	1		$lgyr_{25}$	0	1		$rgyr_{25}$	0	1
0	1	0		0	A	A		0	A	A
1	0	1		1	A	A		1	A	A

$\cdot_{26}$	0	1		$lgyr_{26}$	0	1		$rgyr_{26}$	0	1
0	1	0		0	T	T		0	A	A
1	1	0		1	T	T		1	A	A

$\cdot_{27}$	0	1		$lgyr_{27}$	0	1		$rgyr_{27}$	0	1
0	1	0		0	T	T		0	T	T
1	1	0		1	T	T		1	T	T

$\cdot_{28}$	0	1		$lgyr_{28}$	0	1		$rgyr_{28}$	0	1
0	1	1		0	A	A		0	T	T
1	0	0		1	A	A		1	T	T

$\cdot_{29}$	0	1		$lgyr_{29}$	0	1		$rgyr_{29}$	0	1
0	1	1		0	A	T		0	T	T
1	0	0		1	T	A		1	T	T

$\cdot_{30}$	0	1		$lgyr_{30}$	0	1		$rgyr_{30}$	0	1
0	1	1		0	T	A		0	T	T
1	0	0		1	A	T		1	T	T

$\cdot_{31}$	0	1		$lgyr_{31}$	0	1		$rgyr_{31}$	0	1
0	1	1		0	T	T		0	T	T
1	0	0		1	T	T		1	T	T

$\cdot_{32}$	0	1		$lgyr_{32}$	0	1		$rgyr_{32}$	0	1
0	1	1		0	A	A		0	A	A
1	1	1		1	A	A		1	A	A

Table 5: All bi-gyrosemigroups of order 2

Theorem 3.18. *There does not exist a bi-gyrogroup of order 2.*

Proof. In accordance with Theorem 3.17, it is evident that the 22 gyrosemigroups of order 2 do not constitute bi-gyrogroups. □

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References

[1] A. R. Ashrafi, K. Mavaddat Nezhaad and M. A. Salahshour, Classification of gyrogroups of orders at most 31, *AUT J. Math. Comput.*, 5(1) (2024) 11–18.

[2] F. Chatelin, *Qualitative Computing: A Computational Journey into Nonlinearity*, World Scientific Publishing, Hackensack, NJ, 2012.

[3] A. H. Clifford and G.B. Preston, The algebraic theory of semigroups, Mathematical Surveys, vol. I, *Amer. Math. Soc.*, Providence, RI, 1961.

[4] A. H. Clifford and G.B. Preston, The algebraic theory of semigroups, Mathematical Surveys, vol. II,*Amer. Math. Soc.*, Providence, RI, 1967.

[5] M. Ferreira, Factorizations of Möbius gyrogroups, *Adv. Appl. Clifford Algebras*, 19 (2009) 303–323.

[6] M. Ferreira, *Gyrogroups in Projective Hyperbolic Clifford Analysis, in: Hypercomplex Analysis and Applications*, Trends in Mathematics, I. Sabadini, F. Sommen F. (Eds.), Springer, Basel, 2011.

[7] M. Ferreira and G. Ren, Möbius gyrogroups: A Clifford algebra approach, *J. Algebra*, 328 (2011) 230–253.

[8] J. M. Howie, *An Introduction to Semigroup Theory*, London Math. Soc. Monographs, Academic Press, London, 1976.

[9] J. Lawson, Clifford algebras, Mobius transformations, Vahlen matrices, and B-loops. *Comment. Math. Univ. Carolin.*, 51(2) (2010) 319–331.

[10] S. Mahdavi, A. R. Ashrafi and M. A. Salahshour, Construction of new gyrogroup and the structure of their subgyrogroups, *Algebraic Structures and Their Applications*, 8(2) (2021) 17–30.

[11] S. Mirvakili, R. Manaviyat and K. Hamidizadeh, Exploring the Algebraic Properties of Gyrosemigroups and a Characterization of Gyrosemigroups of Order 2, *Mathematics Interdisciplinary Research*, 9 (3) (2024) 255 – 267.

[12] T. Suksumran and A. A. Ungar, Bi-Gyrogroup: The Group-Like Structure Induced by Bi-Decomposition of Groups, *Mathematics Interdisciplinary Research*, 1 (2016) 111 – 142.

[13] T. Suksumran and K. Wiboonton, Einstein gyrogroup as a B-loop, *Rep. Math. Phys.*, 76 (2015) 63–74.

[14] A. A. Ungar, Parametric realization of the Lorentz transformation group in pseudo-Euclidean spaces,*J. Geom. Symmetry Phys.*, 38 (2015) 39108.

[15] A. A. Ungar, Thomas rotation and parametrization of the Lorentz transformation group, *Found. Phys. Lett.*, 1 (1988) 57–89.

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