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Properties of the Complete Lift of Riemannian Connection for Flat Manifolds

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Abstract. Here, we deals with a special lift $\tilde{g}$ of a Riemannian metric g on a manifold M to the tangent bundle TM of M. This lift is defined as a linear combination of certain well-known lifts of g. The main results of the paper are proved under the condition that the Riemannian manifold $(M\ g)$ is flat, in fact the Riemannian connection of the metric $\tilde{g}$ coincides with the complete lift of the Riemannian connection of the metric g. In addition, the main objectives of this study is to find the necessary and sufficient conditions such that some of the lift vector fields with this general metric to be *parallel*.

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1 Introduction

Let M be an n-dimensional Riemannian manifold with a metric $g = g_{ij}dx^i dx^j$ and TM be its tangent bundle. It turns out that the manifold TM has some Riemannian metrics known in literature as: complete lift metric or g_2, diagonal lift metric or g_1+g_3, lift metric g_2+g_3 and lift metric $g_1 + g_2$, where $g_1 := g_{ij}dx^i dx^j$, $g_2 := 2g_{ij}dx^i \delta y^j$ and $g_3 := g_{ij}\delta y^i \delta y^j$ are all bilinear differential forms defined globally on TM. For vertical, complete, and horizontal lift vector fields, the following results are widely known as mentioned in [11]:

The *vertical distribution* on TM is *parallel* with respect to the Levi-Civita connection of metric g_2.

The *horizontal distribution* is *parallel* with respect to the Levi-Civita connection of metric g_2 if and only if the metric on M is locally Euclidean.

The complete lift of a vector field on M to TM is *concurrent* with respect to the metric g_2 if and only if the vector field on M is *concurrent*.

The tangent bundle TM over a Riemannian manifold M is *locally flat* with respect to metric $g_1 + g_3$ if and only if M is *locally flat*.

In addition in [9] we have:

The vertical, complete, and horizontal lifts of a vector field on M to TM are *parallel* with respect to the metric g_2 and $g_1 + g_3$ if and only if the vector field given on M is *parallel*.

The general Riemannian lift metric $\tilde{g}$ on TM is a combination of the diagonal lift and complete lift metrics and it is, in some senses, more general than those used previously [5]. The use of lifts has led to some results in Riemann-Finsler geometry [13]. Here, we prove that:

Theorem: *The complete, vertical and horizontal distributions on TM are parallel with respect to the Riemannian connection of metric $\tilde{g}$.*

Theorem: *A vector field on M is parallel if and only if its complete(vertical, horizontal) lift to TM is parallel with respect to metric $\tilde{g}$. And the complete lift of a vector field on M is concurrent if and only if it is concurrent.*

Theorem: *The tangent bundle TM is locally flat with respect to the metric $\tilde{g}$ if and only if M is locally flat.*

2 Preliminaries

A Riemannian metric on a smooth manifold M is a covariant tensor field g of type $(0,2)$ which is symmetric($g(X,Y) = g(Y,X)$), and positive definite($g(X,X) > 0$ if $X \neq 0$). A Riemannian metric thus determines an inner product on each tangent space T_pM, which is typically written as $< X, Y >:= g(X,Y)$ for all $X, Y \in T_pM$ where $p \in M$. A manifold together with a given Riemannian metric is called a Riemannian manifold. Let (M, g) be a real n-dimensional Riemannian manifold and (U, x) be a local chart on M, where the induced coordinates of the point $p \in U$ are denoted by its image on $\mathbb{R}^n$, $x(p)$ or briefly (x^i).[2]

Suppose that TM is the tangent bundle of M and π is the natural projection from TM to M. Consider $\pi_{*v} : T_vTM \mapsto T_{\pi(v)}M$ and let us put:

$$\mathrm{Ker}\pi_{*v} = \{z \in T_vTM | \pi_{*v}(z) = 0\}, \qquad \forall v \in TM.$$

The vertical vector bundle or *vertical distribution* on TM is defined by $VTM = \bigcup_{v \in TM} \mathrm{Ker}\pi_{*v}$. A non-linear connection or a *horizontal distribution* on TM is a complementary distribution HTM for VTM on TTM. [1]

Using the induced coordinates (x^i, y^i) on TM, where x^i and y^i are called respectively the position and the direction of a point on TM, the researchers introduce the local field of frames $\{\partial_i, \partial_{\bar{i}}\}$ on TTM where $\partial_i := \frac{\partial}{\partial x^i}$ and $\partial_{\bar{i}} := \frac{\partial}{\partial y^i}$. Let (M, g) be a Riemannian manifold with components $g_{ij} \in C^\infty(M)$, where $C^\infty(M)$ is the set of all C^∞ functions from M to $\mathbb{R}$. If we put $X_h = \partial_h - y^a \Gamma^m_{ah} \partial_{\bar{m}}$ and $X_{\bar{h}} = \partial_{\bar{h}}$ then $\{X_h, X_{\bar{h}}\}$

is the adapted local field of frames of TM and $\{dx^h, \delta y^h\}$ be the dual basis of $\{X_h, X_{\bar{h}}\}$, where $\delta y^h = dy^h + y^a \Gamma^h_{ai} dx^i$ and Γ^k_{ij} are the Christoffel symbols. Here, the indices $i, j, h, \dots$ and $\bar{i}, \bar{j}, \bar{h}, \dots$ in relations run over the range $1, 2, \dots n$.[3]

By means of the above mentioned dual basis, it is known that $g_1 := g_{ij} dx^i dx^j$, $g_2 := 2g_{ij} dx^i \delta y^j$, and $g_3 := g_{ij} \delta y^i \delta y^j$ are all bilinear differential forms defined globally on TM. It turns out that the manifold TM has four Riemannian metrics $g_2 = 2g_{ij} dx^i \delta y^j$, $g_1 + g_2 = g_{ij} dx^i dx^j + 2g_{ij} dx^i \delta y^j$, $g_1 + g_3 = g_{ij} dx^i dx^j + g_{ij} \delta y^i \delta y^j$, and $g_2 + g_3 = 2g_{ij} dx^i \delta y^j + g_{ij} \delta y^i \delta y^j$. [11]
The tensor field

$$\tilde{g} = ag_1 + bg_2 + cg_3,$$

on TM has the components:

$$\begin{pmatrix} ag_{ij} & bg_{ij} \\ bg_{ij} & cg_{ij} \end{pmatrix},$$

with respect to the dual basis of the adapted frame of TM, where a, b, and c are certain positive real numbers. From linear algebra, we have $det\tilde{g} = (ac - b^2)^n detg^2$. Therefore, the tensor field $\tilde{g}$ is a pseudo-Riemannian metric on TM if $ac - b^2 \neq 0$ and is a Riemannian metric on TM if $ac - b^2 > 0$. [5]

Now, suppose that the set of all p-covariant and q-contravariant tensors on M ($\otimes^q_p M$) is denoted by $\otimes M$ and the set of all 1-forms and all vector fields on M are denoted by $\Omega^1(M)$ and $\chi(M)$ respectively.
A section on M is a map $S : M \to TM$ such that $\pi o S = Id$, and the set of all sections on M is denoted by $\Gamma(M)$.

Let $\pi : E \to M$ be a vector bundle over a manifold M and $\Gamma(E)$ denote the space of smooth sections on E. A *connection* on E is a map

$$\nabla : \chi(M) \times \Gamma(E) \to \Gamma(E),$$

written by $(X, Y) \mapsto \nabla_X Y$, satisfying the following properties:

1) $\nabla_{fX_1 + gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y, \qquad f, g \in C^\infty(M),$

2) $\nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2, \qquad a, b \in \mathbb{R}^n,$

3) $\nabla_X(fY) = f\nabla_X Y + X(f)Y.$

A linear or *affine connection* on M is a connection on TM i.e, a map

$$\nabla : \chi(M) \times \chi(M) \to \chi(M),$$

satisfying properties (1)-(3) in the definition given above.
Let M be a Riemannian manifold, then there exists a unique affine connection ∇ on M which is symmetric($\nabla_X Y - \nabla_Y X = [X, Y]$) and compatible with the Riemannian metric($\nabla_X < Y, Z > = < \nabla_X Y, Z > + < Y, \nabla_X Z >$). This affine connection is called Levi-Civita or *Riemannian connection.*
The Christoffel symbols Γ^k_{ij} of ∇ with respect to a local frame $\{\partial_i\}$ is defined by $\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$.

3 Tensor Lift

3.1 Vertical lift

The vertical lift of $f \in C^\infty(M)$ is defined by $f^V = fo\pi$.
Vertical lift of a vector field X on M(with components X^h) to TM has the components:

$$X^V : \begin{pmatrix} 0 \\ X^h \end{pmatrix},$$

with respect to the induced coordinates on TM.
Suppose that ω is a one form on M, the vertical lift ω^V of the 1-form ω is defined by $\omega^V = (\omega_i)^V (dx^i)^V$, with respect to constant coefficients. [11]

The vertical lifts can extend to a unique algebraic isomorphism of the tensor algebra $\otimes M$ into the tensor algebra $\otimes TM$ with respect to constant coefficients, by the conditions: $(P \otimes Q)^V = P^V \otimes Q^V$ and $(P + R)^V = P^V + R^V$, where P, Q, and R are arbitrary elements of $\otimes M$. [11]

The vertical lift of a 2-covariant tensor $g \in \otimes_2^0 M$ with local components g_{ij} has components of the form:

$$g^V : \begin{pmatrix} g_{ij} & 0 \\ 0 & 0 \end{pmatrix},$$

with respect to the induced coordinates on TM.

3.2 Complete lift

For a function f on M, the complete lift of f is regarded in natural way as a function on TM which is denoted by f^C and defined in a coordinate neighborhood U of M, where the local expression $f^C = \partial f := y^i \partial_i f$ with respect to the induced coordinates in $\pi^{-1}(U)$.
The complete lift of a vector field X on M is defined by $X^C.f^C = (X.f)^C$, where $f \in C^\infty(M)$. Thus the complete lift X^C of X(with components X^h on M) has the components:

$$X^C : \begin{pmatrix} X^h \\ \partial X^h \end{pmatrix},$$

with respect to the induced coordinates on TM. A distribution is a subbundle of the tangent bundle. By *complete distribution* on TM we mean a distribution whose sections are complete lifts of vector field on M.
The complete lift of a one form ω on M is defined by $\omega^C(X^C) = (\omega(X))^C$ for all $X \in TM$ and has components of the form $\omega^C = (\partial\omega_i, \omega_i)$ where ω_i are the components of ω.
The complete lifts are extended to a unique algebraic isomorphism of the tensor algebra $\otimes M$ into the tensor algebra $\otimes TM$ with respect to constant coefficients, by the conditions: $(P \otimes Q)^C = P^C \otimes Q^V + P^V \otimes Q^C$ and $(P+R)^C = P^C + R^C$, where P, Q, and R are arbitrary elements of $\otimes M$. [11]
The complete lift of a 2-covariant tensor $g \in \otimes_2^0 M$ with local components g_{ij} has components of the form:

$$g^C : \begin{pmatrix} \partial g_{ij} & g_{ij} \\ g_{ij} & 0 \end{pmatrix},$$

with respect to the induced coordinates on TM.

Lemma 3.1. *The complete lift of a Riemannian metric g coincide with the Riemannian metric g_2.*

Proof.

$$\begin{aligned} g_2 &= 2g_{ij}dx^i\delta y^j \\ &= 2g_{ij}dx^i(dy^j + \Gamma^j_t dx^t) \\ &= 2g_{ij}dx^i dy^j + 2g_{ij}y^k\Gamma^j_{kt}dx^i dx^t \\ &= 2g_{ij}dx^i dy^j + y^k(\Gamma^l_{ki}g_{lj} + \Gamma^l_{kj}g_{il})dx^i dx^j \\ &= 2g_{ij}dx^i dy^j + y^k\partial_k g_{ij}dx^i dx^j \\ &= 2g_{ij}dx^i dy^j + \partial g_{ij}dx^i dx^j, \end{aligned}$$

so g_2 has components of the form:

$$g_2 : \begin{pmatrix} \partial g_{ij} & g_{ij} \\ g_{ij} & 0 \end{pmatrix},$$

with respect to the induced coordinates on TM, where $\Gamma^h_i = y^j\Gamma^h_{ji}$ and $\Gamma^l_{ki}g_{lj} + \Gamma^l_{kj}g_{il} = \partial_k g_{ij}$. [6] □

3.3 Horizontal lift

For an arbitrary type tensor field

$$S = S^{j_1\dots j_q}_{i_1\dots i_p}\partial_{j_1}\otimes\cdots\otimes\partial_{j_q}\otimes dx^{i_1}\otimes\cdots\otimes dx^{i_p},$$

on M, the tensor field ∇S on TM is

$$\nabla S = (y^l\nabla_l S^{j_1\dots j_q}_{i_1\dots i_p})\partial_{\bar{j}_1}\otimes\cdots\otimes\partial_{\bar{j}_q}\otimes dx^{i_1}\otimes\cdots\otimes dx^{i_p}.$$

The horizontal lift S^H is defined by

$$S^H = S^C - \nabla S.$$

The horizontal lift of a 2-covariant tensor $g \in \otimes^0_2 M$ with local components g_{ij} has components of the form:

$$g^H : \begin{pmatrix} \Gamma^t_j g_{ti} + \Gamma^t_i g_{jt} & g_{ij} \\ g_{ij} & 0 \end{pmatrix},$$

with respect to the induced coordinates on TM.
Especially, for all $f \in C^\infty(M)$ we have $f^H = f^C - \nabla f$. On the other hand, since $\nabla f = y^l \nabla_l f = y^l \partial_l f$ we obtain $\nabla f = f^C$, therefore $f^H = 0$.
The horizontal lift of a vector field X on M is defined by $X^H = X^C - \nabla X$, so it has the components:

$$X^H : \begin{pmatrix} X^h \\ -\Gamma_i^h X^i \end{pmatrix},$$

with respect to the induced coordinates on TM.
The horizontal lift of the product of two tensors P and Q in $\otimes M$ is [11]

$$(P \otimes Q)^H = P^H \otimes Q^V + P^V \otimes Q^H.$$

Lemma 3.2. [10] *If g is a Riemannian metric and ∇ the Riemannian connection determined by g on M, then g^C and g^H coincide with respect to ∇.*

3.4 M-lift

We are in position to introduced a new useful mixed lift g^M for a 2-covariant tensor $g \in \otimes_2^0 M$ with local components g_{ij} by

$$g^M = a(g_{ij})^V (dx^i)^V \otimes (dx^j)^V + c(g_{ij})^V (dx^i)^H \otimes (dx^j)^H,$$

where $a, c \in R^+$.

4 Connection Lift

If ∇ is a linear connection on M, the total covariant derivative of a tensor field $S \in \otimes_q^l M$ is a $\binom{l}{q+1}$-tensor field

$$\nabla S : \Omega^1(M) \times \cdots \times \Omega^1(M) \times \chi(M) \times \cdots \times \chi(M) \to C^\infty(M),$$

given by

$$\nabla S(\omega^1, \ldots, \omega^l, Y_1, \ldots, Y_q, X) = \nabla_X S(\omega^1, \ldots, \omega^l, Y_1, \ldots, Y_q),$$

where $Y_i, X \in \chi(M)$ and $\omega^i \in \Omega^1(M)$.
It is necessary to recall that the Riemannian connection is a metric connection i.e $\nabla g = 0$. [6]

- A vector field X on M is *parallel* if and only if its total covariant derivative ∇X vanishes identically.
- A distribution D on M is *parallel* if $\nabla_X \Gamma(D) \subseteq \Gamma(D)$ for any $X \in \Gamma(TM)$.

Let ∇ be a Riemannian connection on M with coefficients Γ^k_{ij}. The *Riemannian curvature tensor* is defined by

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z, \qquad \forall X, Y, Z \in TM.$$

Locally

$$R^m_{ijk} = \partial_i \Gamma^m_{jk} - \partial_j \Gamma^m_{ik} + \Gamma^m_{ia} \Gamma^a_{jk} - \Gamma^m_{ja} \Gamma^a_{ik},$$

where $R(\partial_i, \partial_j)\partial_k = R^m_{ijk} \partial_m$. [6]

- A Riemannian manifold (M, g) is *locally flat* if and only if its Riemannian curvature tensor vanishes identically.

Let ∇ be a linear connection on TM, the *torsion tensor* of ∇ on TM is defined by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \qquad \forall X, Y \in TM.$$

It is obvious by [4] that Riemannian connection is *torsion free* i.e $T = 0$.

Lemma 4.1. *[7] An affine connection ∇ has the following properties.*

$$\nabla_X (T \otimes S) = (\nabla_X T) \otimes S + T \otimes (\nabla_X S),$$

$$\nabla_X f = X.f,$$

$$\nabla_{\partial_j} dx^h = -\Gamma^h_{ji} dx^i,$$

where $f \in C^\infty(M)$, $X \in TM$ and $T, S \in \otimes M$.

4.1 Complete lift

If $\tilde{\nabla}$ is a linear connection on TM, then the Christoffel symbols with respect to the $\tilde{\nabla}$ is defined as follows:

$$\tilde{\nabla}_{\partial_i} \partial_j = \tilde{\Gamma}^m_{ji} \partial_m + \tilde{\Gamma}^{\bar{m}}_{ji} \partial_{\bar{m}}, \quad \tilde{\nabla}_{\partial_{\bar{i}}} \partial_j = \tilde{\Gamma}^m_{j\bar{i}} \partial_m + \tilde{\Gamma}^{\bar{m}}_{j\bar{i}} \partial_{\bar{m}},$$

$$\tilde{\nabla}_{\partial_i}\partial_{\bar{j}} = \tilde{\Gamma}^{m}_{\bar{j}i}\partial_m + \tilde{\Gamma}^{\bar{m}}_{\bar{j}i}\partial_{\bar{m}}, \quad \tilde{\nabla}_{\partial_{\bar{i}}}\partial_{\bar{j}} = \tilde{\Gamma}^{m}_{\bar{j}\bar{i}}\partial_m + \tilde{\Gamma}^{\bar{m}}_{\bar{j}\bar{i}}\partial_{\bar{m}}.$$

There exists a unique affine connection ∇^C on TM which satisfies: [11]

$$\nabla^C_{X^C} Y^C = (\nabla_X Y)^C, \qquad \forall X, Y \in TM,$$

so

$$\tilde{\Gamma}^h_{ji} = \Gamma^h_{ji}, \qquad \tilde{\Gamma}^h_{j\bar{i}} = 0, \qquad \tilde{\Gamma}^h_{\bar{j}i} = 0, \qquad \tilde{\Gamma}^h_{\bar{j}\bar{i}} = 0,$$

$$\tilde{\Gamma}^{\bar{h}}_{ji} = \partial\Gamma^h_{ji}, \qquad \tilde{\Gamma}^{\bar{h}}_{j\bar{i}} = \Gamma^h_{ji}, \qquad \tilde{\Gamma}^{\bar{h}}_{\bar{j}i} = \Gamma^h_{ji}, \qquad \tilde{\Gamma}^{\bar{h}}_{\bar{j}\bar{i}} = 0.$$

It is easy to verify that $\tilde{\Gamma}^A_{CB}$, which is denoted by the preceding relations, determines an affine connection globally on TM. This affine connection is called the *complete lift* of the affine connection ∇ to TM and is denoted by ∇^C.

Proposition 4.2. *[12] If ∇ is the Riemannian connection of a manifold M with respect to a Riemannian metric g, then ∇^C is the Riemannian connection of TM with respect to g^C.*

Proposition 4.3. *[11] The Riemannian connection ∇^C has the following properties.*

$$\nabla^C_{X^V} K^V = 0,$$

$$\nabla^C_{X^V} K^C = (\nabla_X K)^V,$$

$$\nabla^C_{X^C} K^V = (\nabla_X K)^V,$$

$$\nabla^C_{X^C} K^C = (\nabla_X K)^C,$$

for all tensor field K on M and $X \in TM$.

So, in general, $\nabla^C K^V = (\nabla K)^V$ and $\nabla^C K^C = (\nabla K)^C$.

Proposition 4.4. *[12] If T and R are respectively the torsion and the curvature tensors of ∇, then T^C and R^C are respectively the torsion and the curvature tensors of ∇^C.*

4.2 Horizontal lift

The horizontal lift ∇^H of an affine connection ∇ on M to TM is defined by the conditions:

$$\nabla^H_{X^V}Y^V = 0,\ \nabla^H_{X^V}Y^H = 0,\ \nabla^H_{X^H}Y^V = (\nabla_X Y)^V,\ \nabla^H_{X^H}Y^H = (\nabla_X Y)^H,$$

for any $X, Y \in TM$. [11]

Note : It is worth saying that, in general, ∇^H is not unique. [11]

Proposition 4.5. *[11] The horizontal lift ∇^H has the following properties.*

$$\nabla^H_{X^C}(dx^h)^V = -X^j\Gamma^h_{ji}(dx^i)^V,$$

$$\nabla^H_{X^C}(dx^h)^H = -X^j\Gamma^h_{ji}(dx^i)^H,$$

$$\nabla^H_{X^C}K^V = (\nabla_X K)^V,$$

$$\nabla^H_{X^C}K^H = (\nabla_X K)^H,$$

$$\nabla^H_{X^V}K^V = 0,$$

$$\nabla^H_{X^V}K^H = 0.$$

for all tensor field $K \in \otimes M$ and $X \in TM$.

5 Main Results

In the following results, assume that (M, g) is a Riemannian manifold with respect to the Riemannian connection ∇.

Lemma 5.1. *For a tensor field g^M on TM, $\nabla^H_{X^C}g^M = (\nabla_X g)^M$.*

Proof.

$$\begin{aligned}
\nabla^H_{X^C} g^M &= \nabla^H_{X^C}(a(g_{ij})^V (dx^i)^V \otimes (dx^j)^V + c(g_{ij})^V (dx^i)^H \otimes (dx^j)^H) \\
&= a(X.g_{ij})^V (dx^i)^V \otimes (dx^j)^V + a(g_{ij})^V \nabla^H_{X^C}(dx^i)^V \otimes (dx^j)^V \\
&+ a(g_{ij})^V (dx^i)^V \otimes \nabla^H_{X^C}(dx^j)^V \\
&+ c(X.g_{ij})^V (dx^i)^H \otimes (dx^j)^H + c(g_{ij})^V \nabla^H_{X^C}(dx^i)^H \otimes (dx^j)^H \\
&+ c(g_{ij})^V (dx^i)^H \otimes \nabla^H_{X^C}(dx^j)^H \\
&= a(X.g_{ij})^V (dx^i)^V \otimes (dx^j)^V \\
&- a(g_{ij})^V X^k \Gamma^i_{kl} (dx^l)^V \otimes (dx^j)^V \\
&- a(g_{ij})^V X^k \Gamma^j_{kl} (dx^l)^V \otimes (dx^i)^V \\
&+ c(X.g_{ij})^V (dx^i)^H \otimes (dx^j)^H \\
&- c(g_{ij})^V X^k \Gamma^i_{kl} (dx^l)^H \otimes (dx^j)^H \\
&- c(g_{ij})^V X^k \Gamma^j_{kl} (dx^l)^H \otimes (dx^i)^H.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\nabla_X g &= \nabla_{X^k \partial_k} g_{ij} dx^i \otimes dx^j \\
&= X^k \nabla_{\partial_k} g_{ij} dx^i \otimes dx^j \\
&= X^k (\partial_k g_{ij} dx^i \otimes dx^j + g_{ij} \nabla_{\partial_k} dx^i \otimes dx^j + g_{ij} dx^i \otimes \nabla_{\partial_k} dx^j) \\
&= (X.g_{ij}) dx^i \otimes dx^j - g_{ij} X^k \Gamma^i_{kl} dx^l \otimes dx^j - g_{ij} X^k \Gamma^j_{kl} dx^l \otimes dx^i.
\end{aligned}$$

So its M-lift is

$$\begin{aligned}
(\nabla_X g)^M &= ((X.g_{ij}) dx^i \otimes dx^j - g_{ij} X^k \Gamma^i_{kl} dx^l \otimes dx^j \\
&\quad - g_{ij} X^k \Gamma^j_{kl} dx^l \otimes dx^i)^M \\
&= ((X.g_{ij}) dx^i \otimes dx^j)^M - (g_{ij} X^k \Gamma^i_{kl} dx^l \otimes dx^j)^M \\
&- (g_{ij} X^k \Gamma^j_{kl} dx^l \otimes dx^i)^M \\
&= a(X.g_{ij})^V (dx^i)^V \otimes (dx^j)^V + c(X.g_{ij})^V (dx^i)^H \otimes (dx^j)^H \\
&- (a(g_{ij})^V X^k \Gamma^i_{kl} (dx^l)^V \otimes (dx^j)^V \\
&+ c(g_{ij})^V X^k \Gamma^i_{kl} (dx^l)^H \otimes (dx^j)^H) \\
&- (a(g_{ij})^V X^k \Gamma^j_{kl} (dx^i)^V \otimes (dx^l)^V \\
&+ c(g_{ij})^V X^k \Gamma^j_{kl} (dx^l)^H \otimes (dx^i)^H).
\end{aligned}$$

Hence

$$\nabla^H_{X^C} g^M = (\nabla_X g)^M.$$

□

Proposition 5.2. *The horizontal lift ∇^H of $\tilde{g}$ is a metric connection.*

Proof.

$$\begin{aligned}\nabla^H_{X^C}\tilde{g} &= \nabla^H_{X^C}(ag_1 + bg_2 + cg_3)\\ &= \nabla^H_{X^C}(ag_1 + cg_3) + b\nabla^H_{X^C} g_2,\end{aligned}$$

in addition, by Lemmas 3.1 and 3.2, $g_2 = g^C = g^H$, hence

$$\begin{aligned}&= \nabla^H_{X^C} g^M + b\nabla^H_{X^C} g^H\\ &= (\nabla_X g)^M + b(\nabla_X g)^H\\ &= 0,\end{aligned}$$

where the last equality comes from the compatibility of Riemannian connection property. □

Remark 5.3. [11] The complete lift ∇^C and the horizontal lift ∇^H of an affine connection ∇ on M coincide, if and only if ∇ is of zero curvature.

In the following results, assume that the Riemannian connection ∇ with respect to the Riemannian manifold (M, g) is of zero curvature.

Proposition 5.4. *The Riemannian connections of metrics $\tilde{g}$ and g_2 coincide.*

Proof. The Riemannian connection with respect to the metric $\tilde{g}$ is ∇^C because

$$\begin{aligned}\nabla^C_{X^C}\tilde{g} &= \nabla^C_{X^C}(ag_1 + bg_2 + cg_3)\\ &= \nabla^C_{X^C}(ag_1 + cg_3) + b\nabla^C_{X^C} g_2.\end{aligned}$$

Taking Remark 5.3 and Lemma 3.1 into consideration, this is equal to

$$\begin{aligned}&= \nabla^H_{X^C}(ag_1 + cg_3) + b\nabla^C_{X^C} g^C\\ &= \nabla^H_{X^C} g^M + b\nabla^C_{X^C} g^C\\ &= (\nabla_X g)^M + b(\nabla_X g)^C\\ &= 0.\end{aligned}$$

Thus, ∇^C is compatible with respect to the metric $\tilde{g}$. In addition by Proposition 4.4, ∇^C is torsion-free, therefore ∇^C is the Riemannian connection of $\tilde{g}$.
Additionally, based on Proposition 4.2, ∇^C is the Riemannian connection of $g^C = g_2$. Note that the Riemannian connection is unique, thus the Riemannian connection of the metric $\tilde{g}$ and g_2 coincide. $\square$

Theorem 5.5. *The complete, vertical, and horizontal distributions on TM are parallel with respect to the Riemannian connection of metric $\tilde{g}$.*

Proof. For any $X, Y \in TM$ with components X^h and Y^h,

$$
\begin{aligned}
\nabla^C_{Y^C} X^C &= \nabla^C_{Y^j\partial_j + \partial Y^j \partial_{\bar{j}}}(X^h\partial_h + \partial X^h \partial_{\bar{h}}) \\
&= Y^j(\nabla^C_{\partial_j} X^h\partial_h + \nabla^C_{\partial_j}\partial X^h\partial_{\bar{h}}) + \partial Y^j(\nabla^C_{\partial_{\bar{j}}} X^h\partial_h + \nabla^C_{\partial_{\bar{j}}}\partial X^h\partial_{\bar{h}}) \\
&= Y^j(\partial_j X^h\partial_h + X^h\nabla^C_{\partial_j}\partial_h + \partial_j(\partial X^h)\partial_{\bar{h}} + \partial X^h\nabla^C_{\partial_j}\partial_{\bar{h}}) \\
&+ \partial Y^j(\partial_{\bar{j}} X^h\partial_h + X^h\nabla^C_{\partial_{\bar{j}}}\partial_h + \partial_{\bar{j}}(\partial X^h)\partial_{\bar{h}} + \partial X^h\nabla^C_{\partial_{\bar{j}}}\partial_{\bar{h}}) \\
&= Y^j(\partial_j X^h\partial_h + X^h(\Gamma^k_{jh}\partial_k + \partial\Gamma^k_{jh}\partial_{\bar{k}}) + \partial_j(y^t\partial_t X^h)\partial_{\bar{h}} \\
&+ \partial X^h\Gamma^k_{jh}\partial_{\bar{k}}) + \partial Y^j(X^h\Gamma^k_{jh}\partial_{\bar{k}} + \partial_{\bar{j}}(y^t\partial_t X^h)\partial_{\bar{h}}) \\
&= Y^j(\partial_j X^k + X^h\Gamma^k_{jh})\partial_k + (Y^j X^h\partial\Gamma^k_{jh} + Y^j y^t\partial_j\partial_t X^k \\
&+ Y^j\partial X^h\Gamma^k_{jh} + \partial Y^j\partial_{\bar{j}} y^t\partial_t X^k)\partial_{\bar{k}} \\
&= Y^j(\partial_j X^k + X^h\Gamma^k_{jh})\partial_k + (Y^j y^t(X^h\partial_t\Gamma^k_{jh} + \partial_t X^h\Gamma^k_{jh}) \\
&+ Y^j y^t\partial_j\partial_t X^k + \partial Y^j(\partial_j X^k + \Gamma^k_{jh}X^h))\partial_{\bar{k}} \\
&= Y^j(\partial_j X^k + X^h\Gamma^k_{jh})\partial_k + (Y^j y^t\partial_t(X^h\Gamma^k_{jh}) + Y^j y^t\partial_t\partial_j X^k \\
&+ \partial Y^j(\partial_j X^k + \Gamma^k_{jh}X^h))\partial_{\bar{k}} \\
&= Y^j(\partial_j X^k + X^h\Gamma^k_{jh})\partial_k + (Y^j\partial(X^h\Gamma^k_{jh}) + Y^j\partial(\partial_j X^k) \\
&+ \partial Y^j(\partial_j X^k + \Gamma^k_{jh}X^h))\partial_{\bar{k}} \\
&= Y^j(\partial_j X^k + X^h\Gamma^k_{jh})\partial_k + (Y^j\partial(\partial_j X^k + X^h\Gamma^k_{jh}) + \partial Y^j(\partial_j X^k \\
&+ \Gamma^k_{jh}X^h))\partial_{\bar{k}},
\end{aligned}
$$

so, $\nabla^C_{Y^C} X^C$ has components:

$$\begin{pmatrix} Y^j(\partial_j X^k + X^h\Gamma^k_{jh}) \\ \partial(Y^j(\partial_j X^k + X^h\Gamma^k_{jh})) \end{pmatrix},$$

with respect to the induced coordinates. This shows assertion for complete distribution.
For vertical lift X^V,

$$\begin{aligned}
\nabla^C_{Y^C} X^V &= \nabla^C_{Y^k\partial_k + \partial Y^k \partial_{\bar{k}}} X^h \partial_{\bar{h}} \\
&= Y^k \nabla^C_{\partial_k} X^h \partial_{\bar{h}} + \partial Y^k \nabla^C_{\partial_{\bar{k}}} X^h \partial_{\bar{h}} \\
&= Y^k(\partial_k X^h \partial_{\bar{h}} + X^h \Gamma^{\bar{l}}_{k\bar{h}} \partial_{\bar{l}} + X^h \Gamma^l_{k\bar{h}} \partial_l) \\
&+ \partial Y^k (\partial_{\bar{k}} X^h \partial_{\bar{h}} + X^h \Gamma^l_{\bar{k}\bar{h}} \partial_l + X^h \Gamma^{\bar{l}}_{\bar{k}\bar{h}} \partial_{\bar{l}}) \\
&= Y^k(\partial_k X^h \partial_{\bar{h}} + X^h \Gamma^l_{kh} \partial_{\bar{l}}) \\
&= Y^k(\partial_k X^l + X^h \Gamma^l_{kh}) \partial_{\bar{l}} \\
&= (Y^k \nabla_k X^l) \partial_{\bar{l}},
\end{aligned}$$

where $\nabla_k X^l = \partial_k X^l + X^h \Gamma^l_{kh}$ are the components of ∇X. Therefore, $\nabla^C_{Y^C} X^V$ has the components:

$$\begin{pmatrix} 0 \\ Y^k \nabla_k X^l \end{pmatrix},$$

with respect to the induced coordinates, and this is the assertion for vertical distribution.
Take a horizontal vector field X^H on TM, that is, a vector field with

local components $\tilde{X}^A$ satisfying $\Gamma_i^h \tilde{X}^i + \tilde{X}^{\bar{h}} = 0$,

$$
\begin{aligned}
\nabla^C_{Y^C} X^H &= \nabla^C_{Y^j\partial_j + \partial Y^j \partial_{\bar{j}}}(\tilde{X}^h \partial_h + \tilde{X}^{\bar{h}} \partial_{\bar{h}}) \\
&= \nabla^C_{Y^j\partial_j + \partial Y^j \partial_{\bar{j}}}(\tilde{X}^h \partial_h - \Gamma_i^h \tilde{X}^i \partial_{\bar{h}}) \\
&= Y^j(\nabla^C_{\partial_j} \tilde{X}^h \partial_h - \nabla^C_{\partial_j} \Gamma_i^h X^i \partial_{\bar{h}}) \\
&+ \partial Y^j(\nabla^C_{\partial_{\bar{j}}} \tilde{X}^h \partial_h - \nabla^C_{\partial_{\bar{j}}} \Gamma_i^h \tilde{X}^i \partial_{\bar{h}}) \\
&= Y^j(\partial_j \tilde{X}^h \partial_h + \tilde{X}^h \nabla^C_{\partial_j} \partial_h - \partial_j(\Gamma_i^h \tilde{X}^i)\partial_{\bar{h}} - \Gamma_i^h \tilde{X}^i \nabla^C_{\partial_j} \partial_{\bar{h}}) \\
&+ \partial Y^j(\partial_{\bar{j}} \tilde{X}^h \partial_h + \tilde{X}^h \nabla^C_{\partial_{\bar{j}}} \partial_h - \partial_{\bar{j}}(\Gamma_i^h \tilde{X}^i)\partial_{\bar{h}} - \Gamma_i^h \tilde{X}^i \nabla^C_{\partial_{\bar{j}}} \partial_{\bar{h}}) \\
&= Y^j \partial_j \tilde{X}^h \partial_h + Y^j \tilde{X}^h \Gamma^k_{jh} \partial_k - Y^j \partial_j(\Gamma_i^h \tilde{X}^i)\partial_{\bar{h}} \\
&- Y^j \Gamma_i^h \tilde{X}^i \Gamma^k_{jh} \partial_{\bar{k}} + \partial Y^j \tilde{X}^h \Gamma^k_{jh} \partial_{\bar{k}} \\
&= (Y^j \partial_j \tilde{X}^k + Y^j \tilde{X}^h \Gamma^k_{jh})\partial_k \\
&+ (-Y^j \partial_j(\Gamma_i^k \tilde{X}^i) - Y^j \Gamma_i^h \tilde{X}^i \Gamma^k_{jh} + \partial Y^j \tilde{X}^h \Gamma^k_{jh})\partial_{\bar{k}} \\
&= (Y^j \partial_j \tilde{X}^k + Y^j \tilde{X}^h \Gamma^k_{jh})\partial_k \\
&+ (-Y^j \partial_j \Gamma_i^k \tilde{X}^i - Y^j \Gamma_i^k \partial_j \tilde{X}^i - Y^j \Gamma_i^k \tilde{X}^h \Gamma^i_{jh} + \partial Y^j \tilde{X}^h \Gamma^k_{jh})\partial_{\bar{k}} \\
&= (Y^j \partial_j \tilde{X}^k + Y^j \tilde{X}^h \Gamma^k_{jh})\partial_k \\
&- Y^j \Gamma_i^k(\partial_j \tilde{X}^i + \Gamma^i_{jh} \tilde{X}^h)\partial_{\bar{k}} + (\partial Y^j \tilde{X}^h \Gamma^k_{jh} - Y^j \partial_j \Gamma_i^k \tilde{X}^i)\partial_{\bar{k}} \\
&= (Y^j \partial_j \tilde{X}^k + Y^j \tilde{X}^h \Gamma^k_{jh})\partial_k - Y^j \Gamma_i^k(\partial_j \tilde{X}^i + \Gamma^i_{jh} \tilde{X}^h)\partial_{\bar{k}} \\
&+ (y^t \partial_t Y^j \tilde{X}^i \Gamma^k_{ji} - Y^j y^t \partial_j \Gamma^k_{ti} \tilde{X}^i)\partial_{\bar{k}} \\
&= (Y^j \partial_j \tilde{X}^k + Y^j \tilde{X}^h \Gamma^k_{jh})\partial_k - Y^j \Gamma_i^k(\partial_j \tilde{X}^i + \Gamma^i_{jh} \tilde{X}^h)\partial_{\bar{k}} \\
&+ y^t \tilde{X}^i(\partial_t Y^j \Gamma^k_{ji} - Y^j \partial_j \Gamma^k_{ti})\partial_{\bar{k}},
\end{aligned}
$$

so, $\nabla^C_{Y^C} X^H$ has the components:

$$
\begin{pmatrix}
Y^j(\partial_j \tilde{X}^k + \Gamma^k_{jh} \tilde{X}^h) \\
-\Gamma_i^k Y^j(\partial_j \tilde{X}^i + \Gamma^i_{jh} \tilde{X}^h) + y^t R^k_{tji} Y^j \tilde{X}^i
\end{pmatrix},
$$

with respect to the induced coordinates. Since the Riemannian connection ∇ is of zero curvature, $R^k_{tji} = 0$. Therefore the assertion for horizontal distribution is proved. □

Theorem 5.6. *A vector field on M is parallel if and only if its complete(vertical, horizontal) lift to TM is parallel with respect to metric $\tilde{g}$.*

Proof. Suppose that X is a vector field on M with the components X^h, and $\nabla^C_B \tilde{X}^A$ are the components of $\nabla^C X^C$. Along the same line, $\nabla^C_B \hat{X}^A$ are the components of $\nabla^C X^V$, and $\nabla^C_B \bar{X}^A$ are the components of $\nabla^C X^H$. Then by Proposition 4.3 for the complete lift:

$$\begin{aligned}\nabla^C X^C &= (\nabla_j X^h \partial_h \otimes dx^j)^C \\ &= \partial\nabla_j X^h (\partial_h)^V \otimes (dx^j)^V + \nabla_j X^h (\partial_h)^C \otimes (dx^j)^V \\ &+ \nabla_j X^h (\partial_h)^V \otimes (dx^j)^C \\ &= \partial\nabla_j X^h \partial_{\bar{h}} \otimes dx^j + \nabla_j X^h \partial_h \otimes dx^j \\ &+ \nabla_j X^h \partial_{\bar{h}} \otimes dy^j,\end{aligned}$$

so

$$\nabla^C_B \tilde{X}^A = \begin{pmatrix} \nabla_j X^h & 0 \\ \partial\nabla_j X^h & \nabla_j X^h \end{pmatrix}.$$

For vertical lift:

$$\begin{aligned}\nabla^C X^V &= (\nabla_j X^h \partial_h \otimes dx^j)^V \\ &= \nabla_j X^h (\partial_h)^V \otimes (dx^j)^V \\ &= \nabla_j X^h \partial_{\bar{h}} \otimes dx^j,\end{aligned}$$

hence

$$\nabla^C_B \hat{X}^A = \begin{pmatrix} 0 & 0 \\ \nabla_j X^h & 0 \end{pmatrix}.$$

Finally, for horizontal lift, since ∇ is of zero curvature, by Proposition

4.5:

$$\begin{aligned}
\nabla^C X^H &= (\nabla_j X^h \partial_h \otimes dx^j)^H \\
&= (\nabla_j X^h \partial_h)^H \otimes dx^j + (\nabla_j X^h \partial_h)^V \otimes (dx^j)^H \\
&= (\nabla_j X^h)^H (\partial_h)^V \otimes dx^j + (\nabla_j X^h)^V (\partial_h)^H \otimes dx^j \\
&+ \nabla_j X^h (\partial_h)^V \otimes (\Gamma^j_i dx^i + dy^j) \\
&= \nabla_j X^h (\partial_h - \Gamma^t_h \partial_{\bar{t}}) \otimes dx^j \\
&+ \nabla_j X^h (\partial_{\bar{h}}) \otimes (\Gamma^j_i dx^i + dy^j) \\
&= \nabla_j X^h \partial_h \otimes dx^j - (\Gamma^h_t \nabla_j X^t) \partial_{\bar{h}} \otimes dx^j \\
&+ (\Gamma^t_j \nabla_t X^h) \partial_{\bar{h}} \otimes dx^j + \nabla_j X^h \partial_{\bar{h}} \otimes dy^j,
\end{aligned}$$

so

$$\nabla^C_B \bar{X}^A = \begin{pmatrix} \nabla_j X^h & 0 \\ -\Gamma^h_t \nabla_j X^t + \Gamma^t_j \nabla_t X^h & \nabla_j X^h \end{pmatrix}.$$

From these matrices we understand that if a vector field$X$ on M is *parallel*, then $\nabla_j X^h = 0$. Therefore it is obvious that $\nabla^C_B \tilde{X}^A = 0$ ($\nabla^C_B \hat{X}^A = 0,\ \nabla^C_B \bar{X}^A = 0$).

Conversely, if complete(vertical, horizontal) lift of a vector field on TM is *parallel*, then $\nabla^C_B \tilde{X}^A = 0$($\nabla^C_B \hat{X}^A = 0,\ \nabla^C_B \bar{X}^A = 0$) and from the matrices, $\nabla_j X^h = 0$. □

- The vector field X on M is said to be *concurrent* if its component X^h has the following relation:

$$\nabla_j X^h := \nabla_{\partial_j} X^h = k\delta^h_j,$$

where k is constant and δ^h_j is Kronecker delta. [8]

Corollary 5.7. *The complete lift of a vector field on M is concurrent with respect to the metric $\tilde{g}$ if and only if it is concurrent.*

Proof. It is clear by

$$\nabla^C_B \tilde{X}^A = \begin{pmatrix} \nabla_j X^h & 0 \\ \partial \nabla_j X^h & \nabla_j X^h \end{pmatrix},$$

which is motivated by the proof of Theorem 5.6. □

Theorem 5.8. *The tangent bundle TM is locally flat with respect to metric $\tilde{g}$ if and only if M is locally flat.*

Proof. Let R be a Riemannian curvature tensor with components R^h_{kji} and R^C be its complete lift, then the components of R^C have the following relations: [11]

$$\tilde{R}^h_{kji} = R^h_{kji} \ , \ \tilde{R}^{\bar{h}}_{kji} = \partial R^h_{kji} \ , \ \tilde{R}^{\bar{h}}_{kj\bar{i}} = R^h_{kji} \ \ , \ \tilde{R}^{\bar{h}}_{k\bar{j}i} = R^h_{kji} \ , \ \tilde{R}^{\bar{h}}_{\bar{k}ji} = R^h_{kji}.$$

Moreover, suppose that all the others are zero, with respect to the induced coordinates on TM. Since by Proposition 5.4 the connections with respect to the metrics $\tilde{g}$ and g_2 coincide, $R^C = 0$ iff $R = 0$. □

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