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Original Research Paper

Generalized Weak ϕ-Contractions in Metric and Normed Interval Spaces

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Abstract. In this article, we describe the notions of generalized ϕ-contraction mappings and generalized weak ϕ-contraction mappings in metric and normed interval spaces. We also prove some theorems for the existence of the near-fixed points for these mappings in the interval spaces. Moreover, we investigate the uniqueness of the equivalence class in these theorems. Additionally, we provide examples to demonstrate the validity of our results.

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1 Introduction

For many years, Banach's contraction principle was the most important tool to find fixed points. It has been used in various areas of mathematics, and numerous generalizations and extensions have been derived from it. The weak contraction principle, which was extended to metric spaces

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by Rhoades [5, 13], is one such generalization of Banach's contraction principle. Fréchet [1, 3] was the first person to propose the concept of a metric space, and many researchers have since extended it to other spaces, including metric interval spaces. In 2018, Wu [10] presented the notion of the metric interval space (MIS) and the normed interval space (NIS) for a collection of closed and bounded intervals $\mathcal{I}$ in $\mathbb{R}$, which are different from customary metric and normed spaces. Continuing Wu's research, Ullah and et al. [9] presented the near-coincidence point theorems in these spaces via a simulation function. Recently, Sarwar et al. [8] studied some near-fixed point results in MIS and NIS by using α-admissibility and the concept of simulation functions. The interval space $\mathcal{I}$ is not a vector space because the difference of each member of $\mathcal{I}$ from itself is not zero. Therefore, the customary normed space $(\mathcal{I}, \|\cdot\|)$ cannot be considered for the interval space $\mathcal{I}$. For this purpose, NIS has been introduced based on the null set. According to Banach's contraction principle, the fixed point is provided for the mapping T when the space $(\mathcal{I}, d)$ is a metric space, but for some distance function candidates such as $d([k,l], [u,v]) = |(k+l)-(u+v)|$, the space $(\mathcal{I}, d)$ cannot be a metric space. So, MIS has been presented based on the null set [10]. There is no additive inverse member for any non-degenerated closed interval in $\mathcal{I}$. Therefore, $\mathcal{I}$ cannot be a linear space. The algebraic structure of $\mathcal{I}$ is named a quasilinear space (QLS) [6, 12]. The QLS was first stated by Aseev in 1986 [4].

Today, data collection is considered a significant issue for solving practical problems, but in many situations, it is impossible to do it precisely. In this condition, instead of using probability theory, we can use bounded and closed intervals. For example, because it is impossible to measure the level of liquids precisely due to fluctuations, we can consider that it is in a bounded closed interval. As another example, due to the intensity of fluctuations in the trading market, the stock price in a short time interval cannot be recorded as a number. To solve this problem, we can suppose it in a bounded and closed interval. Therefore, interval analysis can be used in various issues, including engineering, economics and social sciences to address problems with uncertainty [11].

The purpose of this article is to provide concepts of generalized ϕ-contraction mappings and generalized weak ϕ-contraction mappings in

MIS and NIS and then demonstrate the near-fixed point theorems for these mappings in interval spaces. Although contraction mappings have been studied in a number of articles [2, 13] in conventional metric spaces, these mappings are stated differently and for the first time in MIS and NIS in this research. Therefore, we state some near-fixed point theorems for generalized ϕ-contraction mappings and generalized weak ϕ-contraction mappings in these new spaces. We also provide some examples.
In section 2, we define the interval space and the null set. In section 3, we introduce metric and normed interval spaces and their properties. In section 4, we state generalized ϕ-contraction and generalized weak ϕ-contraction mappings in metric and normed interval spaces and prove the near-fixed point theorems for such mappings. Finally, in section 5, we give a general conclusion of the article.

2 Preliminaries

Suppose $\mathcal{I}$ is the collection of all closed and bounded intervals $[k, l]$ in $\mathbb{R}$, such that $k \leq l$. The addition and scalar multiplication operations on $\mathcal{I}$ are defined as follows:

$$[k, l] \oplus [u, v] = [k + u, l + v] \quad \text{and} \quad a\,[k, l] = \begin{cases} [ak, al] & \text{if } a \geq 0 \\ [al, ak] & \text{if } a < 0. \end{cases}$$

Note that $\mathcal{I}$ is not a customary vector space considering the mentioned two operations, because there is no additive inverse member for any non-degenerated closed interval. Clearly, we have $[0, 0] \in \mathcal{I}$ as a zero member. However, the subtraction $[k, l] \ominus [k, l] = [k, l] \oplus [-l, -k] = [k - l, l - k]$ does not give a zero element for some $[k, l] \in \mathcal{I}$.
The null set is defined by:

$$\Omega = \left\{ [k, l] \ominus [k, l] : \ [k, l] \in \mathcal{I} \right\}.$$

It is clear that

$$\Omega = \left\{ [-a, a] : \ \ a \geq 0 \right\} = \left\{ a\,[-1, 1] : \ \ a \geq 0 \right\}.$$

For more details, see [10].
Because some members of $\mathcal{I}$ have not an additive inverse member, $\mathcal{I}$

cannot be a linear space. The algebraic structure of $\mathcal{I}$ is named a quasi-linear space (QLS) [6]. The QLS was first stated by Aseev in 1986 [4]. An important example of quasilinear spaces is $\mathcal{I}$, the collection of all closed real intervals, with the inclusion relation " $\subseteq$ ", and with the addition and real-scalar multiplication operations defined as follows:

$$A + B = \{a + b : a \in A, b \in B\} \quad \text{and} \quad \lambda \cdot A = \{\lambda a : a \in A\}.$$

This set is denoted by $\Omega_C(\mathbb{R})$. For more details, see [6, 12].

Remark 2.1. [10] *In any interval space, the following facts are valid:*

- *For any* $[k,l]\,, [u,v]\,, [m,n] \in \mathcal{I}$*, we have*

$$\begin{aligned}[m,n] \ominus ([k,l] \oplus [u,v]) &= [m,n] \oplus (-\,[k,l]) \oplus (-\,[u,v]) \\ &= [m,n] \ominus [k,l] \ominus [u,v]\,.\end{aligned}$$

- *We define* $[k,l] \stackrel{\Omega}{=} [u,v]$ *if and only if*

$$\exists\, \omega_1, \omega_2 \in \Omega \quad \textit{such that} \quad [k,l] \oplus \omega_1 = [u,v] \oplus \omega_2.$$

Clearly, $[k,l] = [u,v]$ *implies* $[k,l] \stackrel{\Omega}{=} [u,v]$ *by choosing* $\omega_1 = \omega_2 = [0,0]$. *However, the inverse is not usually true. The following class based on the relation* $\stackrel{\Omega}{=}$ *for any* $[k,l] \in \mathcal{I}$*, is defined by:*

$$\big\langle\, [k,l]\,\big\rangle = \Big\{\, [u,v] \in \mathcal{I} : [k,l] \stackrel{\Omega}{=} [u,v]\,\Big\}. \tag{1}$$

$\big\langle \mathcal{I} \big\rangle$ *represents the family of all classes* $\big\langle\, [k,l]\,\big\rangle$ *for* $[k,l] \in \mathcal{I}$*.*

Proposition 2.2. *[10] The relation* $\stackrel{\Omega}{=}$ *is a reflexive, symmetric and transitive relation. Therefore,* $\stackrel{\Omega}{=}$ *will be an equivalence relation.*

This proposition shows that the classes (1) provide the equivalence classes. Moreover, $[u,v] \in \big\langle\, [k,l]\,\big\rangle$ implies that $\big\langle\, [k,l]\,\big\rangle = \big\langle\, [u,v]\,\big\rangle$ ([10]).

3 Metric and Normed Interval Spaces

Definition 3.1. [10] *Suppose that* $d : \mathcal{I} \times \mathcal{I} \to \mathbb{R}^+$ *is a mapping. A pair* $(\mathcal{I}, d)$ *is named a metric interval space (MIS) if* d *satisfies the following three conditions for all* $[k,l], [u,v], [m,n] \in \mathcal{I}$*:*
(*i*) $d([k,l], [u,v]) = 0$ *if and only if* $[k,l] \overset{\Omega}{=} [u,v]$
(*ii*) $d([k,l], [u,v]) = d([u,v], [k,l])$;
(*iii*) $d([k,l], [u,v]) \leq d([k,l], [m,n]) + d([m,n], [u,v])$.
We say that the null equalities hold for d *if it satisfies the following equalities for any* $\omega_1, \omega_2 \in \Omega$ *and* $[k,l], [u,v] \in \mathcal{I}$*:*

- $d([k,l] \oplus \omega_1, [u,v] \oplus \omega_2) = d([k,l], [u,v])$;
- $d([k,l] \oplus \omega_1, [u,v]) = d([k,l], [u,v])$;
- $d([k,l], [u,v] \oplus \omega_2) = d([k,l], [u,v])$.

Definition 3.2. [10] *The sequence* $\{ [k_n, l_n] \}_{n=1}^{\infty}$ *in MIS* $(\mathcal{I}, d)$ *converges to* $[k,l] \in \mathcal{I}$ *if and only if* $\lim_{n\to\infty} d([k_n, l_n], [k,l]) = 0$. *The class* $\langle [k,l] \rangle$ *is named the class limit of* $\{ [k_n, l_n] \}_{n=1}^{\infty}$. *In addition, the uniqueness of the class limit in a MIS is easily obtained. see [10] for the definitions of completeness of space and Cauchyness of sequence in MIS.*

Proposition 3.3. *[10] If the sequence* $\{ [k_n, l_n] \}_{n=1}^{\infty}$ *exists in* $\mathcal{I}$ *such that* $\lim_{n\to\infty} d([k_n, l_n], [k,l]) = 0$, *and the null equality holds for* d, *then* $\lim_{n\to\infty} d([k_n, l_n], [u,v]) = 0$ *for any* $[u,v] \in \langle [k,l] \rangle$.

Example 3.4. [10] Let $d : \mathcal{I} \times \mathcal{I} \to \mathbb{R}^+$ be defined by $d([k,l], [u,v]) = |(k+l) - (u+v)|$. Then, it is obvious that $(\mathcal{I}, d)$ is a complete metric interval space (CMIS) and the null equality holds for d.

Definition 3.5. [10] *For a mapping* $\|\cdot\| : \mathcal{I} \to \mathbb{R}^+$ *that* $\mathbb{R}^+$ *is nonnegative real numbers, we present the following features:*
(*i*) $\|\lambda [k,l]\| = |\lambda| \cdot \|[k,l]\|, \ \forall [k,l] \in \mathcal{I}$ *and* $\lambda \in \mathbb{F}$;
(*i°*) $\|\lambda [k,l]\| = |\lambda| \cdot \|[k,l]\|, \ \forall [k,l] \in \mathcal{I}$ *and* $\lambda \in \mathbb{F}$ *with* $\lambda \neq 0$*;*
(*ii*) $\|[k,l] \oplus [u,v]\| \leq \|[k,l]\| + \|[u,v]\|, \ \forall [k,l], [u,v] \in \mathcal{I}$;
(*iii*) $\|[k,l]\| = 0$ *implies* $[k,l] \in \Omega$.

- *It is said that $(\mathcal{I}, \|\cdot\|)$ is a normed interval space (NIS) if it satisfies cases (i), (ii) and (iii).*
- *It is said that the null condition holds for $\|\cdot\|$ if item (iii) is changed to $\|[k,l]\| = 0 \Leftrightarrow [k,l] \in \Omega$.*
- *It is said that the null equality holds for $\|\cdot\|$ if $\|[k,l] \oplus \omega\| = \|[k,l]\|$ for all $[k,l] \in \mathcal{I}$ and $\omega \in \Omega$.*

A complete normed interval space is named a Banach interval space (BIS).

Example 3.6. [10] Assume that the mapping $\|\cdot\| : \mathcal{I} \to \mathbb{R}^+$ is defined by $\|[k,l]\| = |k+l|$. Then, it is evident that $(\mathcal{I}, \|\cdot\|)$ is a BIS, and the null equality holds for $\|\cdot\|$.

For more details about normed interval spaces (NIS), see [10].

Remark 3.7. *Every normed interval space (NIS) may not be a metric interval space (MIS).*

Example 3.8. Suppose that a nonnegative real-valued function $\|\cdot\| : \mathcal{I} \to \mathbb{R}^+$ is defined by $\|\,[k,l]\,\| = |k| + |l|$. Then $(\mathcal{I}, \|\cdot\|)$ satisfies all the conditions of a NIS, but it cannot be considered a MIS. Because condition (i) of Definition 3.1 does not hold. Now we check this claim. Assume that $[k,l] \stackrel{\Omega}{=} [u,v]$. Then $[k,l] \oplus \omega_1 = [u,v] \oplus \omega_2$, where $\omega_1 = [-a_1, a_1] \in \Omega$ and $\omega_2 = [-a_2, a_2] \in \Omega$ with $a_1, a_2 \geq 0$. So, we have $[k - a_1, l + a_1] = [u - a_2, v + a_2]$. Therefore, $k - a_1 = u - a_2$ and $l + a_1 = v + a_2$, i.e., $k = a_1 - a_2 + u$ and $l = a_2 - a_1 + v$. Thus we obtain

$$\begin{aligned} d([k,l]\,,[u,v]) &= \|[k,l] \ominus [u,v]\| = \|[k,l] \oplus [-v,-u]\| = \|[k-v, l-u]\| \\ &= |k-v| + |l-u| = |k-v| + |a_2 - a_1 + v - u| \\ &= |k-v| + |-(a_1 - a_2 + u - v)| \\ &= |k-v| + |a_1 - a_2 + u - v| = |k-v| + |k-v| = 2|k-v|, \end{aligned}$$

where k is not necessarily equal to v. Therefore, we have $d([k,l]\,,[u,v]) \neq 0$. For example, consider $[-1,1] \stackrel{\Omega}{=} [-2,2]$. So, we have $d([-1,1]\,,[-2,2]) = |-1-2| + |1+2| = 6 \neq 0$.

4 Near-Fixed Point Results

Definition 4.1. [10] *A point* $[k,l] \in \mathcal{I}$ *is named a near-fixed point for a self-mapping* $T : \mathcal{I} \to \mathcal{I}$ *if* $T\,[k,l] \stackrel{\Omega}{=} [k,l]$.

We have $T\,[k,l] \stackrel{\Omega}{=} [k,l]$ if and only if there exist $[-b_1, b_1], [-b_2, b_2] \in \Omega$ where $b_1, b_2 \in \mathbb{R}^+$ so that at least one of the following equalities holds:

- $T\,[k,l] \oplus [-b_1, b_1] = [k,l]$;
- $T\,[k,l] = [k,l] \oplus [-b_1, b_1]$;
- $T\,[k,l] \oplus [-b_1, b_1] = [k,l] \oplus [-b_2, b_2]$.

Definition 4.2. [7] *A point* $[k,l] \in \mathcal{I}$ *is a common near-fixed point for functions* $T, S : \mathcal{I} \to \mathcal{I}$ *if* $T\,[k,l] \stackrel{\Omega}{=} S\,[k,l] \stackrel{\Omega}{=} [k,l]$.

Example 4.3. Assume that $T, S : \mathcal{I} \to \mathcal{I}$ are defined by

$$T\,[k,l] = [k-3, l+3] \quad \text{and} \quad S\,[k,l] = [k-4, l+4].$$

We indicate that $[k,l]$ is a common near-fixed point of T and S. For $\omega_1 = [0,0] \in \Omega$ and $\omega_2 = [-3,3] \in \Omega$, we have $T\,[k,l] \stackrel{\Omega}{=} [k,l]$, i.e.,

$$\begin{aligned}[k-3, l+3] \stackrel{\Omega}{=} [k,l] &\iff [k-3, l+3] \oplus [0,0] = [k,l] \oplus [-3,3] \\ &\iff [k-3, l+3] = [k-3, l+3].\end{aligned}$$

Similarly, for $\omega_1 = [0,0] \in \Omega$ and $\omega_2 = [-4,4] \in \Omega$, we obtain $S\,[k,l] \stackrel{\Omega}{=} [k,l]$. According to Proposition 2.2, we have $T\,[k,l] \stackrel{\Omega}{=} S\,[k,l]$. Hence, $T\,[k,l] \stackrel{\Omega}{=} S\,[k,l] \stackrel{\Omega}{=} [k,l]$.

Now, we state several new definitions in MIS and NIS as follows.

Definition 4.4. *A mapping* $T : (\mathcal{I}, d) \to (\mathcal{I}, d)$ *is said to be a weak* ϕ*-contraction on* $\mathcal{I}$ *if for all* $[k,l], [u,v] \in \mathcal{I}$,

$$d(T\,[k,l], T\,[u,v]) \leq d([k,l], [u,v]) - \phi\big(d([k,l], [u,v])\big),$$

where a function $\phi : [0,\infty) \to [0,\infty)$ *is continuous and non-decreasing with* $\phi(t) > 0$ *for* $t \in (0,\infty)$ *and* $\phi(0) = 0$.

Definition 4.5. *Suppose that $(\mathcal{I}, d)$ is a MIS. A pair of mappings $S, T : (\mathcal{I}, d) \to (\mathcal{I}, d)$ is called generalized weak ϕ-contractions on $\mathcal{I}$ if for all $[k,l], [u,v] \in \mathcal{I}$,*

$$d(T[k,l], S[u,v]) \leq M([k,l],[u,v]) - \phi\big(M([k,l],[u,v])\big), \tag{2}$$

where a function $\phi : [0,\infty) \to [0,\infty)$ is continuous and non-decreasing with $\phi(t) > 0$ for $t \in (0,\infty)$ and $\phi(0) = 0$, and

$$M([k,l],[u,v]) = \max\Big\{d([k,l],[u,v]), d([k,l], T[k,l]), d([u,v], S[u,v]), \frac{d([k,l], S[u,v]) + d([u,v], T[k,l])}{2}\Big\}.$$

Theorem 4.6. *Let the pair of mappings $T, S : (\mathcal{I}, d) \to (\mathcal{I}, d)$ be generalized weak ϕ-contractions on $\mathcal{I}$ in CMIS $(\mathcal{I}, d)$ such that the null equality holds for d. Then T and S have a common near-fixed point $[k,l] \in \mathcal{I}$ satisfying $T[k,l] \overset{\Omega}{=} [k,l] \overset{\Omega}{=} S[k,l]$. Moreover, a unique equivalence class $\big\langle [k,l] \big\rangle$ exists for T and S such that if $[k,l]$ and $\big[\bar{k}, \bar{l}\big]$ are the common near-fixed points of T and S, then $[k,l] \overset{\Omega}{=} \big[\bar{k}, \bar{l}\big]$. In addition, every point $\big[\bar{k}, \bar{l}\big] \in \big\langle [k,l] \big\rangle$ is a common near-fixed point for T and S.*

Proof. It is clear that $[k,l] \overset{\Omega}{=} [u,v]$ is a common near-fixed point of T and S if and only if $M\big([k,l],[u,v]\big) = 0$. Indeed, if $[k,l] \overset{\Omega}{=} [u,v]$ is a common near-fixed point for S and T, then by using Proposition 2.2, we have $T[k,l] \overset{\Omega}{=} [k,l] \overset{\Omega}{=} S[k,l] \overset{\Omega}{=} T[u,v] \overset{\Omega}{=} [u,v] \overset{\Omega}{=} S[u,v]$ and

$$M([k,l],[u,v]) = \max\Big\{d([k,l],[u,v]), d([k,l], T[k,l]), d([u,v], S[u,v]), \frac{d([k,l], S[u,v]) + d([u,v], T[k,l])}{2}\Big\} = 0.$$

Now, suppose that $M([k,l],[u,v]) = 0$. Then using $d([k,l],[u,v]) \leq M([k,l],[u,v])$, $d([k,l], T[k,l]) \leq M([k,l],[u,v])$ and $d([u,v], S[u,v]) \leq M([k,l],[u,v])$ and making use of Proposition 2.2, we have

$$T[k,l] \overset{\Omega}{=} [k,l] \overset{\Omega}{=} S[k,l] \overset{\Omega}{=} T[u,v] \overset{\Omega}{=} [u,v] \overset{\Omega}{=} S[u,v].$$

Let $[k_0, l_0] \in \mathcal{I}$. Then, inductively select a sequence $\big\{ [k_n, l_n] \big\}_{n=1}^{\infty}$ in $\mathcal{I}$ such that

$$[k_{2n+2}, l_{2n+2}] = T[k_{2n+1}, l_{2n+1}] \quad \text{and} \quad [k_{2n+1}, l_{2n+1}] = S[k_{2n}, l_{2n}],$$

for all $n \geq 0$. According to the above content, note that if $[k_{n+1}, l_{n+1}] = [k_n, l_n]$ for any $n \geq 0$, then S and T have a common near-fixed point. Thus, assume that $d([k_{n+1}, l_{n+1}], [k_n, l_n]) \neq 0$ for all $n \geq 0$. If $n \in \mathbb{N}$ is an odd number, by using (2) and the triangle inequality, we have

$$\begin{aligned}
&d([k_{n+1}, l_{n+1}], [k_n, l_n]) = d(T[k_n, l_n], S[k_{n-1}, l_{n-1}]) \\
&\leq M([k_n, l_n], [k_{n-1}, l_{n-1}]) - \phi\big(M([k_n, l_n], [k_{n-1}, l_{n-1}])\big) \\
&\leq M([k_n, l_n], [k_{n-1}, l_{n-1}]) \\
&= \max\Big\{d([k_n, l_n], [k_{n-1}, l_{n-1}]), d([k_n, l_n], [k_{n+1}, l_{n+1}]), \\
&d([k_{n-1}, l_{n-1}], [k_n, l_n]), \frac{d([k_n, l_n], [k_n, l_n]) + d([k_{n-1}, l_{n-1}], [k_{n+1}, l_{n+1}])}{2}\Big\} \\
&\leq \max\Big\{d([k_n, l_n], [k_{n-1}, l_{n-1}]), d([k_n, l_n], [k_{n+1}, l_{n+1}]), \\
&\qquad \frac{d([k_{n-1}, l_{n-1}], [k_n, l_n]) + d([k_n, l_n], [k_{n+1}, l_{n+1}])}{2}\Big\} \\
&\leq \max\big\{d([k_n, l_n], [k_{n-1}, l_{n-1}]), d([k_n, l_n], [k_{n+1}, l_{n+1}])\big\}, \qquad (3)
\end{aligned}$$

where

$$\begin{aligned}
&d([k_{n-1}, l_{n-1}], [k_{n+1}, l_{n+1}]) \\
&\qquad \leq d([k_{n-1}, l_{n-1}], [k_n, l_n]) + d([k_n, l_n], [k_{n+1}, l_{n+1}]) \\
&\qquad \leq 2\max\big\{d([k_n, l_n], [k_{n-1}, l_{n-1}]), d([k_n, l_n], [k_{n+1}, l_{n+1}])\big\}.
\end{aligned}$$

Note that if

$$\begin{aligned}
\max\big\{d([k_n, l_n], [k_{n-1}, l_{n-1}]), d([k_n, l_n], [k_{n+1}, l_{n+1}])\big\} \\
= d([k_n, l_n], [k_{n+1}, l_{n+1}]),
\end{aligned}$$

then

$$\begin{aligned}
&d([k_{n+1}, l_{n+1}], [k_n, l_n]) \\
&\qquad \leq d([k_{n+1}, l_{n+1}], [k_n, l_n]) - \phi\big(d([k_{n+1}, l_{n+1}], [k_n, l_n])\big),
\end{aligned}$$

which is a contradiction. So, from (3), we have $d([k_{n+1}, l_{n+1}], [k_n, l_n]) \leq d([k_n, l_n], [k_{n-1}, l_{n-1}])$. Similarly, if $n \in \mathbb{N}$ is an even number, we also obtain $d([k_{n+1}, l_{n+1}], [k_n, l_n]) \leq d([k_n, l_n], [k_{n-1}, l_{n-1}])$.
Hence, $d([k_{n+1}, l_{n+1}], [k_n, l_n]) \leq d([k_n, l_n], [k_{n-1}, l_{n-1}])$ for all $n \geq 0$.

Thus, $\big\{d([k_n, l_n], [k_{n-1}, l_{n-1}])\big\}$ is a nonnegative, decreasing and bounded below sequence. Therefore, it converges to L where $L \geq 0$. Suppose that $L > 0$. Hence from (3), we have

$$d([k_{n+1}, l_{n+1}], [k_n, l_n]) \leq M([k_n, l_n], [k_{n-1}, l_{n-1}]) \leq d([k_n, l_n], [k_{n-1}, l_{n-1}]).$$

Taking $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} M([k_n, l_n], [k_{n-1}, l_{n-1}]) = L. \tag{4}$$

By using (2), we have

$$\begin{aligned} d([k_{n+1}, l_{n+1}], [k_n, l_n]) \leq M([k_n, l_n], [k_{n-1}, l_{n-1}]) \\ - \phi\big(M([k_n, l_n], [k_{n-1}, l_{n-1}])\big). \end{aligned}$$

Therefore, taking $n \to \infty$ in the above inequality and using (4), we obtain

$$L \leq L - \phi(L).$$

Thus $\phi(L) \leq 0$, which contradicts $L > 0$ and $\phi(t) > 0$ for $t > 0$. So $L = 0$, i.e.,

$$\lim_{n\to\infty} d\big(\,[k_n, l_n], [k_{n-1}, l_{n-1}]\,\big) = 0. \tag{5}$$

Now, we indicate that $\big\{\,[k_n, l_n]\,\big\}_{n=1}^{\infty}$ is a Cauchy sequence in MIS $(\mathcal{I}, d)$. Due to (5), it is adequate to demonstrate that $\big\{\,[k_{2n}, l_{2n}]\,\big\}_{n=1}^{\infty}$ is a Cauchy sequence in MIS $(\mathcal{I}, d)$. Suppose this is not true, so there is an $\varepsilon > 0$ that we can detect two sequences of positive integers $\{2m(h)\}$ and $\{2n(h)\}$ with $2n(h) > 2m(h) > h$ such that

$$\begin{aligned} &d\big(\,\big[k_{2m(h)}, l_{2m(h)}\big], \big[k_{2n(h)}, l_{2n(h)}\big]\,\big) \geq \varepsilon \quad \text{and} \\ &d\big(\,\big[k_{2m(h)}, l_{2m(h)}\big], \big[k_{2n(h)-2}, l_{2n(h)-2}\big]\,\big) < \varepsilon, \end{aligned}$$

for all positive integers h. Therefore, we have

$$\begin{aligned} \varepsilon \leq & d\big(\,\big[k_{2m(h)}, l_{2m(h)}\big], \big[k_{2n(h)}, l_{2n(h)}\big]\,\big) \\ \leq & d\big(\,\big[k_{2m(h)}, l_{2m(h)}\big], \big[k_{2n(h)-2}, l_{2n(h)-2}\big]\,\big) \\ & + d\big(\,\big[k_{2n(h)-2}, l_{2n(h)-2}\big], \big[k_{2n(h)-1}, l_{2n(h)-1}\big]\,\big) \\ & + d\big(\,\big[k_{2n(h)-1}, l_{2n(h)-1}\big], \big[k_{2n(h)}, l_{2n(h)}\big]\,\big) \\ < & \varepsilon + d\big(\,\big[k_{2n(h)-2}, l_{2n(h)-2}\big], \big[k_{2n(h)-1}, l_{2n(h)-1}\big]\,\big) \\ & + d\big(\,\big[k_{2n(h)-1}, l_{2n(h)-1}\big], \big[k_{2n(h)}, l_{2n(h)}\big]\,\big). \end{aligned}$$

Taking $h \to \infty$ in the above inequality and making use of (5), we have

$$\lim_{h\to\infty} d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2n(h)}, l_{2n(h)}\right] \big) = \varepsilon. \tag{6}$$

Also,

$$\begin{aligned} d\big(&\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2n(h)}, l_{2n(h)}\right] \big) \\ &\leq d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2m(h)+1}, l_{2m(h)+1}\right] \big) \\ &\quad + d\big(\left[k_{2m(h)+1}, l_{2m(h)+1}\right], \left[k_{2n(h)+1}, l_{2n(h)+1}\right] \big) \\ &\quad + d\big(\left[k_{2n(h)+1}, l_{2n(h)+1}\right], \left[k_{2n(h)}, l_{2n(h)}\right] \big), \end{aligned} \tag{7}$$

and

$$\begin{aligned} d\big(&\left[k_{2m(h)+1}, l_{2m(h)+1}\right], \left[k_{2n(h)+1}, l_{2n(h)+1}\right] \big) \\ &\leq d\big(\left[k_{2m(h)+1}, l_{2m(h)+1}\right], \left[k_{2m(h)}, l_{2m(h)}\right] \big) \\ &\quad + d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2n(h)}, l_{2n(h)}\right] \big) \\ &\quad + d\big(\left[k_{2n(h)}, l_{2n(h)}\right], \left[k_{2n(h)+1}, l_{2n(h)+1}\right] \big). \end{aligned} \tag{8}$$

Taking $h \to \infty$ in the inequalities (7)-(8) and using (5)-(6), we obtain

$$\lim_{h\to\infty} d\big(\left[k_{2m(h)+1}, l_{2m(h)+1}\right], \left[k_{2n(h)+1}, l_{2n(h)+1}\right] \big) = \varepsilon. \tag{9}$$

Moreover,

$$\begin{aligned} d\big(&\left[k_{2n(h)+2}, l_{2n(h)+2}\right], \left[k_{2m(h)+1}, l_{2m(h)+1}\right] \big) \\ &\leq d\big(\left[k_{2n(h)+2}, l_{2n(h)+2}\right], \left[k_{2n(h)+1}, l_{2n(h)+1}\right] \big) \\ &\quad + d\big(\left[k_{2n(h)+1}, l_{2n(h)+1}\right], \left[k_{2m(h)+1}, l_{2m(h)+1}\right] \big), \end{aligned} \tag{10}$$

and

$$\begin{aligned} d\big(&\left[k_{2n(h)+1}, l_{2n(h)+1}\right], \left[k_{2m(h)+1}, l_{2m(h)+1}\right] \big) \\ &\leq d\big(\left[k_{2n(h)+1}, l_{2n(h)+1}\right], \left[k_{2n(h)+2}, l_{2n(h)+2}\right] \big) \\ &\quad + d\big(\left[k_{2n(h)+2}, l_{2n(h)+2}\right], \left[k_{2m(h)+1}, l_{2m(h)+1}\right] \big). \end{aligned} \tag{11}$$

Also,

$$\begin{aligned} d\big(\,[k_{2m(h)}, l_{2m(h)}]\,, [k_{2n(h)+2}, l_{2n(h)+2}]\,\big) &\le d\big(\,[k_{2m(h)}, l_{2m(h)}]\,, [k_{2m(h)+1}, l_{2m(h)+1}]\,\big) \\ &+ d\big(\,[k_{2m(h)+1}, l_{2m(h)+1}]\,, [k_{2n(h)+1}, l_{2n(h)+1}]\,\big) \\ &+ d\big(\,[k_{2n(h)+1}, l_{2n(h)+1}]\,, [k_{2n(h)+2}, l_{2n(h)+2}]\,\big), \end{aligned} \tag{12}$$

and

$$\begin{aligned} d\big(\,[k_{2m(h)+1}, l_{2m(h)+1}]\,, [k_{2n(h)+1}, l_{2n(h)+1}]\,\big) &\le d\big(\,[k_{2m(h)+1}, l_{2m(h)+1}]\,, [k_{2m(h)}, l_{2m(h)}]\,\big) \\ &+ d\big(\,[k_{2m(h)}, l_{2m(h)}]\,, [k_{2n(h)+2}, l_{2n(h)+2}]\,\big) \\ &+ d\big(\,[k_{2n(h)+2}, l_{2n(h)+2}]\,, [k_{2n(h)+1}, l_{2n(h)+1}]\,\big). \end{aligned} \tag{13}$$

Furthermore,

$$\begin{aligned} d\big(\,[k_{2m(h)}, l_{2m(h)}]\,, [k_{2n(h)+1}, l_{2n(h)+1}]\,\big) &\le d\big(\,[k_{2m(h)}, l_{2m(h)}]\,, [k_{2n(h)}, l_{2n(h)}]\,\big) \\ &+ d\big(\,[k_{2n(h)}, l_{2n(h)}]\,, [k_{2n(h)+1}, l_{2n(h)+1}]\,\big), \end{aligned} \tag{14}$$

and

$$\begin{aligned} d\big(\,[k_{2m(h)}, l_{2m(h)}]\,, [k_{2n(h)}, l_{2n(h)}]\,\big) &\le d\big(\,[k_{2m(h)}, l_{2m(h)}]\,, [k_{2n(h)+1}, l_{2n(h)+1}]\,\big) \\ &+ d\big(\,[k_{2n(h)+1}, l_{2n(h)+1}]\,, [k_{2n(h)}, l_{2n(h)}]\,\big). \end{aligned} \tag{15}$$

Letting $h \to \infty$ in the inequalities (10)-(15) and using (5)-(6) and (9), we have

$$\lim_{h\to\infty} d\big(\,[k_{2n(h)+2}, l_{2n(h)+2}]\,, [k_{2m(h)+1}, l_{2m(h)+1}]\,\big) = \varepsilon, \tag{16}$$

$$\lim_{h\to\infty} d\big(\,[k_{2m(h)}, l_{2m(h)}]\,, [k_{2n(h)+2}, l_{2n(h)+2}]\,\big) = \varepsilon, \tag{17}$$

$$\lim_{h\to\infty} d\big(\,[k_{2m(h)}, l_{2m(h)}]\,, [k_{2n(h)+1}, l_{2n(h)+1}]\,\big) = \varepsilon. \tag{18}$$

By putting $[k,l] = \left[k_{2n(h)+1}, l_{2n(h)+1}\right]$ and $[u,v] = \left[k_{2m(h)}, l_{2m(h)}\right]$ in (2), we obtain

$$\begin{aligned}
&d\big(\left[k_{2n(h)+2}, l_{2n(h)+2}\right], \left[k_{2m(h)+1}, l_{2m(h)+1}\right]\big) \\
&\quad = d\big(T\left[k_{2n(h)+1}, l_{2n(h)+1}\right], S\left[k_{2m(h)}, l_{2m(h)}\right]\big) \\
&\quad \leq \max\Big\{d\big(\left[k_{2n(h)+1}, l_{2n(h)+1}\right], \left[k_{2m(h)}, l_{2m(h)}\right]\big), \\
&\qquad d\big(\left[k_{2n(h)+1}, l_{2n(h)+1}\right], \left[k_{2n(h)+2}, l_{2n(h)+2}\right]\big), \\
&\qquad d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2m(h)+1}, l_{2m(h)+1}\right]\big), \\
&\qquad \frac{1}{2}\Big[d\big(\left[k_{2n(h)+1}, l_{2n(h)+1}\right], \left[k_{2m(h)+1}, l_{2m(h)+1}\right]\big) \\
&\qquad\quad + d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2n(h)+2}, l_{2n(h)+2}\right]\big)\Big]\Big\} \\
&\quad - \phi\Big(\max\Big\{d\big(\left[k_{2n(h)+1}, l_{2n(h)+1}\right], \left[k_{2m(h)}, l_{2m(h)}\right]\big), \\
&\qquad d\big(\left[k_{2n(h)+1}, l_{2n(h)+1}\right], \left[k_{2n(h)+2}, l_{2n(h)+2}\right]\big), \\
&\qquad d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2m(h)+1}, l_{2m(h)+1}\right]\big) \\
&\qquad \frac{1}{2}\Big[d\big(\left[k_{2n(h)+1}, l_{2n(h)+1}\right], \left[k_{2m(h)+1}, l_{2m(h)+1}\right]\big) \\
&\qquad\quad + d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2n(h)+2}, l_{2n(h)+2}\right]\big)\Big]\Big\}.
\end{aligned}$$

Taking $h \to \infty$ in the above inequality, making use of (5), (9) and (16)-(18) and using the continuity of ϕ, we have

$$\varepsilon \leq \varepsilon - \phi(\varepsilon),$$

which contradicts $\varepsilon > 0$. Thus, $\left\{ [k_{2n}, l_{2n}] \right\}_{n=1}^{\infty}$ is a Cauchy sequence in MIS $(\mathcal{I}, d)$. Hence according to (5), $\left\{ [k_n, l_n] \right\}_{n=1}^{\infty}$ is a Cauchy sequence in MIS $(\mathcal{I}, d)$. Therefore, due to the completeness of $\mathcal{I}$, there is $[k,l] \in \mathcal{I}$ such that

$$d([k_n, l_n], [k,l]) \to 0. \tag{19}$$

Now, we prove that each point $\left[\bar{k}, \bar{l}\right] \in \left\langle [k,l] \right\rangle$ is a common near-fixed point for T and S. Due to $\left[\bar{k}, \bar{l}\right] \in \left\langle [k,l] \right\rangle$, we have

$$\left[\bar{k}, \bar{l}\right] \oplus \omega_1 = [k,l] \oplus \omega_2 \quad \text{for some } \omega_1, \omega_2 \in \Omega. \tag{20}$$

Since the null equality holds for d, then according to (19), (20) and Proposition 3.3, we have

$$\lim_{n\to\infty} d([k_n, l_n], [\bar{k}, \bar{l}]) = 0, \text{ for any } [\bar{k}, \bar{l}] \in \langle [k, l] \rangle. \tag{21}$$

Therefore, using (2) and the triangle inequality, we obtain

$$\begin{aligned} d([\bar{k}, \bar{l}], S[\bar{k}, \bar{l}]) &\leq d([\bar{k}, \bar{l}], [k_{2n}, l_{2n}]) + d([k_{2n}, l_{2n}], S[\bar{k}, \bar{l}]) \\ &= d([\bar{k}, \bar{l}], [k_{2n}, l_{2n}]) + d(T[k_{2n-1}, l_{2n-1}], S[\bar{k}, \bar{l}]) \\ &= d([\bar{k}, \bar{l}], [k_{2n}, l_{2n}]) + M([k_{2n-1}, l_{2n-1}], [\bar{k}, \bar{l}]) \\ &\quad - \phi\big(M([k_{2n-1}, l_{2n-1}], [\bar{k}, \bar{l}])\big), \end{aligned} \tag{22}$$

where

$$\begin{aligned} &M([k_{2n-1}, l_{2n-1}], [\bar{k}, \bar{l}]) \\ &= \max\Big\{d([k_{2n-1}, l_{2n-1}], [\bar{k}, \bar{l}]), d([k_{2n-1}, l_{2n-1}], T[k_{2n-1}, l_{2n-1}]), \\ &d([\bar{k}, \bar{l}], S[\bar{k}, \bar{l}]), \frac{d([k_{2n-1}, l_{2n-1}], S[\bar{k}, \bar{l}]) + d([\bar{k}, \bar{l}], T[k_{2n-1}, l_{2n-1}])}{2}\Big\} \\ &\leq \max\Big\{d([k_{2n-1}, l_{2n-1}], [\bar{k}, \bar{l}]), \\ &\quad d([k_{2n-1}, l_{2n-1}], [k_{2n}, l_{2n}]), d([\bar{k}, \bar{l}], S[\bar{k}, \bar{l}]), \\ &\quad \frac{d([k_{2n-1}, l_{2n-1}], [\bar{k}, \bar{l}]) + d([\bar{k}, \bar{l}], S[\bar{k}, \bar{l}]) + d([\bar{k}, \bar{l}], [k_{2n}, l_{2n}])}{2}\Big\}\Big). \end{aligned} \tag{23}$$

On the other hand, we have

$$d([\bar{k}, \bar{l}], S[\bar{k}, \bar{l}]) \leq M([k_{2n-1}, l_{2n-1}], [\bar{k}, \bar{l}]). \tag{24}$$

Letting $n \to \infty$ in the inequalities (23)-(24) and using (5) and (21), we have

$$\lim_{n\to\infty} M([k_{2n-1}, l_{2n-1}], [\bar{k}, \bar{l}]) = d([\bar{k}, \bar{l}], S[\bar{k}, \bar{l}]). \tag{25}$$

Taking $n \to \infty$ in (22) and using (21), (25) and the continuity of ϕ, we have

$$d([\bar{k}, \bar{l}], S[\bar{k}, \bar{l}]) \leq d([\bar{k}, \bar{l}], S[\bar{k}, \bar{l}]) - \phi\big(d([\bar{k}, \bar{l}], S[\bar{k}, \bar{l}])\big),$$

which implies that $\phi\big(d(\left[\bar{k},\bar{l}\right], S\left[\bar{k},\bar{l}\right])\big) \le 0$. Hence, $d(\left[\bar{k},\bar{l}\right], S\left[\bar{k},\bar{l}\right]) = 0$. Thus $S\left[\bar{k},\bar{l}\right] \stackrel{\Omega}{=} \left[\bar{k},\bar{l}\right]$ for any point $\left[\bar{k},\bar{l}\right] \in \big\langle [k,l] \big\rangle$. Similarly, $T\left[\bar{k},\bar{l}\right] \stackrel{\Omega}{=} \left[\bar{k},\bar{l}\right]$ for any point $\left[\bar{k},\bar{l}\right] \in \big\langle [k,l] \big\rangle$. Hence due to Proposition 2.2, we have $\left[\bar{k},\bar{l}\right] \stackrel{\Omega}{=} T\left[\bar{k},\bar{l}\right] \stackrel{\Omega}{=} S\left[\bar{k},\bar{l}\right]$ for any point $\left[\bar{k},\bar{l}\right] \in \big\langle [k,l] \big\rangle$.
Now, suppose that there is another common near-fixed point $[\tilde{u},\tilde{v}]$ for S and T such that $[\tilde{u},\tilde{v}] \notin \big\langle [k,l] \big\rangle$, i.e., $[\tilde{u},\tilde{v}] \stackrel{\Omega}{=} T[\tilde{u},\tilde{v}] \stackrel{\Omega}{=} S[\tilde{u},\tilde{v}]$, $[k,l] \stackrel{\Omega}{=} T[k,l] \stackrel{\Omega}{=} S[k,l]$ and $[k,l] \stackrel{\Omega}{\neq} [\tilde{u},\tilde{v}]$. Then

$$\begin{aligned} &[k,l] \oplus \omega_1 = T[k,l] \oplus \omega_2 \quad \text{and} \quad [\tilde{u},\tilde{v}] \oplus \omega_3 = T[\tilde{u},\tilde{v}] \oplus \omega_4 \\ &[\tilde{u},\tilde{v}] \oplus \omega_5 = S[\tilde{u},\tilde{v}] \oplus \omega_6 \quad \text{and} \quad [k,l] \oplus \omega_7 = S[k,l] \oplus \omega_8 \end{aligned} \tag{26}$$

for some $\omega_i \in \Omega$, $i = 1,\ldots,8$. Using (2), (26) and the null equality, we have

$$\begin{aligned} d([k,l],[\tilde{u},\tilde{v}]) &= d([k,l] \oplus \omega_1, [\tilde{u},\tilde{v}] \oplus \omega_5) = d(T[k,l] \oplus \omega_2, S[\tilde{u},\tilde{v}] \oplus \omega_6) \\ &= d(T[k,l], S[\tilde{u},\tilde{v}]) \\ &\le M\big([k,l],[\tilde{u},\tilde{v}]\big) - \phi\big(M\big([k,l],[\tilde{u},\tilde{v}]\big)\big), \end{aligned} \tag{27}$$

where

$$\begin{aligned} M([k,l],[\tilde{u},\tilde{v}]) = \max\Big\{&d([k,l],[\tilde{u},\tilde{v}]), d([k,l],T[k,l]), d([\tilde{u},\tilde{v}],S[\tilde{u},\tilde{v}]), \\ &\frac{d([k,l],S[\tilde{u},\tilde{v}]) + d([\tilde{u},\tilde{v}],T[k,l])}{2}\Big\} = d([k,l],[\tilde{u},\tilde{v}]). \end{aligned} \tag{28}$$

Note that according to the null equality, we have

$$\begin{aligned} d([k,l],S[\tilde{u},\tilde{v}]) &= d([k,l],S[\tilde{u},\tilde{v}] \oplus \omega_6) = d([k,l],[\tilde{u},\tilde{v}] \oplus \omega_5) \\ &= d([k,l],[\tilde{u},\tilde{v}]), \end{aligned}$$

$$\begin{aligned} d([\tilde{u},\tilde{v}],T[k,l]) &= d([\tilde{u},\tilde{v}],T[k,l] \oplus \omega_2) = d([\tilde{u},\tilde{v}],[k,l] \oplus \omega_1) \\ &= d([\tilde{u},\tilde{v}],[k,l]). \end{aligned}$$

Therefore, from (27) and (28), we obtain

$$d([k,l],[\tilde{u},\tilde{v}]) \le d([k,l],[\tilde{u},\tilde{v}]) - \phi\big(d([k,l],[\tilde{u},\tilde{v}])\big),$$

which implies that $\phi\big(d([k,l],[\tilde{u},\tilde{v}])\big) \leq 0$. Hence $d([k,l],[\tilde{u},\tilde{v}]) = 0$, i.e., $[k,l] \overset{\Omega}{=} [\tilde{u},\tilde{v}]$, which contradicts $[\tilde{u},\tilde{v}] \notin \big\langle [k,l] \big\rangle$. Thus, any $[\tilde{u},\tilde{v}] \notin \big\langle [k,l] \big\rangle$ cannot be a common near-fixed point for T and S. In fact, if $[\tilde{u},\tilde{v}]$ is a common near-fixed point of T and S, then $[\tilde{u},\tilde{v}] \in \big\langle [k,l] \big\rangle$. □

Corollary 4.7. *Let the mapping $T : (\mathcal{I}, d) \to (\mathcal{I}, d)$ be a weak ϕ-contraction (by Definition 4.4) on $\mathcal{I}$ in CMIS $(\mathcal{I}, d)$ such that the null equality holds for d. Then T has a near-fixed point $[k,l] \in \mathcal{I}$ satisfying $T[k,l] \overset{\Omega}{=} [k,l]$. Moreover, a unique equivalence class $\big\langle [k,l] \big\rangle$ exists for T such that if $[k,l]$ and $\big[\bar{k},\bar{l}\big]$ are the near-fixed points of T, then $[k,l] \overset{\Omega}{=} \big[\bar{k},\bar{l}\big]$.*

Example 4.8. Let $T, S : \mathcal{I} \to \mathcal{I}$ and $\phi : [0,\infty) \to [0,\infty)$ in CMIS $(\mathcal{I}, d)$ be defined by $T[k,l] = \big[-1 + \frac{3}{8}k, 1 + \frac{3}{8}l\big]$, $S[k,l] = \big[-2 + \frac{3}{8}k, 2 + \frac{3}{8}l\big]$ and $\phi(t) = \dfrac{t}{8}$, respectively. Define $d : \mathcal{I} \times \mathcal{I} \to \mathbb{R}^+$ by $d([k,l],[u,v]) = |(k+l) - (u+v)|$ for all $[k,l],[u,v] \in \mathcal{I}$. Then by (2), we have

$$\begin{aligned}
d\big(T[k,l], S[u,v]\big) &= d\bigg(\Big[-1 + \frac{3}{8}k, 1 + \frac{3}{8}l\Big], \Big[-2 + \frac{3}{8}u, 2 + \frac{3}{8}v\Big]\bigg) \\
&= \frac{3}{8}\big|(k+l) - (u+v)\big| \leq \frac{7}{8}\big|(k+l) - (u+v)\big| = \frac{7}{8}d([k,l],[u,v]) \\
&\leq \frac{7}{8}M([k,l],[u,v]) = M([k,l],[u,v]) - \frac{1}{8}M([k,l],[u,v]) \\
&= M([k,l],[u,v]) - \phi\big(M([k,l],[u,v])\big).
\end{aligned}$$

Thus, this example satisfies all conditions of Theorem 4.6. Hence, T and S have a unique equivalence class of common near-fixed points $\big\langle [-1,1] \big\rangle$ in $\mathcal{I}$.

Definition 4.9. *A pair of mappings $S, T : (\mathcal{I}, \|\cdot\|) \to (\mathcal{I}, \|\cdot\|)$ is named generalized weak ϕ-contractions on $\mathcal{I}$ if it satisfies the following condition for all $[k,l],[u,v] \in \mathcal{I}$:*

$$\|T[k,l] \ominus S[u,v]\| \leq M\big([k,l],[u,v]\big) - \phi\big(M\big([k,l],[u,v]\big)\big), \qquad (29)$$

where

$$\begin{aligned}
M([k,l],[u,v]) = \max\Big\{&\|[k,l] \ominus [u,v]\|, \|[k,l] \ominus T[k,l]\|, \|[u,v] \ominus S[u,v]\|, \\
&\frac{\|[k,l] \ominus S[u,v]\| + \|[u,v] \ominus T[k,l]\|}{2}\Big\},
\end{aligned}$$

and a function $\phi : [0, \infty) \to [0, \infty)$ is continuous and non-decreasing with $\phi(t) > 0$ for $t \in (0, \infty)$ and $\phi(0) = 0$.

Corollary 4.10. *Let the pair of mappings $T, S : (\mathcal{I}, \|\cdot\|) \to (\mathcal{I}, \|\cdot\|)$ be generalized weak ϕ-contractions on $\mathcal{I}$ in BIS $(\mathcal{I}, \|\cdot\|)$ such that the null equality and null condition hold for $\|\cdot\|$. Then T and S have a common near-fixed point $[k, l] \in \mathcal{I}$ satisfying $T[k, l] \stackrel{\Omega}{=} [k, l] \stackrel{\Omega}{=} S[k, l]$. Moreover, a unique equivalence class $\langle [k, l] \rangle$ exists for T and S such that if $[\bar{k}, \bar{l}]$ is another common near-fixed point of T and S, then $\langle [k, l] \rangle = \langle [\bar{k}, \bar{l}] \rangle$.*

Example 4.11. Suppose that $T, S : \mathcal{I} \to \mathcal{I}$ and $\phi : [0, \infty) \to [0, \infty)$ in BIS $(\mathcal{I}, \|\cdot\|)$ are defined by $T[k, l] = \left[-1 + \frac{2}{9}k, 1 + \frac{2}{9}l\right]$, $S[k, l] = \left[-2 + \frac{2}{9}k, 2 + \frac{2}{9}l\right]$ and $\phi(t) = \dfrac{t}{9}$, respectively. Define $\|\cdot\| : \mathcal{I} \to \mathbb{R}^+$ by $\|[k, l]\| = |k + l|$. Note that for all $[k, l], [u, v] \in \mathcal{I}$, we have

$$\begin{aligned}\|[k, l] \ominus [u, v]\| &= \|[k, l] \oplus [-v, -u]\| = \|[k - v, l - u]\| \\ &= |(k - v) + (l - u)| = |(k + l) - (u + v)|.\end{aligned}$$

Then for all $[k, l], [u, v] \in \mathcal{I}$ by (29), we have

$$\begin{aligned}&\|T[k, l] \ominus S[u, v]\| = \left\| \left[-1 + \frac{2k}{9}, 1 + \frac{2l}{9}\right] \ominus \left[-2 + \frac{2u}{9}, 2 + \frac{2v}{9}\right] \right\| \\ &= \frac{2}{9}\big|(k + l) - (u + v)\big| \leq \frac{8}{9}\big|(k + l) - (u + v)\big| = \frac{8}{9}\|[k, l] \ominus [u, v]\| \\ &\leq \frac{8}{9}M([k, l], [u, v]) = M([k, l], [u, v]) - \frac{1}{9}M([k, l], [u, v]) \\ &= M([k, l], [u, v]) - \phi\big(M([k, l], [u, v])\big).\end{aligned}$$

Thus, this example satisfies all conditions of Corollary 4.10. Hence, T and S have a unique equivalence class of common near-fixed points $\langle [-2, 2] \rangle$ in $\mathcal{I}$.

Theorem 4.12. *Suppose that $(\mathcal{I}, d)$ is a CMIS and the null equality holds for d. Assume that $T, S : (\mathcal{I}, d) \to (\mathcal{I}, d)$ are self-mappings of $\mathcal{I}$ such that for all $[k, l], [u, v] \in \mathcal{I}$,*

$$d(S[k, l], T[u, v]) \leq M([k, l], [u, v]) - \phi\big(M([k, l], [u, v])\big), \tag{30}$$

where a function $\phi : [0,\infty) \to [0,\infty)$ is continuous and nondecreasing with the property $\phi(t) > 0$ for $t \in (0,\infty)$ and $\phi(0) = 0$, and

$$M([k,l],[u,v]) = \alpha d([k,l],[u,v]) + \beta\big[d([k,l],S[k,l]) + d([u,v],T[u,v])\big] + \gamma\big[d([k,l],T[u,v]) + d([u,v],S[k,l])\big], \tag{31}$$

with $\alpha,\beta > 0, \gamma \geq 0$ and $\alpha + 2\beta + 2\gamma \leq 1$. Then a common near-fixed point $[k,l] \in \mathcal{I}$ exists for T and S satisfying $T[k,l] \overset{\Omega}{=} [k,l] \overset{\Omega}{=} S[k,l]$. Moreover, a unique equivalence class $\langle [k,l] \rangle$ exists for T and S such that if $[\bar{k},\bar{l}]$ is another common near-fixed point of T and S, then $\langle [k,l] \rangle = \langle [\bar{k},\bar{l}] \rangle$ and $[k,l] \overset{\Omega}{=} [\bar{k},\bar{l}]$. In addition, any near-fixed point of T is a near-fixed point of S and conversely.

Proof. It is clear that $[k,l] \overset{\Omega}{=} [u,v]$ is a common near-fixed point of T and S if and only if $M\big([k,l],[u,v]\big) = 0$. Indeed if $[k,l] \overset{\Omega}{=} [u,v]$ is a common near-fixed point of S and T, then by using Proposition 2.2, we have $T[k,l] \overset{\Omega}{=} [k,l] \overset{\Omega}{=} S[k,l] \overset{\Omega}{=} T[u,v] \overset{\Omega}{=} [u,v] \overset{\Omega}{=} S[u,v]$ and

$$M\big([k,l],[u,v]\big) = \alpha d([k,l],[u,v]) + \beta\big[d([k,l],S[k,l]) + d([u,v],T[u,v])\big] + \gamma\big[d([k,l],T[u,v]) + d([u,v],S[k,l])\big] = 0$$

Note that according to the null equality, we have

$d([u,v],S[k,l]) = d([k,l],S[k,l])$ and $d([k,l],T[u,v]) = d([u,v],T[u,v])$.

Now, suppose that $M([k,l],[u,v]) = 0$. Then, due to (31) and $\alpha,\beta > 0$, we have $d([k,l],[u,v]) = 0$, $d([k,l],T[k,l]) = 0$ and $d([u,v],S[u,v])] = 0$, i.e.,

$$T[k,l] \overset{\Omega}{=} [k,l] \overset{\Omega}{=} S[k,l] \overset{\Omega}{=} T[u,v] \overset{\Omega}{=} [u,v] \overset{\Omega}{=} S[u,v].$$

Let $[k_0,l_0] \in \mathcal{I}$ be arbitrary. Then, inductively select a sequence $\{ [k_n,l_n] \}_{n=1}^{\infty}$ in $\mathcal{I}$ such that

$$[k_{2n+2},l_{2n+2}] = T[k_{2n+1},l_{2n+1}] \quad \text{and} \quad [k_{2n+1},l_{2n+1}] = S[k_{2n},l_{2n}],$$

for all $n \geq 0$. According to the above content, note that if $[k_{n+1},l_{n+1}] = [k_n,l_n]$ for any $n \geq 0$, then S and T have a common near-fixed point.

Thus, assume that $d([k_{n+1}, l_{n+1}], [k_n, l_n]) \neq 0$ for all $n \geq 0$. Hence, we have

$$M\big([k_n, l_n], [k_{n+1}, l_{n+1}]\big) > 0, \tag{32}$$

for all $n \in \mathbb{N}$. Therefore, from (30), we have

$$\begin{aligned} d\big([k_{2n+1}, l_{2n+1}], [k_{2n+2}, l_{2n+2}]\big) &= d(S[k_{2n}, l_{2n}], T[k_{2n+1}, l_{2n+1}]) \\ &\leq M\big([k_{2n}, l_{2n}], [k_{2n+1}, l_{2n+1}]\big) - \phi\big(M\big([k_{2n}, l_{2n}], [k_{2n+1}, l_{2n+1}]\big)\big), \end{aligned} \tag{33}$$

where due to (31), we obtain

$$\begin{aligned} M\big([k_{2n}, l_{2n}], [k_{2n+1}, l_{2n+1}]\big) &= \alpha d\big([k_{2n}, l_{2n}], [k_{2n+1}, l_{2n+1}]\big) \\ &+ \beta\big[d\big([k_{2n}, l_{2n}], [k_{2n+1}, l_{2n+1}]\big) + d\big([k_{2n+1}, l_{2n+1}], [k_{2n+2}, l_{2n+2}]\big)\big] \\ &+ \gamma\big[d\big([k_{2n}, l_{2n}], [k_{2n+2}, l_{2n+2}]\big) + d\big([k_{2n+1}, l_{2n+1}], [k_{2n+1}, l_{2n+1}]\big)\big]. \end{aligned}$$

Since

$$\begin{aligned} &d([k_{2n}, l_{2n}], [k_{2n+2}, l_{2n+2}]) \\ &\quad \leq d([k_{2n}, l_{2n}], [k_{2n+1}, l_{2n+1}]) + d([k_{2n+1}, l_{2n+1}], [k_{2n+2}, l_{2n+2}]), \end{aligned}$$

it results that

$$\begin{aligned} M\big([k_{2n}, l_{2n}], [k_{2n+1}, l_{2n+1}]\big) &\leq \alpha d\big([k_{2n}, l_{2n}], [k_{2n+1}, l_{2n+1}]\big) \\ &+ \beta\big[d\big([k_{2n}, l_{2n}], [k_{2n+1}, l_{2n+1}]\big) + d\big([k_{2n+1}, l_{2n+1}], [k_{2n+2}, l_{2n+2}]\big)\big] \\ &+ \gamma\big[d\big([k_{2n}, l_{2n}], [k_{2n+1}, l_{2n+1}]\big) + d\big([k_{2n+1}, l_{2n+1}], [k_{2n+2}, l_{2n+2}]\big)\big] \\ &= (\alpha + \beta + \gamma) d\big([k_{2n}, l_{2n}], [k_{2n+1}, l_{2n+1}]\big) \\ &+ (\beta + \gamma) d\big([k_{2n+1}, l_{2n+1}], [k_{2n+2}, l_{2n+2}]\big). \end{aligned} \tag{34}$$

Suppose that

$$d\big([k_{2n+1}, l_{2n+1}], [k_{2n+2}, l_{2n+2}]\big) > d\big([k_{2n}, l_{2n}], [k_{2n+1}, l_{2n+1}]\big),$$

for some $n \in \mathbb{N}$. Then from (33) and (34), we obtain

$$\begin{aligned}
&d\big([k_{2n+1}, l_{2n+1}] , [k_{2n+2}, l_{2n+2}] \big)\\
&\quad \leq M\big([k_{2n}, l_{2n}] , [k_{2n+1}, l_{2n+1}] \big) - \phi\big(M\big([k_{2n}, l_{2n}] , [k_{2n+1}, l_{2n+1}] \big)\big)\\
&\quad = (\alpha+\beta+\gamma) d\big([k_{2n}, l_{2n}] , [k_{2n+1}, l_{2n+1}] \big)\\
&\quad\ + (\beta+\gamma) d\big([k_{2n+1}, l_{2n+1}] , [k_{2n+2}, l_{2n+2}] \big)\\
&\quad\ - \phi\big(M\big([k_{2n}, l_{2n}] , [k_{2n+1}, l_{2n+1}] \big)\big)\\
&\quad \leq (\alpha+2\beta+2\gamma) d\big([k_{2n+1}, l_{2n+1}] , [k_{2n+2}, l_{2n+2}] \big)\\
&\quad\ - \phi\big(M\big([k_{2n}, l_{2n}] , [k_{2n+1}, l_{2n+1}] \big)\big)\\
&\quad \leq d\big([k_{2n+1}, l_{2n+1}] , [k_{2n+2}, l_{2n+2}] \big)\\
&\quad\ - \phi\big(M\big([k_{2n}, l_{2n}] , [k_{2n+1}, l_{2n+1}] \big)\big) \text{ (since } (\alpha+2\beta+2\gamma) \leq 1),
\end{aligned}$$

which results that $\phi\big(M\big([k_{2n}, l_{2n}] , [k_{2n+1}, l_{2n+1}] \big)\big) \leq 0$. Thus we have $M\big([k_{2n}, l_{2n}] , [k_{2n+1}, l_{2n+1}] \big) = 0$, which is a contradiction according to (32). Hence, $d\big([k_{2n+1}, l_{2n+1}] , [k_{2n+2}, l_{2n+2}] \big) \leq d\big([k_{2n}, l_{2n}] , [k_{2n+1}, l_{2n+1}] \big)$ for all $n \in \mathbb{N}$. Similarly, we also obtain $d\big([k_{2n+2}, l_{2n+2}] , [k_{2n+3}, l_{2n+3}] \big) \leq d\big([k_{2n+1}, l_{2n+1}] , [k_{2n+2}, l_{2n+2}] \big)$ for all $n \in \mathbb{N}$. So, $\big\{d([k_n, l_n] , [k_{n+1}, l_{n+1}])\big\}$ is a monotone decreasing sequence of non-negative real numbers. Therefore, there is an $r \geq 0$ such that

$$\lim_{n\to\infty} d([k_n, l_n] , [k_{n+1}, l_{n+1}]) = r. \tag{35}$$

Assume that $r > 0$. Thus, taking $n \to \infty$ in (34), we have

$$\lim_{n\to\infty} M\big([k_{2n}, l_{2n}] , [k_{2n+1}, l_{2n+1}] \big) \leq (\alpha+2\beta+2\gamma) r. \tag{36}$$

Therefore, according to the above facts and (33)-(34), we obtain

$$\begin{aligned}
&d\big([k_{2n+1}, l_{2n+1}] , [k_{2n+2}, l_{2n+2}] \big)\\
&\quad \leq M\big([k_{2n}, l_{2n}] , [k_{2n+1}, l_{2n+1}] \big) - \phi\big(M\big([k_{2n}, l_{2n}] , [k_{2n+1}, l_{2n+1}] \big)\big)\\
&\quad = (\alpha+\beta+\gamma) d\big([k_{2n}, l_{2n}] , [k_{2n+1}, l_{2n+1}] \big)\\
&\quad\ + (\beta+\gamma) d\big([k_{2n+1}, l_{2n+1}] , [k_{2n+2}, l_{2n+2}] \big)\\
&\quad\ - \phi\big(M\big([k_{2n}, l_{2n}] , [k_{2n+1}, l_{2n+1}] \big)\big)\\
&\quad \leq (\alpha+2\beta+2\gamma) d\big([k_{2n}, l_{2n}] , [k_{2n+1}, l_{2n+1}] \big)\\
&\quad\ - \phi\big(M\big([k_{2n}, l_{2n}] , [k_{2n+1}, l_{2n+1}] \big)\big),
\end{aligned} \tag{37}$$

Taking $n \to \infty$ in the inequality (37) and making use of (35) and the continuity of ϕ, we have

$$r \leq (\alpha + 2\beta + 2\gamma)r - \phi\big(\lim_{n\to\infty} M\big([k_{2n}, l_{2n}], [k_{2n+1}, l_{2n+1}] \big)\big),$$

since $\alpha + 2\beta + 2\gamma \leq 1$, it implies that

$$r \leq r - \phi\big(\lim_{n\to\infty} M\big([k_{2n}, l_{2n}], [k_{2n+1}, l_{2n+1}] \big)\big),$$

which implies that $\phi\big(\lim_{n\to\infty} M\big([k_{2n}, l_{2n}], [k_{2n+1}, l_{2n+1}] \big)\big) \leq 0$. Therefore, by the property of ϕ, we have $\lim_{n\to\infty} M\big([k_{2n}, l_{2n}], [k_{2n+1}, l_{2n+1}] \big) = 0$. Which is a contradiction, since $r, \alpha > 0$ and from (32) and (36), we have $0 < \lim_{n\to\infty} M\big([k_{2n}, l_{2n}], [k_{2n+1}, l_{2n+1}] \big) \leq (\alpha + 2\beta + 2\gamma)r$. Thus

$$\lim_{n\to\infty} d([k_n, l_n], [k_{n+1}, l_{n+1}]) = 0. \tag{38}$$

To show that $\{ [k_n, l_n] \}_{n=1}^{\infty}$ is a Cauchy sequence, we can proceed similar to the proof of Theorem 4.6. So, $\{ [k_n, l_n] \}_{n=1}^{\infty}$ is a Cauchy sequence in MIS $(\mathcal{I}, d)$. Therefore, due to the completeness of the $\mathcal{I}$, there is $[k, l] \in \mathcal{I}$ such that

$$d([k_n, l_n], [k, l]) \to 0. \tag{39}$$

Now, we prove that any point $\big[\bar{k}, \bar{l}\big] \in \big\langle [k, l] \big\rangle$ is a common near-fixed point for T and S. For this purpose, suppose that $d\big(\big[\bar{k}, \bar{l}\big], T\big[\bar{k}, \bar{l}\big] \big) > 0$. Due to $\big[\bar{k}, \bar{l}\big] \in \big\langle [k, l] \big\rangle$, we have

$$\big[\bar{k}, \bar{l}\big] \oplus \omega_1 = [k, l] \oplus \omega_2, \quad \text{for some } \omega_1, \omega_2 \in \Omega. \tag{40}$$

Since the null equality holds for d, according to (39), (40) and Proposition 3.3, we have

$$\lim_{n\to\infty} d([k_n, l_n], \big[\bar{k}, \bar{l}\big]) = 0, \ \text{ for any } \big[\bar{k}, \bar{l}\big] \in \big\langle [k, l] \big\rangle. \tag{41}$$

Therefore, by setting $[k, l] = [k_{2n}, l_{2n}]$ and $[u, v] = \big[\bar{k}, \bar{l}\big]$ in (30) and using the null equality and the triangle inequality, we obtain

$$\begin{aligned} d\big(\big[\bar{k}, \bar{l}\big], T\big[\bar{k}, \bar{l}\big] \big) &\leq d\big(\big[\bar{k}, \bar{l}\big], [k_{2n+1}, l_{2n+1}] \big) + d\big([k_{2n+1}, l_{2n+1}], T\big[\bar{k}, \bar{l}\big] \big) \\ &= d\big(\big[\bar{k}, \bar{l}\big], [k_{2n+1}, l_{2n+1}] \big) + d\big(S[k_{2n}, l_{2n}], T\big[\bar{k}, \bar{l}\big] \big) \\ &\leq d\big(\big[\bar{k}, \bar{l}\big], [k_{2n+1}, l_{2n+1}] \big) + M\big([k_{2n}, l_{2n}], \big[\bar{k}, \bar{l}\big] \big) \\ &\quad - \phi\big(M\big([k_{2n}, l_{2n}], \big[\bar{k}, \bar{l}\big] \big)\big), \end{aligned} \tag{42}$$

where according to (31), we have

$$\begin{aligned}
M\big(\left[k_{2n}, l_{2n}\right], \left[\bar{k}, \bar{l}\right]\big) &= \alpha d\big(\left[k_{2n}, l_{2n}\right], \left[\bar{k}, \bar{l}\right]\big) \\
&+ \beta\big[d\big(\left[k_{2n}, l_{2n}\right], \left[k_{2n+1}, l_{2n+1}\right]\big) + d\big(\left[\bar{k}, \bar{l}\right], T\left[\bar{k}, \bar{l}\right]\big)\big] \\
&+ \gamma\big[d\big(\left[k_{2n}, l_{2n}\right], T\left[\bar{k}, \bar{l}\right]\big) + d\big(\left[\bar{k}, \bar{l}\right], \left[k_{2n+1}, l_{2n+1}\right]\big)\big] \\
&\leq \alpha d\big(\left[k_{2n}, l_{2n}\right], \left[\bar{k}, \bar{l}\right]\big) \\
&+ \beta\big[d\big(\left[k_{2n}, l_{2n}\right], \left[k_{2n+1}, l_{2n+1}\right]\big) + d\big(\left[\bar{k}, \bar{l}\right], T\left[\bar{k}, \bar{l}\right]\big)\big] \\
&+ \gamma\big[d\big(\left[k_{2n}, l_{2n}\right], \left[\bar{k}, \bar{l}\right]\big) + d\big(\left[\bar{k}, \bar{l}\right], T\left[\bar{k}, \bar{l}\right]\big) \\
&+ d\big(\left[\bar{k}, \bar{l}\right], \left[k_{2n+1}, l_{2n+1}\right]\big)\big].
\end{aligned}$$

Taking $n \to \infty$ in the above inequality and using (38) and (41), we have

$$\lim_{n\to\infty} M\big(\left[k_n, l_n\right], \left[\bar{k}, \bar{l}\right]\big) \leq (\beta+\gamma) d(\left[\bar{k}, \bar{l}\right], T\left[\bar{k}, \bar{l}\right]). \tag{43}$$

Letting $n \to \infty$ in (42), making use of the continuity of ϕ and using (41) and (43), we obtain

$$d(\left[\bar{k}, \bar{l}\right], T\left[\bar{k}, \bar{l}\right]) \leq (\beta+\gamma) d(\left[\bar{k}, \bar{l}\right], T\left[\bar{k}, \bar{l}\right]) - \phi\big(\lim_{n\to\infty} M\big(\left[k_n, l_n\right], \left[\bar{k}, \bar{l}\right]\big)\big),$$

Due to $\beta + \gamma \leq 1$, it follows that

$$d(\left[\bar{k}, \bar{l}\right], T\left[\bar{k}, \bar{l}\right]) \leq d(\left[\bar{k}, \bar{l}\right], T\left[\bar{k}, \bar{l}\right]) - \phi\big(\lim_{n\to\infty} M\big(\left[k_n, l_n\right], \left[\bar{k}, \bar{l}\right]\big)\big),$$

which implies that $\phi\big(\lim_{n\to\infty} M\big(\left[k_n, l_n\right], \left[\bar{k}, \bar{l}\right]\big)\big) \leq 0$. So, by the property of ϕ, we have $\lim_{n\to\infty} M\big(\left[k_n, l_n\right], \left[\bar{k}, \bar{l}\right]\big) = 0$. Which is a contradiction, since $\beta > 0$ and $d\big(\left[\bar{k}, \bar{l}\right], T\left[\bar{k}, \bar{l}\right]\big) > 0$.
Therefore, $d(\left[\bar{k}, \bar{l}\right], T\left[\bar{k}, \bar{l}\right]) = 0$. Thus, $T\left[\bar{k}, \bar{l}\right] \stackrel{\Omega}{=} \left[\bar{k}, \bar{l}\right]$ for any point $\left[\bar{k}, \bar{l}\right] \in \big\langle \left[k, l\right] \big\rangle$. Similarly, $S\left[\bar{k}, \bar{l}\right] \stackrel{\Omega}{=} \left[\bar{k}, \bar{l}\right]$ for any $\left[\bar{k}, \bar{l}\right] \in \big\langle \left[k, l\right] \big\rangle$. Hence, according to Proposition 2.2, we have $T\left[\bar{k}, \bar{l}\right] \stackrel{\Omega}{=} \left[\bar{k}, \bar{l}\right] \stackrel{\Omega}{=} S\left[\bar{k}, \bar{l}\right]$ for any point $\left[\bar{k}, \bar{l}\right] \in \big\langle \left[k, l\right] \big\rangle$.
Now, assume that T and S have two common near-fixed points $[k, l]$ and $[\tilde{u}, \tilde{v}]$ such that $[k, l] \stackrel{\Omega}{\neq} [\tilde{u}, \tilde{v}]$, i.e., $[\tilde{u}, \tilde{v}] \stackrel{\Omega}{=} T[\tilde{u}, \tilde{v}] \stackrel{\Omega}{=} S[\tilde{u}, \tilde{v}]$, $[k, l] \stackrel{\Omega}{=} T[k, l] \stackrel{\Omega}{=} S[k, l]$ and $[\tilde{u}, \tilde{v}] \notin \big\langle [k, l] \big\rangle$. Then

$$[\tilde{u}, \tilde{v}] \oplus \omega_1 = T[\tilde{u}, \tilde{v}] \oplus \omega_2 \quad \text{and} \quad [k, l] \oplus \omega_5 = T[k, l] \oplus \omega_6 \tag{44}$$

$$[k, l] \oplus \omega_3 = S[k, l] \oplus \omega_4 \quad \text{and} \quad [\tilde{u}, \tilde{v}] \oplus \omega_7 = S[\tilde{u}, \tilde{v}] \oplus \omega_8 \tag{45}$$

for some $\omega_i \in \Omega$, $i = 1, \ldots, 8$. Note that $d([\tilde{u}, \tilde{v}], T[\tilde{u}, \tilde{v}]) = 0$ and $d([k, l], S[k, l]) = 0$.
Then using (30), (44) and (45) and making use of the null equality and the triangle inequality, we have

$$\begin{aligned} d\big([k,l],[\tilde{u},\tilde{v}]\big) &= d\big([k,l]\oplus\omega_3,[\tilde{u},\tilde{v}]\oplus\omega_1\big) \\ &= d\big(S[k,l]\oplus\omega_4,T[\tilde{u},\tilde{v}]\oplus\omega_2\big) = d\big(S[k,l],T[\tilde{u},\tilde{v}]\big) \\ &\leq M\big([k,l],[\tilde{u},\tilde{v}]\big) - \phi\big(M\big([k,l],[\tilde{u},\tilde{v}]\big)\big), \end{aligned}$$

where due to (31), we obtain

$$\begin{aligned} M([k,l],[\tilde{u},\tilde{v}]) &= \alpha d([k,l],[\tilde{u},\tilde{v}]) + \beta\big[d([k,l],S[k,l]) + d([\tilde{u},\tilde{v}],T[\tilde{u},\tilde{v}])\big] \\ &\quad + \gamma\big[d([k,l],T[\tilde{u},\tilde{v}]) + d([\tilde{u},\tilde{v}],S[k,l])\big] \\ &\leq \alpha d([k,l],[\tilde{u},\tilde{v}]) + \beta\big[d([k,l],S[k,l]) + d([\tilde{u},\tilde{v}],T[\tilde{u},\tilde{v}])\big] \\ &\quad + \gamma\big[d([k,l],[\tilde{u},\tilde{v}]) + d([\tilde{u},\tilde{v}],T[\tilde{u},\tilde{v}]) \\ &\qquad + d([k,l],[\tilde{u},\tilde{v}]) + d([k,l],S[k,l])\big] \\ &= (\alpha+2\gamma)d([k,l],[\tilde{u},\tilde{v}]). \end{aligned} \tag{46}$$

Therefore

$$d\big([k,l],[\tilde{u},\tilde{v}]\big) \leq (\alpha+2\gamma)d\big([k,l],[\tilde{u},\tilde{v}]\big) - \phi\big(M([k,l],[\tilde{u},\tilde{v}])\big),$$

Due to $\alpha + 2\gamma \leq 1$, it follows that

$$d\big([k,l],[\tilde{u},\tilde{v}]\big) \leq d\big([k,l],[\tilde{u},\tilde{v}]\big) - \phi\big(M([k,l],[\tilde{u},\tilde{v}])\big),$$

which implies that $\phi\big(M([k,l],[\tilde{u},\tilde{v}])\big) \leq 0$. So, by the property of ϕ, we have $M([k,l],[\tilde{u},\tilde{v}]) = 0$. Which is a contradiction, since $\alpha > 0$ and $d([k,l],[\tilde{u},\tilde{v}]) > 0$. Hence $d([k,l],[\tilde{u},\tilde{v}]) = 0$, i.e., $[k,l] \stackrel{\Omega}{=} [\tilde{u},\tilde{v}]$. Thus, any $[\tilde{u},\tilde{v}] \notin \big\langle [k,l] \big\rangle$ cannot be a common near-fixed point for S and T. In fact, if $[\tilde{u},\tilde{v}]$ is another common near-fixed point for T and S, then $[\tilde{u},\tilde{v}] \in \big\langle [k,l] \big\rangle$ and $[k,l] \stackrel{\Omega}{=} [\tilde{u},\tilde{v}]$.

Now, assume that $[k,l]$ is a near-fixed point of S and $[k,l] \stackrel{\Omega}{\neq} T[k,l]$. Then by using (30), (45) and the null equality, we have

$$\begin{aligned} d\big([k,l],T[k,l]\big) &= d\big([k,l]\oplus\omega_3,T[k,l]\big) = d\big(S[k,l]\oplus\omega_4,T[k,l]\big) \\ &= d\big(S[k,l],T[k,l]\big) \\ &\leq M\big([k,l],[k,l]\big) - \phi\big(M\big([k,l],[k,l]\big)\big), \end{aligned}$$

where according to (31), we obtain

$$\begin{aligned} M([k,l],[k,l]) &= \alpha d([k,l],[k,l]) + \beta\big[d([k,l],S[k,l]) + d([k,l],T[k,l])\big] \\ &\quad + \gamma\big[d([k,l],T[k,l]) + d([k,l],S[k,l])\big] \\ &= (\beta+\gamma)d([k,l],T[k,l]). \end{aligned}$$

Thus

$$d([k,l],T[k,l]) \le (\beta+\gamma)d([k,l],T[k,l]) - \phi\big((\beta+\gamma)d([k,l],T[k,l])\big),$$

since $\beta+\gamma \le 1$, it implies that

$$d([k,l],T[k,l]) \le d([k,l],T[k,l]) - \phi\big((\beta+\gamma)d([k,l],T[k,l])\big),$$

which is a contradiction since $\beta > 0$ and $d([k,l],T[k,l]) > 0$. Hence $d([k,l],T[k,l]) = 0$, i.e., $[k,l] \stackrel{\Omega}{=} T[k,l]$. Similarly, any near-fixed point for T is also a near-fixed point for S. $\square$

Corollary 4.13. *Suppose that $(\mathcal{I}, \|\cdot\|)$ is a BIS and the null equality and null condition hold for $\|\cdot\|$. Assume that $S,T : (\mathcal{I}, \|\cdot\|) \to (\mathcal{I}, \|\cdot\|)$ are self-mappings of $\mathcal{I}$ such that for all $[k,l],[u,v] \in \mathcal{I}$,*

$$\|S[k,l] \ominus T[u,v]\| \le M\big([k,l],[u,v]\big) - \phi\big(M\big([k,l],[u,v]\big)\big),$$

where a function $\phi : [0,\infty) \to [0,\infty)$ is continuous and nondecreasing with the property $\phi(t) > 0$ for $t \in (0,\infty)$ and $\phi(0) = 0$, and

$$\begin{aligned} M([k,l],[u,v]) &= \alpha\|[k,l] \ominus [u,v]\| \\ &\quad + \beta\big[\|[k,l] \ominus S[k,l]\| + \|[u,v] \ominus T[u,v]\|\big] \\ &\quad + \gamma\big[\|[k,l] \ominus T[u,v]\| + \|[u,v] \ominus S[k,l]\|\big], \end{aligned}$$

with $\alpha, \beta > 0, \gamma \ge 0$ and $\alpha + 2\beta + 2\gamma \le 1$. Then a common near-fixed point $[k,l] \in \mathcal{I}$ exists for T and S satisfying $T[k,l] \stackrel{\Omega}{=} [k,l] \stackrel{\Omega}{=} S[k,l]$. Moreover, a unique equivalence class $\big\langle [k,l] \big\rangle$ exists for T and S such that if $\big[\bar{k},\bar{l}\big]$ is another common near-fixed point of T and S, then $\big\langle [k,l] \big\rangle = \big\langle \big[\bar{k},\bar{l}\big] \big\rangle$.

Example 4.14. Let $\phi, \psi : [0, \infty) \to [0, \infty)$ and $S, T : \mathcal{I} \to \mathcal{I}$ in CMIS $(\mathcal{I}, d)$ be defined by $\phi(t) = \dfrac{t}{4}$, $S[k,l] = \left[-1 + \frac{k}{4}, 1 + \frac{l}{4}\right]$ and $T[k,l] = \left[-2 + \frac{k}{4}, 2 + \frac{l}{4}\right]$, respectively. Define a mapping $d : \mathcal{I} \times \mathcal{I} \to \mathbb{R}^+$ by $d([k,l], [u,v]) = |(k+l) - (u+v)|$ for all $[k,l], [u,v] \in \mathcal{I}$. Suppose that $\alpha = \frac{1}{2}$, $\beta = \frac{1}{8}$ and $\gamma = \frac{1}{8}$. Then by (30), we have

$$\begin{aligned}
&d\big(S[k,l], T[u,v]\big) = d\left(\left[-1 + \frac{k}{4}, 1 + \frac{l}{4}\right], \left[-2 + \frac{u}{4}, 2 + \frac{v}{4}\right] \right) \\
&= \frac{1}{4}\big|(k+l) - (u+v)\big| \leq \frac{3}{4}\Big(\alpha\big|(k+l) - (u+v)\big| \\
&\quad + \beta\left[\frac{3}{4}\big|k+l\big| + \frac{3}{4}\big|u+v\big|\right] \\
&\quad + \gamma\left[\Big|(k+l) - \frac{1}{4}(u+v)\Big| + \Big|(u+v) - \frac{1}{4}(k+l)\Big|\right]\Big) \\
&= \frac{3}{4}M\big([k,l], [u,v]\big) = M\big([k,l], [u,v]\big) - \frac{1}{4}M\big([k,l], [u,v]\big) \\
&= M\big([k,l], [u,v]\big) - \phi\big(M\big([k,l], [u,v]\big)\big).
\end{aligned}$$

Thus, this example satisfies all conditions of Theorem 4.12. Hence, T and S have a unique equivalence class of common near-fixed points $\big\langle [-2, 2] \big\rangle$ in $\mathcal{I}$.

5 Conclusion

Today, fixed points are used in various areas of science and are considered a fascinating research field. In 2018, Wu proposed a concept of a fixed point called a near-fixed point. He introduced the null set and presented MIS and NIS [10]. Following the generalizations of Banach's contraction principle, we proved near-fixed point theorems for generalized ϕ-contraction mappings and generalized weak ϕ-contraction mappings in MIS and NIS. We also provided examples to demonstrate the correctness of the results.

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