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A Theoretical Development on Fuzzy Distance Measure

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Abstract. Fuzzy distance measure is one of the most useful tools in applications. Many distance methods have been proposed so far. However, there is no method that can always give a satisfactory solution to every situation. In this paper, we propose a new algorithm to determine distance between two fuzzy numbers. The proposed method could be developed in n-dimentions. Finally, some numerical examples demonstrate the advantages of this method.

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1. Introduction

All the time we receive lots of information from our environment and most of them are approximate. Depending the nature or shape of membership function a fuzzy number can be classified in different ways,

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such as triangular fuzzy number (TFN), trapezoidal fuzzy number and etc. Triangular fuzzy numbers (TFNs) are frequently used in applications.

Distance measures have become important due to the significant applications in diverse fields like remote sensing, data mining, pattern recognition [16] and multivariate data analysis. Also, one of the most important fields in fuzzy set theory is the distance between two triangular fuzzy numbers, because of its applications in decision-making, data mining and image processing. In 1965, Zadeh [43] introduced the concept of fuzzy set theory to get together with those problems. In 1978, Dubois and Prade [15] defined any of the fuzzy numbers as a fuzzy subset of the real line. After that, many researchers used fuzzy numbers to build the mathematical model of linguistic variable under fuzzy environment.

In 1991, Kaufmann et al. [30] considered a distance measure of two fuzzy numbers combined by the interval of r-cuts of fuzzy numbers. In 1992, Liu [34] discussed three concepts of distance measures (normal distance, σ-distance (sub-σ distance) of fuzzy sets. In 1997, Heilpern [27] proposed three definitions of the distance between two fuzzy numbers. These include that mean distance method is generated by expect values of fuzzy numbers. Distance method was combined by a Minkowski distance and the h-levels of the closed intervals of fuzzy numbers, and geometrical distance method is based on the geometrical operation of trapezoidal fuzzy numbers. In 1998, Chen and Hsieh [10, 28] defined the distance of two generalized fuzzy numbers as following:

Let A and B be two generalized fuzzy numbers. $P(A)$ and $P(B)$ are graded mean integration represent (GMIR) of A and B respectively. Then the distance between A and B is defined as $|P(A) - P(B)|$. Then Bloch [7] used r-cuts and extension principle to define the fuzzy distance based on mathematical morphology. Saha et al. [38] also proposed the notion of a fuzzy distance, by first defining the length of a path on a fuzzy subset and then finding the infimum of the length of all paths between two points. Tran and Duckstein [41] defined the distance as a weighted integral of the distance between two intervals at all levels from 0 to 1. In 2007, Pedrycz [35] defined fuzzy C-means. All of the mentioned methods used crisp number to calculate the distance of two trapezoidal fuzzy

numbers.
Now, we state other methods for fuzzy distance, which introduce a fuzzy distance for normal fuzzy numbers. The first one was introduced by voxman [42] through using r-cut. In 2006, Chakraborty and Chakraborty [8] introduced a fuzzy value distance measured by using r-cut of two generalized fuzzy numbers. Then Guha and Chakraborty [18] showed that their previous method could be developed to a new similarity measure with the help of the fuzzy distance measure. It is obvious that, the distance between the fuzzy number A and zero is more suitable to be A. However, the distance between the fuzzy number A and zero is not A by Chakrabortys distance.
Hajjari [26] presented a fuzzy Euclidean distance for fuzzy data. Hajjari first described positive fuzzy number, negative fuzzy number and fuzzy zero to represent some new definitions, and then she discussed fuzzy absolute, equality and inequality of fuzzy numbers based on these concepts and some useful properties too. The aforementioned concepts used to produce the distance between two fuzzy numbers as a trapezoidal fuzzy number.
Abbasbandy and Hajighasemi [3] introduced a symmetric triangular fuzzy number (TFN) as a fuzzy distance based on r-cut concept. Salahshour et al. [39] developed a new approach for ranking of fuzzy numbers. This approach considers not only the left and right spreads of fuzzy numbers, but also their COG points have been associated in order to construct the fuzzy maximizing-minimizing points.
Allahviranloo et. al [6] proposed a new fuzzy distance measure for fuzzy numbers. They introduced an interval distance measure on fuzzy numbers. Moreover, Salahshour et. al [40] in 2012, proposed a novel approach for ranking triangular intuitionistic fuzzy numbers (TIFN). They converted each TIFN to two related triangular fuzzy numbers (TFNs) based on the their membership functions and non-membership functions. Then, they suggested a new defuzzification for obtained TFNs using their values and ambiguities. Adabitabar et. al [5] developed a distance measure between fuzzy numbers as an interval number. They used this metric to define a similarity measure on fuzzy numbers as an interval number. In 2013, Abbasbandy and salahshour [4] discussed the

Voxman's fuzzy distance measure [42], Chakraborty et al. [8] and Guha et. al [18] fuzzy distance measure. They found out some shortcomings for these distances.
Recently, Huchang Liao et. al [33] presented a new distance and similarity measures for hesitant fuzzy linguistic term set and their application in multi-criteria decision making.
All the above-mentioned distance methods used the r-cut concept to calculate the fuzzy distance. There are some other distance methods, which have used other fuzzy concept. Many distance methods for fuzzy numbers have been discussed in [18, 22, 34]. Most of these approaches applied r-cut concept and can consider general fuzzy numbers in one dimension space.
In this paper, we will propose an algorithm to determine distance between two triangular fuzzy numbers. The proposed algorithm could be developed in n-dimensions. In this work, we attempt to determine a new fuzzy distance measure, which is far simple and easier than previous distance methods.
The rest of the paper is organized as follows: Section 2 contains the basic definitions and notations that will be used in the remaining parts of the paper. In Section 3 we review some different fuzzy distance methods. Section 4 includes a new approach to determine fuzzy distance measure and some properties. Some numerical examples demonstrate the advantages of the reviewed methods and compared results in Section 5. The paper is concluded in Section 6.

2. Preliminaries

In general, a generalized fuzzy number A is described as any fuzzy subset of real line R, whose membership $\mu_A(x)$ can be defined as:

$$\mu_A(x) = \begin{cases} L_A(x), & a \leqslant x \leqslant b, \\ \omega, & b \leqslant x \leqslant c, \\ R_A(x), & c \leqslant x, \\ 0, & otherwise, \end{cases} \tag{1}$$

where $0 \leqslant \omega \leqslant 1$ is a constant, $L_A(x) : [a, b] \to [0, \omega]$ and $R_A(x)$:

$[c, d] \to [0, \omega]$ are two strictly monotonic and continuous mapping [15]. If $\omega = 1$, then A is a normal fuzzy number. If $L_A(x) = \omega(x-a)/(b-a)$, and $R_A(x) = \omega(d-x)/(d-c)$ then it is a trapezoidal fuzzy number and is usually denoted by $A = (a, b, c, d; \omega)$ or $A = (a, b, c, d)$ if $\omega = 1$. In particular, when $b = c$, the trapezoidal fuzzy number is reduced to a triangular fuzzy number denoted by $A = (a, b, c, d; \omega)$ or $A = (a, b, c, d)$ if $\omega = 1$. Therefore, triangular fuzzy numbers are special cases of trapezoidal fuzzy numbers.
Since $L_A(x)$ and $R_A(x)$ are both strictly monotonic and continuous functions, their inverse functions exist and should be also continuous and strictly monotonic. Let $\mu_L^{-1} : [0, \omega] \to [a, b]$ and $\mu_R^{-1} : [0, \omega] \to [c, d]$ be the inverse functions of $L_A(x)$ and $R_A(x)$, respectively. Then $L_A^{-1}(r)$ and $R_A^{-1}(r)$ should be integrable on the close interval $[0, \omega]$. In other words, both $\int_0^\omega L_A^{-1}(r)dr$ and $\int_0^\omega R_A^{-1}(r)dr$ should exist. In the case of trapezoidal fuzzy number, the inverse functions $L_A^{-1}(r)$ and $R_A^{-1}(r)$ can be analytically expressed as

$$L_A^{-1}(r) = a + (b-c)r/\omega, \qquad 0 \leqslant \omega \leqslant 1, \tag{2}$$

$$R_A^{-1}(r) = d - (d-c)y/\omega, \qquad 0 \leqslant \omega \leqslant 1. \tag{3}$$

The set of all elements that have a nonzero degree of membership in A is called the *support* of A, i.e.

$$supp(A) = \{x \in X \mid A(x) > 0\}. \tag{4}$$

The set of elements having the largest degree of membership in A is called the *core* of A, i.e.

$$core(A) = \{x \in X \mid A(x) = \sup_{x \in X} A(x)\}. \tag{5}$$

In the following, we will always assume that A is continuous and bounded support $supp(A) = (a, d)$. The strong support of A should be $\overline{supp}\,(A) = [a, d]$.

Definition 2.1. *A function s: $[0, 1] \longrightarrow [0, 1]$ is a reducing function if s is increasing and $s(0) = 0$ and $s(1) = 1$. We say that s is a regular function if $\int_0^1 s(r)\mathrm{d}r = \frac{1}{2}$.*

Definition 2.2. *If A is a fuzzy number with r-cut representation, $\left[L_A^{-1}(r), R_A^{-1}(r)\right]$, and s is a reducing function then the value of A (with respect to s) is defined by*

$$Val(A) = \int_0^1 s(r)\left[L_A^{-1}(r) + R_A^{-1}(r)\right]\mathrm{d}r. \tag{6}$$

Definition 2.3. *If A is a fuzzy number with r-cut representation, $\left[L_A^{-1}(r), R_A^{-1}(r)\right]$, and s is a reducing function then the ambiguity of A (with respect to s) is defined by*

$$Amb(A) = \int_0^1 s(r)\left[R_A^{-1}(r) - L_A^{-1}(r)\right]\mathrm{d}r. \tag{7}$$

Definition 2.4. *Let A is a fuzzy number. The absolute value of the fuzzy number A is denoted by $|A|$ and defined as follows [17]:*

$$\mid A(x) \mid = \begin{cases} 0, & x < 0, \\ A(x) \vee A(-x), & x \geqslant 0, \end{cases} \tag{8}$$

and for all $r \in [0.1]$,

$$[|A_x|]_r = \begin{cases} [A]_r, & L_A^{-1}(r) \geqslant 0, \\ [0, |L_A^{-1}(r)| \vee R_A^{-1}(r)], & L_A^{-1}(r) \leqslant 0 \leqslant R_A^{-1}(r), \\ [-R_A^{-1}(r), -L_A^{-1}(r)], & L_A^{-1}(r) \leqslant R_A^{-1}(r) \leqslant 0, \end{cases} \tag{9}$$

where $[A]_r = \lfloor L_A^{-1}(r), R_A^{-1}(r) \rfloor$ is the r-cut representation of A and $[|Ax|]_r$ is the r-cut representation of $|A|$, respectively.

3. Some Existing Fyzzy Distance Methods

In this section, we briefly review some existing fuzzy distance measures. Different authors have constructed different fuzzy distance measures between two fuzzy numbers. In this paper, some of them are discussed.

3.1 Voxman's fuzzy distance measure

Here, we briefly describe the fuzzy distance measure by Voxman [42]. The fuzzy distance function of F,

$$\begin{cases} \Delta : F \times F \longrightarrow F, \\ \Delta(A,B)(z) = sup_{|x-y|=z} min\{\mu_A(x), \mu_B(y)\}. \end{cases}$$

For each pair of fuzzy number A, B let $\Delta_{A,B}$ denoted the fuzzy number $\Delta(A,B)$.
If the r- cut representation of $\Delta_{AB} = (L(r), R(r))$ is given by

$$L(r) = \begin{cases} max\{\underline{\mu_B}(r) - \overline{\mu_A}(r), 0\}, \text{if} \\ \qquad (\underline{\mu_A}(1) + \overline{\mu_A}(1)) \leqslant \underline{\mu_B}(1) + \overline{\mu_B}(1), \\ max\{\underline{\mu_A}(r) - \overline{\mu_B}(r), 0\}, \text{if} \\ \qquad (\underline{\mu_B}(1) + \overline{\mu_B}(1)) \leqslant \underline{\mu_A}(1) + \overline{\mu_A}(1), \end{cases} \tag{10}$$

and

$$R(r) = max\{\overline{\mu_A}(r) - \underline{\mu_B}(r), \overline{\mu_B}(r) - \underline{\mu_A}(r)\}. \tag{11}$$

3.2 Chakraborty et al.'s fuzzy distance measure

Consider two generalized fuzzy numbers as $A = (a_1, a_2; \beta_1, \gamma_1)$ and $A_2 = (b_1, b_2; \beta_2, \gamma_2)$. Therefore, the r- cut of A_1 and A_2 represents two intervals, respectively $[A_1]_r = [A_1^L(r),\ A_1^R(r)]$ and $[A_2]_r = [A_2^L(r),\ A_2^R(r)]$, for all $r \in [0,1]$.
Moreover, [8] employed the interval-difference operation for the intervals $[A_1]_r = [A_1^L(r), A_1^R(r)]$ and $[A_2]_r = [A_2^L(r), A_2^R(r)]$ to formulate the fuzzy distance between A_1 and A_2. So, the distance between $[A_1]_r$ and $[A_2]_r$ for every $r \in [0,1]$ can be one of the following: either

$$(\mathbf{a}) \quad [A_1]_r - [A_2]_r \quad if \quad \frac{A_1^L(1) + A_1^R(1)}{2} \geqslant \frac{A_2^L(1) + A_2^R(1)}{2},$$

or

$$(\mathbf{b}) \quad [A_2]_r - [A_1]_r \quad if \quad \frac{A_1^L(1) + A_1^R(1)}{2} < \frac{A_2^L(1) + A_2^R(1)}{2}.$$

To consider both notations together an indicator variable λ is used such that

$$\lambda([A_1]_r - [A_2]_r) + (1-\lambda)([A_1]_r - [A_2]_r) = [d_r^L, d_r^R],$$

where

$$\lambda = \begin{cases} 1, & if \quad \dfrac{A_1^L(1) + A_1^R(1)}{2} \geqslant \dfrac{A_2^L(1) + A_2^R(1)}{2}, \\ 0, & if \quad \dfrac{A_1^L(1) + A_1^R(1)}{2} < \dfrac{A_2^L(1) + A_2^R(1)}{2}, \end{cases} \tag{12}$$

therefore, the fuzzy distance measure between A_1 and A_2 is defined by

$$d(A_1, A_2) = (d_{r=1}^L, d_{r=1}^R; \theta, \sigma), \tag{13}$$

where

$$\theta = d_{r=1}^L - max\left\{ \int_0^1 d_r^L \ dr, 0 \right\}, \sigma = \int_0^1 d_r^R \ dr - d_r^R,$$

also in [8], the following metric properties are studied as following:

$(a1) : d(A_1, A_2) = (d_{r=1}^L, d_{r=1}^R; \theta, \sigma)$is a positive fuzzy numbers,

$(a2) : d(A_1, A_2) = d(A_2, A_1)$,

$(a3) : d(A_1, A_3) \leqslant d(A_1, A_2) + d(A_2, A_3)$.

3.3 Fuzzy distance given by Shan-Huo Chen et al.

Let A and B be two trapezoidal fuzzy numbers. Then the fuzzy distance given by Shan-Huo Chen ([12]) of A and B is:

$$d(A, B) = |A - B| \tag{14}$$

3.4 Fuzzy distance given by Chen and Wang

In 2007, Chen and Wang [11] introduced a fuzzy distance of two trapezoidal fuzzy numbers by using the GMIR and the spread of the fuzzy numbers. This new idea has two advantages. With this new idea the

fuzzy distance is easy to calculate and easy to understand. Now, we describe the definition as follows.
Let $A = (a_1, a_2, a_3, a_4)$ and $B = (b_1, b_2, b_3, b_4)$ be two trapezoidal fuzzy numbers, and their GMIR are $P(A)$, $P(B)$ respectively. Assume

$$\begin{aligned}
&P(A) = \frac{a_1 + 2a_2 + 2a_3 + a_4}{6},\\
&P(B) = \frac{b_1 + 2b_2 + 2b_3 + b_4}{6},\\
&s_i = (a_i - P(A) + b_i - P(B))/2, \quad i = 1, 2, 3, 4,\\
&c_i = |P(A) + P(B) + s_i|/i, \quad i = 1, 2, 3, 4,
\end{aligned}$$

then the fuzzy distance of A and B is

$$C = (c_1, c_2, c_3, c_4). \tag{15}$$

3.5 Fuzzy distance given by Hajjari

Let A and B be two trapezoidal fuzzy numbers. The fuzzy distance given by Hajjari ([26]) between A, B denoted by $dist(A, B)$ and it was defined as:

$$dist(A, B) = absf(A - B), \tag{16}$$

where $Absf(A)$ is the fuzzy absolute of the fuzzy number A, i.e.,

$$Asbf(A) = \begin{cases} A, & A \succ 0,\\ 0, & A \approx 0,\\ -A, & A \prec 0, \end{cases} \tag{17}$$

and 0, $A \approx 0$ and $A \prec 0$, stand fuzzy zero number, positive fuzzy number and negative fuzzy number respectively as follows:

$$\begin{aligned}
&\forall A \in E : A \succ 0 \Leftrightarrow \textstyle\int_a^\omega (L_A^{-1}(r) + R_A^{-1}(r))\ dr \succ 0,\\
&\forall A \in E : A \approx 0 \Leftrightarrow \textstyle\int_a^\omega (L_A^{-1}(r) + R_A^{-1}(r))\ dr = 0,\\
&\forall A \in E : A \prec 0 \Leftrightarrow \textstyle\int_a^\omega (L_A^{-1}(r) + R_A^{-1}(r))\ dr \prec 0,
\end{aligned}$$

also ω be a constant such that $0 \leqslant \omega \leqslant 1$.

3.6 Fuzzy distance given by Guha and Chakraborty

Let us consider two generalized trapezoidal fuzzy number as follows:
$A = (a_1, a_2; \beta_1, \gamma_1; \omega_1)$, $B = (b_1, b_2; \beta_2, \gamma_2; \omega_2)$
r-cut representation of A is denoted by:

$$[A]_r = \lfloor L_A^{-1}(r), R_A^{-1}(r) \rfloor, \quad \text{for} \quad 0 \prec \omega_1 \prec 1.$$

r-cut representation of B is denoted by:

$$[B]_r = \lfloor L_B^{-1}(r), R_B^{-1}(r) \rfloor, \quad \text{for} \quad 0 \prec \omega_2 \prec 1.$$

The r-cut representation of the distance measure between two fuzzy numbers A and B is denoted by $[d_r^L, a_r^R]$ for $r \in [0, \omega]$, $\omega = min(\omega_1, \omega_2)$. Now the distance given by Guha and Chakraborty ([18]) between A and B was defined by:

$$\begin{aligned} &d(A, B) = (d^L_{r=\omega}, d^R_{r=\omega}; \theta, \sigma), \\ &\omega = min(\omega_1, \omega_2). \end{aligned} \tag{18}$$

Where θ and σ is defined in the following:

$$\begin{aligned} \theta &= d^L_{r=\omega} - max\big\{ \textstyle\int_0^\omega d_r^L \, dr, 0 \big\}, \\ \sigma &= \Bigg| \bigg[\textstyle\int_0^\omega d_r^R \, dr - d^R_{r=\omega} \bigg] \Bigg|. \end{aligned}$$

3.7 Improved centroid distance method

To overcome the drawback of Cheng's distance method [14] Abbasbandy and Hajjari [2] improved the centroid distance method. They first introduced a sign function as follows:

Definition 3.7.1. *Let E stands the set of non-normal fuzzy numbers, ω be a constant provided that $0 < \omega \leqslant 1$ and $\gamma : E \to \{-\omega, 0, \omega\}$ be a function that is defined as:*

$$\forall A \in E : \gamma(A) = sign\big[\int_0^\omega (L_A^{-1}(r) + R_A^{-1}(r)) \, dx \big],$$

i.e.

$$\gamma(A) = \begin{cases} 1, & \int_0^{\omega} (L_A^{-1}(r) + R_A^{-1}(r))\ dx, \\ 0, & \int_0^{\omega} (L_A^{-1}(r) + R_A^{-1}(r))\ dx, \\ -1, & \int_0^{\omega} (L_A^{-1}(r) + R_A^{-1}(r))\ dx. \end{cases} \tag{19}$$

It is clear that for normal fuzzy numbers $\omega = 1$.
Then they combined Cheng's distance method [14] by corrected formulae with sign function and presented the improved centroid distance method as follows:

$$IR(A) = \gamma(A)R(A), \tag{20}$$

in other words

$$IR(A) = \gamma(A)\sqrt{x_A^2 + y_A^2}. \tag{21}$$

3.8 Sadi-nezhad et al.'s fuzzy distance measure

Sadi-Nezhad et al. [37] proposed a fuzzy distance measure for two triangular fuzzy numbers based on left and right points. They denoted fuzzy distance between $A = (x_1, x_2, x_3)$ and $B = (y_1, y_2, y_3)$ as $D_{AB} = (d_1, d_2, d_3)$ such that:

$$d_1 = \begin{cases} max\{x_1 - y_3, 0\} & x_2 \geqslant y_2, \\ max\{y_1 - x_3, 0\} & x_2 \leqslant y_2, \end{cases}$$

$$d_2 = |x_2 - y_2|,$$
$$d_3 = \{max(y_3 - x_1, x_3 - y_1)\}.$$

Consider triangular fuzzy number $A = (0, 1, 2)$ then by applying Sadi-Nezhad's method, the distance between A and A is $(0, 0, 2)$ which is unreasonable.

3.9 Adabitabar Firozja et al.'s fuzzy distance measure

Here, We briefly describe the fuzzy distance measure by Adabitabar Firozja et al. [5]. Consider two fuzzy numbers A and B with $[A]_r =$

$[\underline{A}(r), \overline{A}(r)]$ and $[B]_r = [\underline{B}(r), \overline{B}(r)]$ respectively. The distance between these two fuzzy numbers is defined by $d(A, B) = [\underline{d}, \overline{d}] \in I(R)$ where

$$\underline{d} = min\{min_{r\in[0,1]}|\underline{A}(r) - \underline{B}(r)|, min_{r\in[0,1]}|\overline{A}(r) - \overline{B}(r)|\},$$

and

$$\overline{d} = max\{max_{r\in[0,1]}|\underline{A}(r) - \underline{B}(r)|, max_{r\in[0,1]}|\overline{A}(r) - \overline{B}(r)|\},$$

The proposed interval distance between two fuzzy numbers satisfies the metric properties and also satisfies the following properties:

Proposition 3.9.1. *If $A, B, C \in F(R)$ and $a, b, \tau \in R$*
1. $d(\tau A, \tau B) = |\tau| d(A, B)$;
2. $d(A + \tau, B + \tau) = d(A, B)$;
3. $d(A + C, B + C) = d(A, B)$;
4. $d([a, b], [c, d]) = [min\{|a - c|, |b - d|\}, max\{|a - c|, |b - d|\}$;
5. $d(a, b) = |a - b|$;

Proposition 3.9.2. *If $A = (a_1, a_2, a_3, a_4)$ and $B = (b_1, b_2, b_3, b_4)$ are two trapezoidal fuzzy numbers, then $d(A, B) = [\underline{d}, \overline{d}]$, so that if*

$$0 \prec \frac{b_1 - a_1}{a_2 - a_1 - b_2 + b_1} \prec 1,$$

or

$$0 \prec \frac{a_3 - b_3}{a_3 - a_2 - b_3 + b_2} \prec 1,$$

then $\underline{d} = 0$ otherwise $\underline{d} = min\{|a_1 - b_1|, |a_2 - b_2|, |a_3 - b_3|\}$ and $\overline{d} = max\{|a_1 - b_1|, |a_2 - b_2|, |a_3 - b_3|\}$.

Proposition 3.9.3. *Suppose $d(A, B) = [\underline{d}, \overline{d}]$, if $\overline{d} = 0$ then $\underline{d} = 0$. They presented a new similarity measure based interval distance measure as following: The relation which has been used here is real interval*

$$S(A, B) = \left[\frac{1}{1 + \overline{d}}, 1 - \frac{\underline{d}}{1 + \overline{d}}\right], \tag{22}$$

where $[\underline{d}, \overline{d}]$ *is the interval distance measure between* A *and* B.
They shown this new similarity measure has the following properties:
1. $[0, 0] \prec S(A, B) \leqslant [1, 1]$.
2. $S(A, B) = S(B, A)$.
3. If $A = B$*, then* $S(A, B) = [1, 1]$.
4. If $A \leqslant B \leqslant C$*, then* $S(A, C) \leqslant min\{S(A, B), S(B, C)\}$.

4. New Fuzzy Distance Measure

In this section we present a new algorithm for fuzzy distance measure between two triangular fuzzy numbers. Let $A = (x_1, x_2, x_3)$ and $B = (y_1, y_2, y_3)$ are two triangular fuzzy numbers and distance between A and B is denoted by $Dist_{AB}$, where $Dist_{AB} = (d_1, d_2, d_3)$. We calculate d_1, d_2 and d_3 by following steps:

Step 1

1. Compare x_1 and y_1.
2. If $x_1 = y_1$, then $d_1 = 0$, go to step 2. Otherwise,
3. $d_1 = |x_1 - y_1|$, go to step 2.

Step 2

1. $d_* = |x_2 - y_2|$.
2. Consider $d_2 = max\{d_*, d_1\}$. Go to step 3.

Step 3

1. $d_{**} = |x_3 - y_3|$.
2. Consider $d_3 = max\{d_{**}, d_2\}$.

Remark 4.1. *The proposed method is a metric.*

1. $Dist_{AB} \geqslant 0$,

2. $Dist_{AB} = Dist_{BA}$,
3. $Dist_{AC} \leqslant Dist_{AB} + Dist_{BC}$.

Proof.

1. $Dist_{AB}$ is a positive fuzzy number. Based on algorithm, the parameters d_1, d_2 and d_3 are non-negative.
2. $Dist_{AB} = Dist_{BA}$, Based on new method it is clear.
3. The triangular inequality: $Dist_{AC} \leqslant Dist_{AB} + Dist_{BC}$:

Let $A = (x_1, x_2, x_3)$, $B = (y_1, y_2, y_3)$ and $C = (z_1, z_2, z_3)$ are three triangular fuzzy numbers and the distance $Dist_{AC}$, $Dist_{AB}$ and $Dist_{BC}$ we should prove that:

$$Dist_{AC} \leqslant Dist_{AB} + Dist_{BC}, \tag{23}$$

Let $Dist_{AC}$, $Dist_{AB}$ and $Dist_{BC}$ be three triangular fuzzy numbers and $Dist_{AB} + Dist_{BC}$ is a triangular fuzzy number too. $\square$

Theorem 4.2. *Let A is a triangular fuzzy number and O is original point $Dist_{AO} = Dist(A, O) = A$.*

Proof. Let $A = (x_1, x_2, x_3)$ and $O = (0, 0, 0)$, based on new fuzzy distance measure, proof is clear.

The proposed method can be developed in n-dimension. Let A and B are two points in n-Dimensional space with triangular fuzzy number values in each dimensions. The points A and B can be shown as:

$$A = \left(\left(x_1^n, x_2^n, x_3^n\right), n = 1, 2, ..., k\right),$$

$$B = \left(\left(y_1^n, y_2^n, y_3^n\right), n = 1, 2, ..., k\right),$$

$Dist_{AB} = (d_1^n, d_2^n, d_3^n)$ is the distance of the nth component of A from the nth component of B and d_1^n, d_2^n and d_3^n are related to from the left point, the centre and the right point of this distance respectively. The distance between each component can be calculated by the same method. Then we define the total fuzzy distance between and as following:

$$\begin{aligned} DistAB &= DistAB^1 + DistAB^2 + ... + DistAB^k \\ &= (d_1^1 + d_1^2 + ... + d_1^k, d_2^1 + d_2^2 + d_2^k, d_3^1 \\ &+ d_3^2 + ... + d_3^k). \end{aligned} \tag{24}$$

5. Numerical Examples

This section uses several numerical examples to compare the distance results of proposed method with some other existing distance methods. The distance between zero and fuzzy number A by Chakraborty and Chakraborty [8] and Guha and Chakraborty [18] has a drawback, A, i.e., $d(A, 0) \neq A$, which is not a satisfactory result. But our method can overcome the shortcoming of aforementioned methods:

$$Dist_{A0} = A.$$

Example 5.1. Consider the data used in Abbasbandy et al. [4] i.e. $[A_1]_\alpha = [1 + \alpha, 3 - \alpha]$, then by Guha et al.'s method [8] we have:

$$d(A_1, A_1) = (0, 0; 0, 1), \tag{25}$$

also by voxman's fuzzy distance measure [42]:

$$d(A_1, A_1) = (0, 0; 0, 2), \tag{26}$$

which is an unreasonable result. Since we expect that the distance between two identical fuzzy numbers should be zero. By proposed method distance between A_1 and A_1 is zero, which is a satisfactory result.

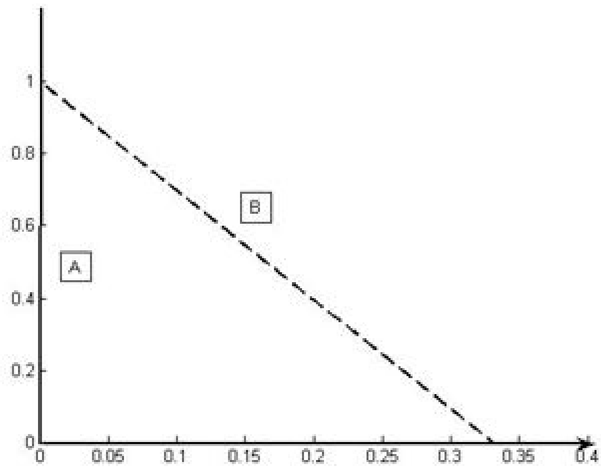

Figure 1: Two fuzzy numbers A and B in Example 5.3.

Example 5.2. Consider the data used in Adabitabar et al. [5], which are given in Table 1. We compare the new similarity measure with five following methods: Chen [9], Lee [32], Chen and Chen [13], Guha and Chakraborty [18] and Adabitabar Firozja et al. [5]. Comparison between the results of the proposed similarity measure and other methods are shown in Table 2.
We can see that our proposed method in some cases gives better result, for example in set 1, the distance between A and B by Chen [9], Lee [32], Chen and Chen [13] and GuhaChakraborty [18] is not zero, which is unreasonable, But in this method the distance between A and B is zero because A and B are the same.

Example 5.3. Let $A = (0, 0, 0)$ and $B = (0, 0, 0.33)$ be two normalized fuzzy numbers, which are indicated in Fig. 1. The distance measure between A and B by Chakraborty and Chakraborty [8] is $d(A.B) = (0, 0, 0)$, this result is far from reality. Also Guha and Chakraborty [18] obtained: $d(A, B) = (0, 0, 0.33/2)$. From Sadi-Nezhad's method [37] $D_{AB} = (0, 0, 0.33)$. By proposed method we get $Dist_{AB} = (0, 0, 0.33)$. Clearly the result is reasonable.

Table 1: The sets of triangular fuzzy numbers in Example 5.2.

	A	B
Set 1	$(0.3, 0.3, 0.3)$	$(0.3, 0.3, 0.3)$
Set 2	$(0.1, 0.2, 0.3)$	$(0.3, 0.3, 0.3)$
Set 3	$(0.2, 0.2, 0.2)$	$(0.3, 0.3, 0.3)$

Table 2: Comparison of the proposed method with some other methods in Example 5.2.

Sets	1	2	3
Chen [9]	1	0.9	0.9
Lee [32]	*	0.5	0
Chen and Chen [13]	1	0.54	0.9
Guha- Chakraborty [18]	(1,1,1)	(0.85,0.9,0.95)	(0.9,0.9,0.9)
Adabitabar Firozja et al. [5]	[1,1]	[0.833,1]	[0.9,0.9]
Proposed Method	(0,0,0)	(0.1,0.1,0.1)	(0.1,0.1,0.1)

Example 5.4. Consider the data used in Sadi-Nezhad et al. [37] i.e. $A = (0, 2, 3)$ and $B = (-1, 0, 1)$. Fig. 2 shows the graphs of the two fuzzy numbers. The results obtained by the proposed method and some others methods are shown in Table 3. In this case for $a_1 = 0.5$ the membership degree is 0.25 i.e. $\mu_A(a_1) = 0.25$, and for $a_2 = 0$ the membership degree is 1 i.e. $\mu_B(a_2) = 1$; therefore, the distance between a_1 and a_2 is 0.5, but from the Table 3 in both Voxman [42] and Chakraborty and Chakraborty [8] methods, the membership is zero. This example shows the strong discrimination power of the proposed approach and its advantages.

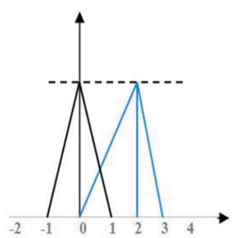

Figure 2: Fuzzy numbers $A = (0, 2, 3)$ and $B = (-1, 0, 1)$ in Example 5.4.

Table 3: Comparative results of Examples 5.4.

Distance approaches	Distance between A and B=$dist(A, B)$
Voxman [42]	$(2, 2, 4)$
Chakraborty and Chakraborty [8]	$(1/2, 2, 3)$
Sadi nejad et al. [37]	$(0, 2, 4)$
Proposed Method	$(1, 2, 4)$

Table 4: Comparative results of Examples 5.5.

	Chen method [9]	Lee method [32]	Chen and Chen [13]
Set 1	0.9	0.5	0.54
Set 2	0.9	0.6667	0.81

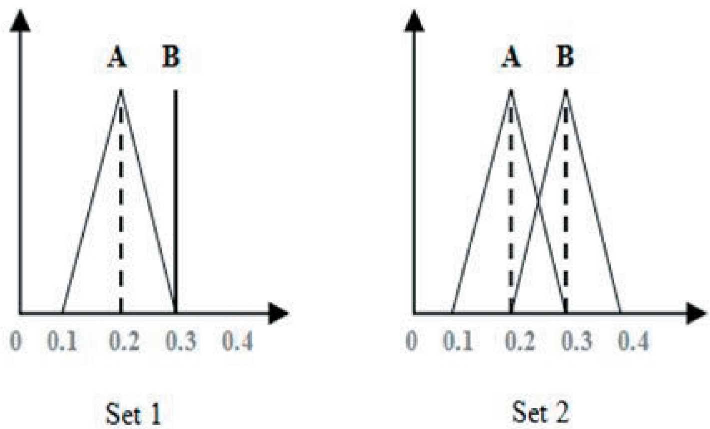

Figure 3: 2 sets of fuzzy numbers A and B in Example 5.5.

Example 5.5. Consider the data used in Guha and Chakraborty [18] i.e. 2 sets of fuzzy numbers are given in Fig. 3 and they are used to compare the distance based fuzzy similarity measure with four methods presented by Chen [9], Lee [32] and Chen and Chen [13]. A comparison between the results of the proposed method and results of the existing methods is shown in Table 4 and Table 5. From Fig. 3, we can see some drawbacks of existing methods and some advantages of the proposed method. From Fig. 3 it is clear that set 1 and set 2 are different sets, but Chen [9] gives the same similarity.

Example 5.6. Consider the data used in Sadi-Nezhad et al. [37] i.e. the two points in a two dimensional space $A = \big((2,3,4),(3,4,5)\big)$ and $B = \big((7,7.5,9),(7,8,9)\big)$ as shown in Fig. 4. The distance results by sadi-nezhads approach [37] is the same with our method:

$$Dist_{AB} = Dist^1_{AB} + Dist^2_{AB} = (3,4.5,7) + (2,4,6) = (5,8.5,13), \quad (27)$$

Table 5: Comparative results of Examples 5.5.

	Guha and Chakraborty method[18]	Proposed method
Set 1	(0.9,0.9,0.9)	(0.2,0.2,0.2)
Set 2	(0.8,0.9,1)	(0.1,0.1,0.1)

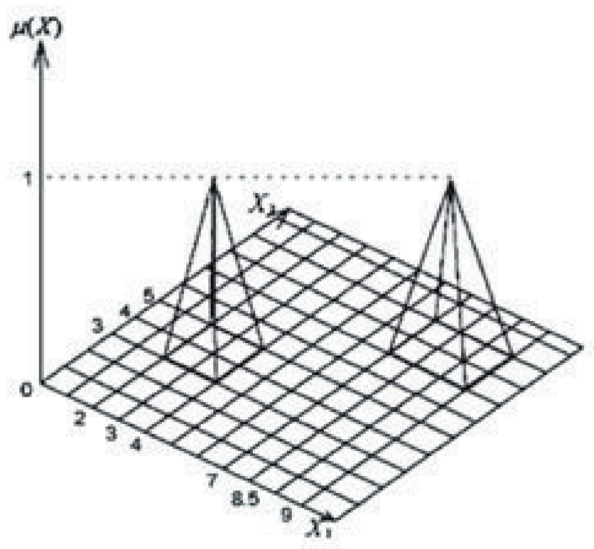

Figure 4: Two-dimensional fuzzy numbers $A = \big((2,3,4),(3,4,5)\big)$ and $B = \big((7,7.5,9),(7,8,9)\big)$ in Example 5.6.

6. Conclusion

Fuzzy distance numbers are vastly used for indicating uncertain and vague information in decision making, linguistic controllers, expert systems, data mining, operation research, etc. In this paper, we reviewed on some fuzzy distance methods, then we proposed a distance for the two triangular fuzzy numbers. The presented method could be developed in n-dimentions. We used the left, centre and right point. The advantage of the proposed method is that the calculation is easier than previous ones and it overcomes the drawbacks of some other methods.

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