Journal of Mathematical Extension
Journal Pre-proof
ISSN: 1735-8299
URL: http://www.ijmex.com
Original Research Paper

The Relationship Between Extreme Efficient and Most Efficient Unit(s) in Data Envelopment Analysis

Z. Noori
Lahijan Branch, Islamic Azad University

H. Zhiani Rezai*
Mashhad Branch, Islamic Azad University

A. Davoodi
Neyshabur Branch, Islamic Azad University

S. Kordrostami
Lahijan Branch, Islamic Azad University

Abstract. Although data envelopment analysis models are able to divide decision-making units (DMUs) into efficient and inefficient sets, choosing the best efficient unit has always been a challenge in decision-making issues. Also, various methods have been introduced to find the most efficient unit, most of which are based on solving linear and nonlinear problems, and also according to the related logic, different units are found as the most efficient unit. The relationship between the most efficient and the extreme efficient units has not been discussed yet. In this paper, we show that each extreme efficient unit can be taken as the most efficient one and vice versa. As a result, the properties of an extreme efficient unit are the same as the most efficient ones. This finding is significant because methods for identifying extreme efficient

Received: September 2023; Accepted: May 2024
*Corresponding Author

units are generally simpler to implement. Therefore, by leveraging these simpler methods, we can effectively find the most efficient DMU within a dataset. To illustrate the application of this approach, we demonstrate its use on a well-known example commonly employed in DEA studies.

AMS Subject Classification: 90C08; 90C05
Keywords and Phrases: Data Envelopment Analysis, Most Efficient Unit, Extreme Efficient Unit.

1 Introduction

Data Envelopment Analysis (DEA) includes techniques and methods for evaluating the efficiency of Decision-Making Units (DMUs) [4, 6, 7]. DEA is actually the generalization of Farrell's research in devising a nonparametric method to find the efficiency of DMUs with multiple inputs and outputs [12]. Through using inputs and outputs of DMUs and some principles, Farrell developed a set called Production Possibility Set (PPS), and its frontier is called the efficient frontier. If the removal of an efficient unit changes the frontier, it is known as the extreme efficient [9].
In most cases, a decision-maker needs to select one efficient DMU from a set of efficient DMUs. Thus, the question is which efficient DUM is the most efficient one and performs better among all efficient Units. Since 2005, the following mentioned researches were all done to find the answer to this question.
In 2005, Karsak and Ahiska used some benchmarks to introduce a multiple criteria decision-making model and claimed that their proposed model led to the most efficient DMU [15]. However, Amin et al. showed that this model fails to discriminate against the most efficient DMU in some specific situations [2]. Ertay and Ruan suggested a cross-efficiency approach to determine the most efficient unit [10]. Ertay et al. proposed the Min-Max model, in which its objective function included the K-parameter and had to be solved through the trial-and-error method [11]. Amin and Toloo proposed a model in which the K-parameter of the Ertay model did not exist [3]. In the improved model of Amin, it was determined the most efficient unit in the constant returns to scale technology [1]. The nonlinear model proposed by Foroughi required the selection of a feasible region for each input and output weight [13, 14]. Toloo and Nalchigar proposed a model to solve the related models under variable returns to scale [22] and [24]. Another model developed by Wang and Jiang's was easier to solve than Foroughi's model but included fundamental restrictions [28]. The proposed model by Lam had an objective function similar to the super-efficiency model [16]. In 2015, Toloo developed his previous models by proposing a new definition [23]. Moreover, it is shown that some of the proposed models may be infeasible or may introduce more than one as the most efficient instead of finding a single most efficient unit, although this is not their function [27]. Toloo and Salahi extended a nonlinear model to deal with improving the discriminating power of DEA models and showed that the proposed model could identify the most efficient unit [25]. Another approach considering user subjective opinions was developed by Toloo et al to find the most efficient information system projects [26]. Özsoy et al. introduced another

model based on mixed-integer programming to determine the most efficient DMU in two-stage systems and sub-stages [19]. They, also developed an epsilon-free approach to choose the most efficient unit [20]. Matin Nejati et al introduced a method based on Genetic Algorithm to find the most efficient Unit [17]. Ravanos and Karagiannis presented a method to find the most preferred solution in value efficiency analysis [21]. Finally, Zhiani Rezai et al. introduced a model-free approach to find the most efficient unit [18].

It is necessary to note that, in all these studies and their respective models, the unit taken as the most efficient one is actually an extreme efficient unit. Therefore, it looks that the most efficient unit and the extreme efficient are similar. Here, we first state the necessity of discussing this subject. Then, by proving some theorems and illustrating some examples, it will be shown that the extreme efficient unit and the most efficient unit are equivalent. There will be a new direction in research in this field.

The structure of the paper is as follows: In the next section, motivations are discussed. Section 3 deals with preliminaries. The relationship between the most efficient and the extreme efficient unit will be presented in section 4. Some examples are provided in section 5 to show the capability of this equivalence. The conclusion appears in section 6.

2 Motivation

Toloo's definition of the most efficient unit does not include uniqueness property. That is, each various proposed model introduces an extreme efficient DMU as the most efficient one, which is not necessarily unique [23]. Results from previous research show that all efficient units are capable of being selected as the most efficient ones. Take the example of finding the most efficient Facility Layout Designs (FLDs), which was discussed for the first time by Ertay et al. as a numerical example [11]. The results of this example are presented in Table 1. The first row indicates the set of (CCR) efficient DMUs, and the first column shows the list of methods applied to finding the most efficient DMU [7] and [23]. As it can be seen, different models select different DMUs as the most efficient unit; this is precise because each model evaluates the most efficient DMU by means of an external criterion other than the data. Since this criterion is different in various models, different DMUs are selected as the most efficient unit.

Dissimilarities in the selection of the most efficient unit are the result of slight differences in calculations adopted in the proposed methods. This leads to divergent selection methods when searching among various extreme points of the PPS. Therefore, when finding the most efficient unit, whatever model and the computational process is involved, the proposed methods have no practical superiority over each other. In section 3, by some theorems and numerical examples, it will be shown that any model which finds extreme efficient units can also be used to find the most efficient one.

Table 1: The most efficient unit selected by different methods

Model	Most Efficienct FLD								
	5	7	10	12	14	15	16	17	19
Ertay et al (2006) [11]							*		
Amin & Toloo (2007) [3]							*		
Amin (2009) [1]							*		
Toloo & Nalchigar (2009) [24]					*				
Foroughi (2011) [13]			*						
Toloo (2012) [22]					*				
Wang & Jiang (2012) [28]			*						
Foroughi (2013) [14]			*						
Lam (2014) [16]			*						
Toloo (2015) [23]			*						
Toloo and Salahi (2018) [25]			*						

3 Preliminaries

Suppose the PPS contains n DMUs in the form of $DMU_j \equiv (x_j, y_j)$, where $x_j \in \mathbb{R}^m$ and $y_j \in \mathbb{R}^s$ are nonnegative input and output vectors, respectively. Evaluating $DMU_p$is done by solving the following two-step model:

$$\begin{array}{lll} \min & \theta & \\ \text{s.t.} & \sum_j x_{ij}\lambda_j \leq \theta x_{ip}, & i = 1, \dots, m, \\ & \sum_j y_{rj}\lambda_j \geq y_{rp}, & r = 1, \dots, s \\ & \lambda_j \geq 0, & j = 1, \dots, n \end{array} \tag{1a}$$

and

$$
\begin{array}{lll}
\max & \sum_i s_i^- + \sum_r s_r^+ & \\
\text{s.t.} & \sum_j x_{ij}\lambda_j + s_i^- = \theta^* x_{ip}, & i = 1, \dots, m, \\
& \sum_j y_{rj}\lambda_j - s_r^+ = y_{rp}, & r = 1, \dots, m, \\
& \lambda_j \geq 0, & j = 1, \dots, n, \\
& s_i^- \geq 0, & i = 1, \dots, m \\
& s_r^+ \geq 0, & r = 1, \dots, s,
\end{array}
\tag{1b}
$$

where θ^* in (1b) is the optimal objective function of (1a).
The above model is named the envelopment form of the CCR model. According to the above model, the efficiency of a DMU can be defined as follows.

Definition 3.1. Suppose that θ^* is the optimal objective function of (1a) and that (s^{-*}, s^{+*}) is the optimal solution of (1b).

(a). DMU_p is strong efficient if $\theta^* = 1$ in (1a) and $\sum_i s_i^{-*} + \sum_r s_r^{+*} = 0$ in (1b).

(b). DMU_p is weak efficient if $\theta^* = 1$ in (1a) and $\sum_i s_i^{-*} + \sum_r s_r^{+*} > 0$ in (1b).

(c). DMU_p is inefficient if $\theta^* < 1$.

There is another way for evaluating efficiency using the dual of model (1a)– Multiplier form of CCR model – as follows:

$$
\begin{array}{lll}
\max & uy_p & \\
\text{s.t.} & vx_p = 1, & \\
& uy_j - vx_j \leq 0, & j = 1, \dots, n, \\
& u \geq 0, & v \geq 0.
\end{array}
\tag{2}
$$

Based on model (2), there is an equivalent definition for efficiency [9].

Definition 3.2. (a). DMU_p is strong efficient if there exists at least one optimal solution of (2) like (u^*, v^*) such that

(i). $u^* y_p = 1$,

(ii). $u^* > 0$, $v^* > 0$.

(b). DMU_p is weak efficient if for each optimal solution of (2):

(i). $u^* y_p = 1$,

(ii). At least one component of (u^*, v^*) is zero in all optimal solutions.

(c). DMU_p is inefficient if $u^* y_p < 1$.

According to Definition 3.1, if DMU_P is inefficient, then it is dominated by a nonnegative combination of efficient DMUs; the set of such efficient DMUs is called the Reference Set. Traditionally, the Reference set is defined for *inefficient* DMUs; however, the definition can be extended to all DMUs of the PPS as follows:

Definition 3.3. For each DMU in the PPS, the Reference set is defined based on a two-step model ((1a) and (1b)) as follows:

$$E_p = \left\{ j | \lambda_j^* > 0 \text{ in some solution of (1b)} \right\}, \quad p \in \{1, \ldots, n\}.$$

It can be shown that all members of E_p are strong efficient [9][3]).

Theorem 3.4. *If DMU_p is strong efficient then $p \in E_p$.*

Proof. Trivial. $\square$

Using the above definitions and Theorem 3.4, all efficient DMUs in a PPS can be classified in the following three types.

Definition 3.5. (a). DMU_p is extreme (and strong) efficient if $E_p = \{p\}$.

(b). DMU_p is nonextreme (and strong) efficient if $p \in E_p$ and E_p is not singleton.

(c). DMU_p is weak efficient if $\theta^* = 1$ and $p \notin E_p$.

If DMU_p is an extreme (and strong) DMU then model ((1a) and (1b)) has a unique optima solution with $\theta^* = 1$ and $s^{-*} = 0$ and $s^{+*} = 0$; and the E_p is singleton. On the other hand, if DMU_p is a non-extreme (and strong) DMU then model ((1a) and (1b)) has alternative optimal solution with $\theta^* = 1$ and $s^{-*} = 0$ and $s^{+*} = 0$; but the E_p is not singleton.

As an example, Farell's frontier of 6 DMUs with two inputs and one output is depicted in Figure 1. In Table 2, the types of efficiency of each DMU and its Reference set are presented.

Table 2: Efficiency type and reference set

DMU	Efficiency type	Reference set
A	Extreme	$\{A\}$
B	Extreme	$\{B\}$
C	Extreme	$\{C\}$
D	Nonextreme	$\{B, C, D\}$
E	Weak	$\{C\}$
F	Inefficient	$\{A, B\}$

[3]Definition 3.3 is an extension to the definition 3·4 in [9].

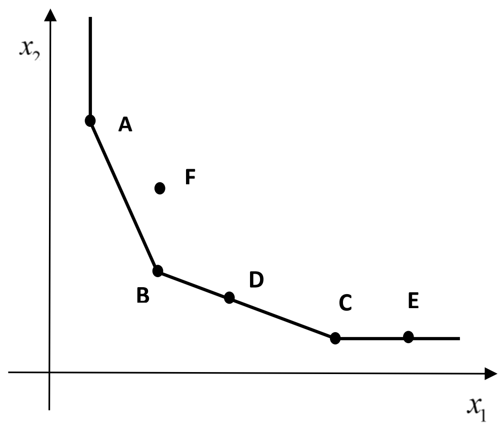

Figure 1: Classification of DMUs

The following definition indicates the conditions of the most efficient DMU.

Definition 3.6. (Toloo 2015 [23]) DMU_p is the most efficient if there exists a vector $(u, v) > 0$ such that $uy_p - vx_p = 0$ and $uy_j - vx_j < 0$ for all $j = 1, \ldots, n; j \neq p$.
It must be noted that if, for all $p = 1, \ldots, n$, the system

$$\begin{cases} uy_p - vx_p = 0, & \\ uy_j - vx_j < 0 & , \quad \begin{matrix} j = 1, \ldots, n \\ j \neq p \end{matrix} \\ u > 0, v > 0, & \end{cases} \tag{3}$$

does not have a solution, then the most efficient DMU does not exist. From Definition 3.6, it is evident that the most efficient DMU for a specific vector (u^*, v^*) is unique. Therefore, for each different solution of system (3), different most efficient units may be recognized.

Definition 3.6 and the above results are the bases of our argument about the equivalence of extreme efficient and most efficient DMUs. The details will be presented in the next section.

4 The Relationship Between Extreme Efficient and Most Efficient DMUs

In what follows, the relationship between extreme efficient and most efficient DMUs is going to be shown.

Theorem 4.1. *Any extreme efficient DMU is the most efficient and vice versa.*

Proof. Suppose that DMU_p is extreme efficient; then, the two-step model ((1a) and (1b)) has a unique optimal solution such that

$$\begin{cases} \theta^* = 1, & \\ {s^-}^* = 0, {s^+}^* = 0, & \\ \lambda_j^* = 0, & j \neq p, \\ \lambda_p^* = 1. & \end{cases}$$

According to the Strong Complementary Slackness theorem ([5]), model (2) has an optimal solution (u^*, v^*) such that

$$\begin{cases} u^* y_p - v^* x_p = 0, & \\ u^* y_j - v^* x_j < 0, & j \neq p, \\ u^* > 0, v^* > 0. & \end{cases} \tag{4}$$

By Definition 3.6, it is clear that DMU_p is the most efficient.
Suppose that DMU_p is the most efficient. First, we are going to show that it is strong efficient. By Definition 3.6, there exists a vector (u^*, v^*) such that (4) holds. Let $v^* x_p = k$. Then $(\frac{u^*}{k}, \frac{v^*}{k})$ satisfies (4). Therefore, according to Definition 3.2(a), $(\frac{u^*}{k}, \frac{v^*}{k})$ is the optimal solution to model (2) and DMU_p is strong efficient. Suppose that DMU_p is not extreme. Then using Definitions 3.3 and 3.5(b), there exists $\lambda_j > 0$ $j = 1, \ldots, n;$ $j \neq p)$ (with at least one positive λ_j) such that

$$\begin{pmatrix} x_p \\ y_p \end{pmatrix} = \sum_{\substack{j=1 \\ j \neq p}}^{n} \begin{pmatrix} x_j \\ y_j \end{pmatrix} \lambda_j. \tag{5}$$

So, the λ_j can be considered as a feasible solution to model (1a) with $\theta = 1$. Since $(u^*, v^*) > 0$ in equations (4), the slack variables in (1b) are all zero (Complementary Slackness theorem).
Since DMU_p is the most efficient, there is a supporting hyperplane $u^* y - v^* x = 0$, which is binding at DMU_p; nonetheless, other DMUs do not lie on it. As a result, we have

$$u^* y_p - v^* x_p = 0, \tag{6}$$
$$u^* y_j - v^* x_j < 0, \qquad j \neq p. \tag{7}$$

By multiplying (7) in λ_j and summing up, we get:

$$\sum_{\substack{j=1 \\ j \neq p}}^{n} \lambda_j (u^* y_j - v^* x_j) < 0.$$

This means $u^*(\sum_{\substack{j=1 \\ j \neq p}}^{n} \lambda_j y_j) - v^*(\sum_{\substack{j=1 \\ j \neq p}}^{n} \lambda_j x_j) < 0$. Now, it follows from (4) that

$$u^* y_p - v^* x_p < 0.$$

This contradicts with (6). Therefore, DMU_p is extreme efficient. $\square$
The above theorem shows that there is a one-to-one correspondence between extreme efficient and most efficient DMUs. Therefore, any method that can find an extreme efficient unit has actually found a most efficient unit.

Result 4.2. If the PPS does not contain any extreme efficient DMUs, then it does not contain any most efficient ones.

The following example illustrates this case in two and three dimensions.

Example 4.3. Consider four DMUs with one input and one output, whose data are shown in Table 3; their corresponding PPS is shown in Figure 2. As it can be seen in Figure 2, the PPS does not contain any extreme efficient DMU and there is no DMU satisfying Definition 3.5. Therefore, the most efficient DMU does not exist.

Table 3: Data of Example 4.3 with one input and one output

	DMU	**A**	**B**	**C**	**D**
x	(input)	1	2	2	3
y	(output)	2	4	1	3

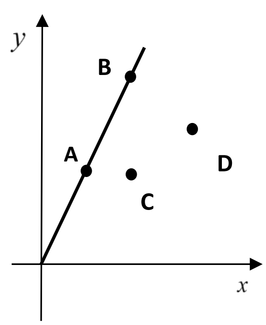

Figure 2: PPS of Table 3

Example 4.4. Consider two DMUs with two inputs and one output whose data are shown in Table 4 and whose corresponding PPS is shown in Figure 3.

Table 4: Data of Example 4.3 with two inputs and one output

	DMU	**A**	**B**
x_1	(input)	1	2
x_1	(input)	1	2
y	(output)	1	2

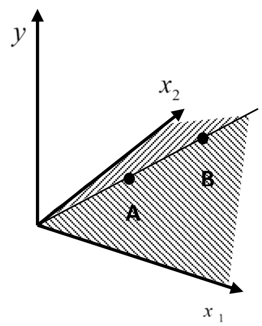

Figure 3: PPS of Table 4

As it can be seen in Figure 3, the PPS does not contain any extreme efficient DMU and there is no *DMU* satisfying Definition 3.5. Therefore, the most efficient DMU does not exist.

Now, we can show how to identify a most efficient unit by presenting a flowchart. The number of iterations of the approach depend on number of DMUs which is bounded. Moreover, in each iteration of the proposed method, some linear programming must be solved. We know that solving LPs is convergent [5], so the method can find the most efficient unit (if any exists) in finite steps.

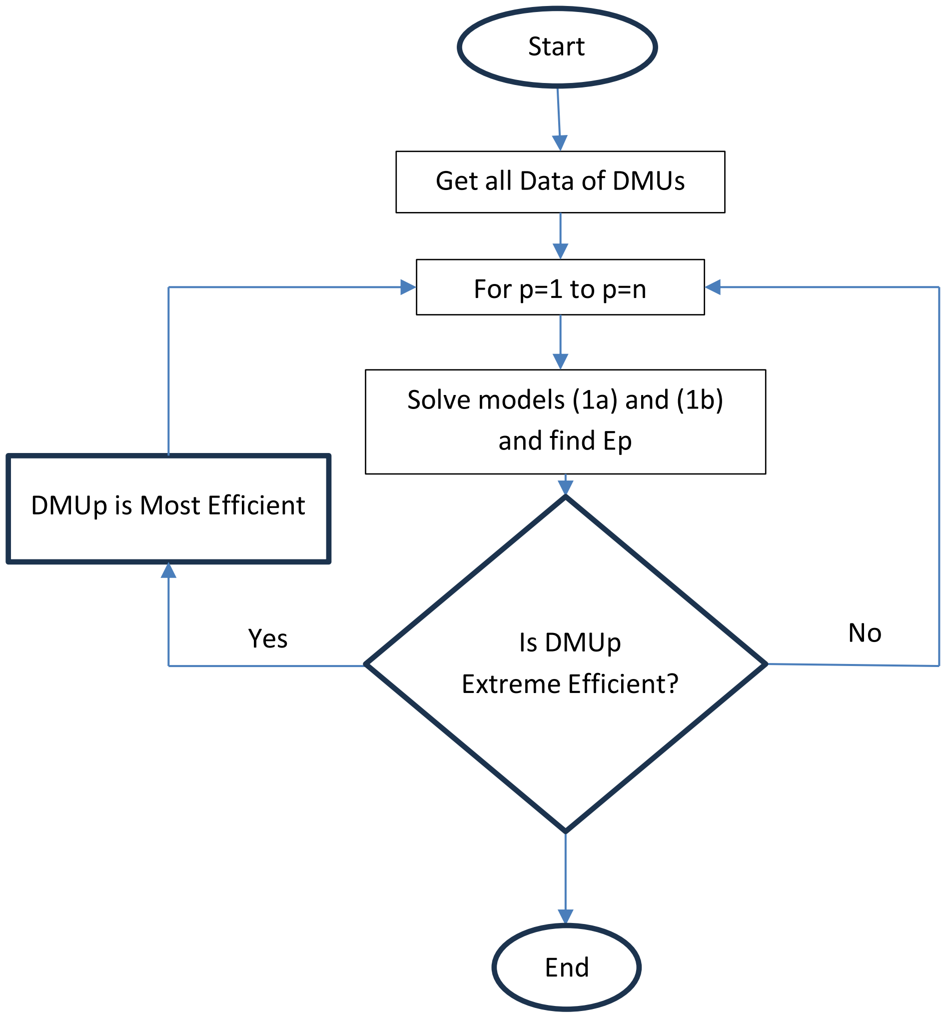

Figure 4: Finding most efficient DMUs

5 Numerical Examples

Example 5.1. Respective data from Figure 1 are depicted in Table 5. By Theorem 4.1 and Definition 3.6, we find (u, v)'s such that each of 3 extreme DMUs (A, B, and

C) can be determined as the most efficient. The related hyperplanes satisfying (3) are shown in the last column of Table 5. Therefore, the selection of the most efficient unit among the units A, B, and C is dependent on the used model, and there is no superiority between them.

Table 5: Data of Figure 1

DMU	Data			Weights			**Hyperplane**
	y	x_1	x_2	u	v_1	v_1	
A	1	1	4	1	0.5	0.125	$y - \frac{1}{2}x_1 - \frac{1}{8}x_2 = 0$
B	1	2	2	1	0.5	0.5	$y - \frac{1}{2}x_1 - \frac{1}{2}x_2 = 0$
C	1	4	1	1	0.125	0.5	$y - \frac{1}{8}x_1 - \frac{1}{2}x_2 = 0$
D	1	3	1.5	-	-	-	-
E	1	5	1	-	-	-	-
F	1	2	3	-	-	-	-

Example 5.2. This example deals with the data set from [11]. The data set is depicted in Table 6. The efficiency of DMUs was calculated by Model ((1a) and (1b)). Moreover, we used the method in [26] to classify the efficient DMUs into extreme and nonextreme, shown in the last column.

Here we show that each extreme DMU can be determined as the most efficient by finding a set of weights, which satisfy Definition 3.6. We use the conventional multiplier model to produce these weights shown in Table 7, which are not necessarily unique. It means that there might exist other weights satisfying (3). It indicates that all extreme efficient DMUs, including DMUs 5, 7, 10, 12, 14, 15, 16, and 17, can be chosen as the most efficient without having any superiority over each other. The determination of the most efficient DMU practically depends on an optimization method chosen to solve the proposed model. Finally, although DMU 19 is determined as efficient, and since it is not extreme, there is no (u, v) satisfying (3) to select it as the most efficient. Moreover, to make a comparison, Table 1 clearly shows that each method could only find one most efficient unit, while in this paper we have shown that all DMUs in the list have the potential to be selected as the most efficient one.

Table 6: Data set from Ertay et al [11]

c	**x__1**	**x__2**	**y__1**	**y__2**	**y__3**	**y__4**	**Efficiency**	**Classifying**
FLD1	20309.56	6405	0.4697	0.0113	0.041	30.89	0.985	
FLD2	20411.22	5393	0.438	0.0337	0.0484	31.34	0.988	
FLD3	20280.28	5294	0.4392	0.0308	0.0653	30.26	0.997	
FLD4	20053.2	4450	0.3776	0.0245	0.0638	28.03	0.949	
FLD5	19998.75	4370	0.3526	0.0856	0.0484	25.43	1	Extreme
FLD6	20193.68	4393	0.3674	0.0717	0.0361	29.11	0.973	
FLD7	19779.73	2862	0.2854	0.0245	0.0846	25.29	1	Extreme
FLD8	19831	5473	0.4398	0.0113	0.0125	24.8	0.857	
FLD9	19608.43	5161	0.2868	0.0674	0.0724	24.45	0.889	
FLD10	20038.1	6078	0.6624	0.0856	0.0653	26.45	1	Extreme
FLD11	20330.68	4516	0.3437	0.0856	0.0638	29.46	0.998	
FLD12	20155.09	3702	0.3526	0.0856	0.0846	28.07	1	Extreme
FLD13	19641.86	5726	0.269	0.0337	0.0361	24.58	0.776	
FLD14	20575.67	4639	0.3441	0.0856	0.0638	32.2	1	Extreme
FLD15	20687.5	5646	0.4326	0.0337	0.0452	33.21	1	Extreme
FLD16	20779.75	5507	0.3312	0.0856	0.0653	33.6	1	Extreme
FLD17	19853.38	3912	0.2847	0.0245	0.0638	31.29	1	Extreme
FLD18	19853.38	5974	0.4398	0.0337	0.0179	25.12	0.852	
FLD19	20355	17402	0.4421	0.0856	0.0217	30.02	1	Nonextreme

Table 7: Weights for most efficient DMUs

DMU	v_1	v_2	u_1	u_2	u_3	u_4
FLD5	0.005	0.001	0.001	12000	0.001	0.23
FLD7	0.02	1	1	4	1	1
FLD10	20	2	100	20	1	4
FLD12	0.005	0.001	60	12000	950	0.23
FLD14	0.005	0.001	6	900	0.001	0.75
FLD15	0.0001	0.00001	42	70.78269	0.001	2.40422
FLD16	2.0	0.5	8	8	0.5	4
FLD17	1	1	1	1	1	1

6 Conclusion

In Data Envelopment Analysis (DEA), a core objective lies in identifying the most efficient Decision Making Units (DMUs). These DMUs serve as benchmarks for evaluating the relative performance of others. However, determining the absolute best performer, the most efficient DMU, can be a complex task.

Traditionally, various methods based on linear, integer, and nonlinear programming have been employed to identify this most efficient unit. However, each approach may lead to different most efficient unit. Additionally, these methods can struggle when the efficient frontier, also known as the Production Possibility Set (PPS), has more than one most efficient unit.

Contributions of the present paper has shed light on a crucial equivalence: the most efficient unit and the extreme efficient unit coincide. This paves the way for a more streamlined approach to finding the most efficient performer. By focusing on identifying the extreme efficient unit, we can effectively locate the most efficient DMU as well.

A major advantage of the result obtained in this paper is that any method that identifies an extreme efficient DMU also identifies a most efficient unit. Therefore, the applicability of this concept is not model-dependent. In this paper, we show that the extreme efficiency can be easily identified based on the optimal solutions of the multiplier model, which is a linear model, and this will lead to finding the most efficient unit.

In conclusion, the focus on identifying extreme efficient units through linear models

presents a compelling approach for pinpointing the most efficient DMU in DEA. This method simplifies the process, leverages readily solvable models, and offers valuable insights for performance evaluation within a set of DMUs.
Two main topics can be considered as directions for future research in this subject. First, given that identifying extreme efficient DMUs solves the problem of finding the most efficient units, developing a direct method with appropriate complexity for identifying all extreme efficient units would be a valuable contribution. This method should be computationally efficient and applicable to a wide range of DEA models and problem sizes. Second, according to the definition of the most efficient unit, there is a possibility of multiple solutions. Therefore, ranking and determining the best unit among the most efficient units is another important research question. This could involve developing new ranking methods that consider various factors, such as the relative efficiency of the units, their input and output levels, and their impact on the overall efficiency of the system.

References

[1] G. R. Amin, Comments on finding the most efficient DMUs in DEA: An improved integrated model, *Computers and Industrial Engineering*, 56(4)(2009), 1701-1702.

[2] G. R. Amin, M. Toloo, and B. Sohrabi, An improved MCDM DEA model for technology selection, *International Journal of Production Research*, 44(13)(2006), 2681-2686.

[3] G. R. Amin, and M. Toloo, Finding the most efficient DMUs in DEA: An improved integrated model, *Computers and Industrial Engineering*, 52(1)(2007), 71-77.

[4] R. D. Banker, A. Charnes, and W. W. Cooper, Some models for estimating technical and scale inefficiency in data envelopment analysis, *Management Science*, 30(1984), 1078-1092.

[5] M. Bazaraa, J. Jarvis, and H. Sherali, *Linear Programming and Network Flows*, Wiley-Blackwell, (2010).

[6] A. Charnes, and W. W. Cooper, Programming with linear fractional functionals, *Naval Research Logistics Quarterly*, 9(1962), 67-88.

[7] A. Charnes, W. W. Cooper, and E. Rhodes, Measuring the efficiency of decision making units, *European Journal of Operational Research*, 2(6)(1978), 429-444.

[8] A. Charnes, W. W. Cooper, and R. M. Thrall, Classifying and characterizing efficiencies and inefficiencies in data development analysis, *Operations Research Letters*, 5(3)(1986), 105-110.

[9] W. W. Cooper, L. M. Seiford, and K. Tone, Data envelopment analysis: A comprehensive text with models, applications, references and DEA-solver software, *Springer Science & Business Media*, (2007).

[10] T. Ertay, and D. Ruan, Data envelopment analysis based decision model for optimal operator allocation in CMS, *European Journal of Operational Research*, 164(3)(2005), 800-810.

[11] T. Ertay, D. Ruan, and U. R. Tuzkaya, Integrating data envelopment analysis and analytic hierarchy for the facility layout design in manufacturing systems, *Information Sciences*, 176(3)(2006), 237-262.

[12] M. J. Farrell, The measurement of productive efficiency, *Journal of the Royal Statistical Society*, Series A (General), 120(3)(1957), 253-290.

[13] A. A. Foroughi, A new mixed integer linear model for selecting the best decision making units in data envelopment analysis, *Computers and Industrial Engineering*, 60(4)(2011), 550-554.

[14] A. A. Foroughi, A revised and generalized model with improved discrimination for finding most efficient DMUs in DEA, *Applied Mathematical Modeling*, 37(6)(2013), 4067-4074.

[15] E. E. Karsak, and S. S. Ahiska, Practical common weight multi-criteria decision-making approach with an improved discriminating power for technology selection, *International Journal of Production Research*, 43(8)(2005), 1537-1554.

[16] K. F. Lam, In the determination of the most efficient decision making unit in data envelopment analysis, *Computers and Industrial Engineering*, 79(2014), 76-84.

[17] M. Nejati, M. Rostamy-Malkhalifeh, R. Kargar, and R. Mehrjoo, The most efficient unit in data envelopment analysis using genetic algorithm, *Theory of Approximation and Applications*, 12(2)(2019), 73-81.

[18] Z. Noori, H. Zhiani Rezai, A. Davoodi, and S. Kordrostami, Finding the most efficient DMU in DEA: A model-free procedure, *Journal of Control and Optimization in Applied Mathematics*, (2022).

[19] V. S. Özsoy, M. Örkcü, and H. H. Örkcü, A minimax approach for selecting the overall and stage-level most efficient unit in two stage production processes, *Annals of Operations Research*, 300(2021), 137-169.

[20] V. S. Özsoy, H. H. Örkcü, and M. Örkcü, A simplistic approach without epsilon to choose the most efficient unit in data envelopment analysis, *Expert Systems with Applications*, (168)(2021), 114472.

[21] P. Ravanos, and G. Karagiannis, In search for the most preferred solution in value efficiency analysis, *Journal of Productivity Analysis*, 58(2022), 203-220.

[22] M. Toloo, On finding the most BCC-efficient DMU: A new integrated MIP–DEA model, *Applied Mathematical Modeling*, 36(11)(2012), 5515-5520.

[23] M. Toloo, Alternative minimax model for finding the most efficient unit in data envelopment analysis, *Computers and Industrial Engineering*, 81(2015), 186-194.

[24] M. Toloo, and S. Nalchigar, A new integrated DEA model for finding most BCC efficient DMU, *Applied Mathematical Modeling*, 33(1)(2009), 597-604.

[25] M. Toloo, and M. Salahi, A powerful discriminative approach for selecting the most efficient unit in DEA, *Computers and Industrial Engineering*, 115(2018), 269-277.

[26] M. Toloo, S. Nalchigar, and B. Sohrabi, Selecting most efficient information system projects in presence of user subjective opinions: a DEA approach, *Central European Journal of Operations Research*, 26(2018), 1027-1051.

[27] J. Vakili, Finding the most efficient decision-making unit in data envelopment analysis, *Journal of New Researches in Mathematics*, 2(8)(2017), 37-44 (In Persian).

[28] Y. M. Wang, and P. Jiang, Alternative mixed integer linear programming models for identifying the most efficient decision-making unit in data envelopment analysis, *Computers and Industrial Engineering*, 62(2)(2012), 546-553.

Zahra Noori
Ph.D Student of Applied Mathematics
Department of Mathematics
Lahijan Branch
Islamic Azad University
Lahijan, Iran
E-mail: armita@yahoo.com

Hamed Zhiani Rezai
Assistant Professor of Applied Mathematics
Department of Mathematics
Mashhad Branch
Islamic Azad University
Mashhad, Iran
E-mail: hz_rezai@hotmail.com

Alireza Davoodi
Associate Professor of Applied Mathematics
Department of Mathematics
Neyshabur Branch
Islamic Azad University
Neyshabur, Iran
E-mail: alizd@yahoo.com

Sohrab Kordrostami
Full Professor of Applied Mathematics
Department of Mathematics
Lahijan Branch
Islamic Azad University

Lahijan, Iran
E-mail: kordrostami@liau.ac.ir