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Two-stage Network Models in DEA and DEA-R with Desirable and Undesirable Outputs

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Abstract. This paper proposes two-stage network models within the frameworks of Data Envelopment Analysis (DEA) and DEA-R, designed to accommodate both desirable and undesirable outputs. These non-radial models are developed under the assumption of constant returns to scale. By employing a multi-objective linear programming approach within non-radial additive DEA and DEA-R models, this study introduces a novel method for identifying suitable benchmarks for decision-making units, even in the presence of undesirable outputs [10] to [12]. The proposed models evaluate decision-making units based on the level of inefficiency within two-stage networks, with the calculation of total

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inefficiency in DEA and DEA-R serving as a criterion for assessing units in two-phase networks with undesirable outputs. The paper concludes with a case study on storage centers for electric power supply equipment in Fars province, illustrating the practical application and effectiveness of the proposed models.

AMS Subject Classification: 90C05; 90C08.

Keywords and Phrases: Two-stage Network Models; Undesirable Outputs; Desirable Outputs; DEA-R.

1 Introduction

In data envelopment analysis (DEA), we first calculate the efficiency of our decision-making units (DMUs) using envelopment and multiplicative forms of linear programming models, and then obtain suitable benchmarks using the mentioned models again. Due to its specific structure, the DEA technique, which is a non-parametric method and one that measures the efficiency of DMUs in relation to other units as "relative efficiency", has gained popularity as a research method.

Charnes et al. [2] introduced the CCR model in 1978, and Banker et al. [1] extended the model to variable returns to scale (VRS) technology in 1984. Thus, non-radial additive models were also proposed to identify efficient and inefficient units.

Since classical DEA models produce multiple outputs by consuming multiple inputs (It is necessary that the DMUs are homogenous, and the number of inputs and outputs is dependent on the number of DMUs [1]), the models can be considered as a black box. Network DEA models were proposed based on the idea presented by Fare and Grosskopf. Hence, the relationship between the efficiency of the first and second network stages and the overall network efficiency became a focus of investigation [2]. Chen et al. [3] propose an additive efficiency decomposition method for evaluating two-stage processes using Data Envelopment Analysis (DEA). In two-stage systems, outputs from the first stage serve as inputs for the second. The paper introduces a model that breaks down overall inefficiency into the inefficiencies of each stage, providing more detailed insights into performance. This approach is helpful for complex systems like supply chains or healthcare, where understanding inefficiencies at each stage can guide improvements [4] also, Du, J., Liang, L., Chen, Y., Cook, W.D., & Zhu, J. (2011) present a bargaining game model to measure the performance of two-stage network structures using DEA. The model applies the Nash bargaining solution to allocate efficiency scores between two interconnected stages, ensuring a fair distribution of performance gains. It is beneficial for analyzing systems where stages depend on each other, such as supply chains or multi-department processes. This approach offers a balanced evaluation of efficiency for both stages [5].

In the section discussing DEA-R, several key references are introduced to provide context and background for the development and application of the DEA-R model. These references typically highlight the evolution of ratio-based approaches in Data Envelopment Analysis and the introduction of innovative methodologies to improve efficiency evaluation. Some notable references include:
Despic, O., Despic, M., & Paradi, J.C. (2007) introduce the DEA-R model, a ratio-based approach to evaluating efficiency in Data Envelopment Analysis (DEA). This model uses input-output ratios to compare the efficiency of decision-making units, making it more interpretable than traditional DEA. The paper highlights the mathematical relationship between DEA-R and standard DEA. It demonstrates its applicability in sectors like banking, where ratio-based assessments are common [4] and Gerami, J., Mozaffari, M. R., Wanke, P.F., & Correa, H. [15] propose a novel slacks-based model for measuring efficiency and super-efficiency within the DEA-R framework. This approach refines the traditional DEA-R model by incorporating slacks, allowing for a more accurate representation of inefficiencies. The model also extends to handle super-efficiency, which is important for differentiating efficient units. The paper offers advancements in evaluating efficiency, with applications in fields where ratio-based analysis is crucial. See [13, 14], and [23].

Considering that resources are almost always limited, this vital issue requires attention [19]. The paper by Omrani, Yang, and Teplova presents a significant contribution to the field of efficiency analysis, particularly in the context of the road transport sector. The integration of PCA with a two-stage network DEA model that combines top-down and bottom-up approaches offers a powerful tool for evaluating and improving efficiency in sectors characterized by shared resources and undesirable outputs. This study not only advances the methodological capabilities of DEA models but also provides practical insights that can be used to enhance the sustainability and efficiency of the road transport sector [30]. The paper by Xiao Shi, Emrouznejad, and Wenqi Yu introduces a powerful and innovative SBM network DEA model that effectively measures the overall efficiency of operational processes involving both series and parallel components while also accounting for undesirable outputs. This model represents a significant advancement in the field of efficiency analysis, offering both methodological innovations and practical applications [9]. Its relevance to industries concerned with sustainability and environmental impact makes it a valuable tool for policymakers and managers alike, engaging the audience in the topic. Overall, this study provides a comprehensive framework for assessing efficiency in complex, multi-stage processes, contributing to the ongoing development of DEA methodologies [32].

In many organizations, managers are faced with uncontrollable outputs that require management. For example, in the case of outputs such as industrial waste, a higher production rate would result in a higher rate of waste production, which makes it critical to be able to control such undesirable outputs. Researchers have also considered undesirable outputs in two-stage network DEA models [26].

In the last two decades, DEA-R models, which combine DEA and Ratio Analysis, have been used to evaluate the performance of DMUs.

DEA models with ratio data have been modified by Emrouznejad and Amin in 2007 and later by Hatami and Toloo [8]. In these models, the input and output data are in the form of ratios, but the numerators and denominators of these fractions are defined and available. Meanwhile, DEA with ratio data was extended to cases with deterministic ratio data (where only the results of the fractions are available rather than the numerator and denominator values) by Podinovski et al. in 2015 and 2019.

In recent years, DEA-R models have emerged as an important extension of traditional Data Envelopment Analysis (DEA). Unlike traditional DEA models, which often utilize ratio data directly, DEA-R models focus on evaluating the performance of decision-making units (DMUs) through the ratios of inputs and outputs. This distinction allows DEA-R models to offer a more detailed assessment of efficiency, particularly in contexts where performance metrics are expressed as ratios.

Several significant studies have contributed to the development and application of DEA-R models. Notably, Despic et al. [4] and Wei et al. [34] have made foundational contributions in this area. Wei et al. [35, 36] initially addressed the issue of pseudo-inefficiency in DEA models and introduced input-oriented DEA-R models to tackle this problem [29, 31, 33]. They further advanced the field by proposing measures of comparative efficiency and comparative super-efficiency within DEA-R, illustrating these methods through an applied study of medical centers in Taiwan.

In the same year, Liu et al. [16] presented DEA models that do not require explicit inputs. Their research on 15 Chinese research centers provided a basis for subsequent studies by Lotfi et al. (2011) and Mozaffari et al. (2011), who developed cost and revenue efficiency models within the DEA-R framework [17, 18, 20]. Mozaffari et al. [26] explored the hybrid nature of DEA-R models, focusing on two-stage networks and supply chains using genetic algorithms [21]. Furthermore, Wanke et al. [33] made significant contributions by studying multi-stage networks with stochastic data.

Other studies have examined various applications of DEA-R models and their integration with traditional DEA models in network structures, as highlighted in references [23, 25, 28, 29]. Notably, the paper by Mozaffari et al. [22] delves into advanced topics within the DEA-R framework, specifically exploring ”efficient surfaces.” This study extends traditional DEA by providing insights into evaluating the efficiency of DMUs when dealing with ratio data, enhancing our understanding and improvement of operational performance.

Ratio-based DEA models differ from traditional DEA models in that they focus on the efficiency evaluation of inputs and outputs presented as ratios. This approach often provides more nuanced insights into performance, particularly in industries such as finance, healthcare, and production, where ratios are commonly used to measure efficiency. For instance, financial ratios like return on assets or healthcare metrics such as cost per patient offer a clearer picture of performance compared to absolute values alone.

Mozaffari et al. [24] introduce an innovative interactive method for identifying benchmarks within ratio-based DEA models. This approach actively involves decision-makers in the refinement process of efficiency analysis. By incorporating feedback from these key stakeholders, the method ensures that the benchmarks are not only theoretically robust but also practically relevant. This interactive process aligns the benchmarks more closely with the operational realities and specific contexts of the decision-making units (DMUs), leading to more actionable and relevant performance assessments.

The paper makes a significant contribution to the field of DEA by introducing an interactive, ratio-based approach to benchmark identification. This method not only advances the theoretical understanding of ratio-based DEA models but also provides a practical tool for improving efficiency in real-world applications. The interactive aspect of the approach ensures that the benchmarks identified are relevant to the specific context of each DMU, making it a valuable resource for decision-makers seeking to enhance operational performance. Overall, the study offers a novel and practical framework that can be widely applied across different sectors where ratio-based performance metrics are used.

Studies have been conducted on DEA-R regarding the SBM model, cost and revenue efficiency in DEA and DEA-R, and sustainability factors in DEA-R.

The general purpose of the current study is first to formulate two-stage network models in DEA and DEA-R and then propose non-radial additive models while considering undesirable outputs, which is of great significance for DMU performance evaluation. In this respect, the study's contribution lies in the use of non-radial models in a two-stage network DEA-R with undesirable outputs. Although the control of undesirable outputs depends on the timely and reasonable decisions of managers, the models proposed in DEA and DEA-R can play an important role.

The general aim of this study is to determine the overall inefficiency of decision-making units (DMUs) in a two-stage network using DEA and DEA-R models. Since non-radial models are crucial for distinguishing between efficient and inefficient units, this study first identifies the inefficient units within a two-stage network and then sets targets for improvement. The key contribution of this study lies in assessing overall inefficiency in a two-stage network while accounting for undesirable outputs in both DEA and DEA-R. DEA-R models possess a hybrid property that is particularly effective for calculating inefficiency in two-stage networks.

The use of DEA-R models in the last two decades has been based on the idea proposed by Despic et al. in 2007. On the other hand, the ratio analysis models proposed by Fernandez et al. in 1994 became the basis for efficiency calculation based on output-to-input ratios or vice versa [7].

The rest of the article is structured as follows:
Section two presents the preliminaries of DEA and DEA-R. In section three, we model a two-stage network in DEA and DEA-R with undesirable outputs, and section four provides an applied study on storage centers for electric power supply equipment in Fars province.

2 Preliminaries of DEA and DEA-R

In this section, assume n DMUs consume m inputs to produce s outputs. These units are denoted by DMU_j, where $j = 1, \{\dots, n\}$, and they consume the input vector $X_j = (x_{1j}, ..., x_{mj})$ to produce the output vector$Y_j = (y_{1j}, ..., y_{sj})$. Now, the additive model (1) for the unit under evaluation $O \in \{1, ..., n\}$ is presented as follows:

$$
\begin{aligned}
&\max \left\{ \sum_{i=1}^{m} \alpha_i + \sum_{r=1}^{s} \beta_r \right\} \\
&s.t. \ \sum_{j=1}^{n} \mu_j x_{ij} + \alpha_i = x_{i0}, \quad i = 1, \dots, m, \\
&\sum_{j=1}^{n} \mu_j y_{rj} - \beta_r = y_{ro}, \qquad r = 1, \dots, s, \\
&\mu_j \geq 0, \alpha_i \geq 0, \beta_r \geq 0, \ j = 1, \dots, n.
\end{aligned} \tag{1}
$$

Theorem 2.1. *The model (1) is always feasible.*

Proof. To demonstrate the feasibility of model (1), we proceed as follows:

- I. **Consideration of Model (1):** We begin by analyzing model (1). Let us denote the decision-making unit DMU_o as $DMU_o = (X_o, Y_o)$, $o \in \{1, ..., n\}$ that X_o and Y_o represent the input and output vectors, respectively.
- II. **Assumption of Variables:** Assume the variables for model (1) are defined as follows:

  $$(\mu_j = e_A, \alpha = 0, \beta = 0)\,.$$

  Here, μ_j represents a vector, α and β are vectors associated with the inputs and outputs of DMU_o, respectively.
- III. **Verification of Constraints:** We must ensure that the constraints of model (1) are met. Specifically: $x_{iA} = x_{iA}$ and $y_{rA} = y_{rA}$.
- IV. **Conclusion:** Since the constraints are inherently satisfied, model (1) is feasible.

□

Definition 2.2. DMU_o, is additive-efficient in the model (1) whenever Z_1^*=0.

The model (1) is a linear programming problem used to evaluate DMU_o in "constant returns to scale" (CRS) technology.

Next, we present a non-radial DEA-R model for the evaluation of DMU_o as follows:

$$\begin{aligned}
&\max \left\{ \sum_{i=1}^{m} \sum_{r=1}^{s} \rho_{ir} \right\} \\
&s.t. \quad \sum_{j=1}^{n} \mu_j (y_{rj}/x_{ij}) - \rho_{ir} = (y_{ro}/x_{io}),\ i = 1, \ldots, m,\ r = 1, \ldots, s, \\
&\sum_{j=1}^{n} \mu_j = 1,\ \mu_j \geq 0, \quad j = 1, \ldots, n, \\
&\rho_{ir} \geq 0,\ i = 1, \ldots, m, r = 1, \ldots, s.
\end{aligned} \tag{2}$$

Theorem 2.3. *The model (2) is always feasible.*

Proof. To demonstrate the feasibility of model (2), we proceed as follows:

I. **Consideration of Model (2):** We begin by analyzing model (1). Let us denote the decision-making unit DMU_o as $DMU_o = (X_o/Y_o)$, $o \in \{1, ..., n\}$ that X_o and Y_o represent the input and output vectors, respectively.

II. **Assumption of Variables:** Assume the variables for model (2) are defined as follows:

$$(\mu_j = e_o, \rho = 0).$$

Here, μ_j represents a vector, ρ are vectors associated with the inputs and outputs of DMU_o, respectively.

III. **Verification of Constraints:** We must ensure that the constraints of model (2) are met. Specifically: $Y_j/X_j = Y_o/X_o$. So, the constraints satisfy, therefore Model (2) is feasible.

□

Definition 2.4. *DMU_o is additive-efficient in the model (2) whenever Z_2^*=0.*

Since DEA-R models are divided into the two categories of radial and non-radial, the model (2) is formulated based on an output-oriented DEA-R model that calculates the inefficiency of DMU_O with the aim of maximizing the slack variables.

Generally, the relationship between efficiency in output-oriented DEA and efficiency in input-oriented DEA-R models is of great importance.

For further reading on the subject of DEA and DEA-R, refer to the following: [6].

3 Non-radial Two-stage Network in DEA and DEA-R

In this section, based on non-radial models in DEA and DEA-R, a two-stage network structure will be proposed while considering both desirable and undesirable outputs.

3.1 Non-radial Two-stage Network in DEA

While considering undesirable outputs in the first stage of a two-stage network, the model (3), as a non-radial model for calculating inefficiency, is proposed as follows:

$$\begin{aligned}
&\max \left\{ \sum_{i=1}^{m} \alpha_i^1 + \sum_{d=1}^{D} \beta_d^1 \right\} \\
&s.t. \sum_{j=1}^{n} \lambda_j^1 x_{ij} + \alpha_i^1 = x_{i0}, \quad i = 1, \ldots, m, \\
&\sum_{j=1}^{n} \lambda_j^1 z_{dj} - \beta_d^1 = z_{do}, \quad d = 1, \ldots, D, \\
&\lambda_j^1 \geq 0, \alpha_i^1 \geq 0, \beta_d^1 \geq 0, j = 1, \ldots, n, i = 1, \ldots, m, d = 1, \ldots, D.
\end{aligned} \tag{3}$$

Definition 3.1. DMU_o is efficient in the two-stage DEA network whenever $NE_1^* = 0$.

The model (3) is a linear programming problem that calculates inefficiency in the first network stage.

Similarly, the model (4) is proposed as follows for calculating inefficiency in the second stage of the DEA network.

$$
\begin{aligned}
&\max \left\{ \sum_{d=1}^{D} \alpha_d^2 + \sum_{r=1}^{s} \beta_s^2 \right\} \\
&s.t. \quad \sum_{j=1}^{n} \lambda_j^2 z_{dj} + \alpha_d^2 = z_{d0}, \quad d = 1, \dots, D, \\
&\sum_{j=1}^{n} \lambda_j^2 y_{rj} - \beta_r^2 = y_{ro}, \quad r = 1, \dots, s, \\
&\lambda_j^2 \geq 0, \alpha_d^2 \geq 0, \beta_r^2 \geq 0, j = 1, \dots, n, d = 1, \dots, D, r = 1, \dots, s.
\end{aligned} \tag{4}
$$

The variable λ_j^2 corresponds to the second network stage. By dividing the desirable and undesirable outputs, we can determine the inefficient units in the two-stage DEA network.

Considering that in some problems, more than one objective function needs to be optimized, using multi-objective problem-solving methods is necessary [27]. To determine the inefficient units in the overall DEA network, we propose the following two-objective linear programming model.

$$
\begin{aligned}
&\max \quad \left\{ (\sum_{i=1}^{m} \alpha_i^1 + \sum_{d=1}^{l} \beta_d^1), (\sum_{d=1}^{l} \alpha_i^2 + \sum_{r=1}^{s} \beta_r^2) \right\} \\
&s.t. \quad \sum_{j=1}^{n} \lambda_j^1 x_{ij} + \alpha_i^1 = x_{i0}, \quad i = 1, \dots, m, \\
&\sum_{j=1}^{n} \lambda_j^1 z_{dj} - \beta_d^1 = z_{do}, \quad d = 1, \dots, l, \\
&\sum_{j=1}^{n} \lambda_j^2 z_{dj} + \alpha_d^2 = z_{do}, \quad d = 1, \dots, l, \\
&\sum_{j=1}^{n} \lambda_j^2 y_{rj} - \beta_r^2 = y_{ro}, \quad r = 1, \dots, s, \\
&\lambda_j^1 \geq 0, \alpha_i^1 \geq 0, \beta_d^1 \geq 0, j = 1, \dots, n, i = 1, \dots, m, d = 1, \dots, l, \\
&\lambda_j^2 \geq 0, \alpha_d^2 \geq 0, \beta_r^2 \geq 0, j = 1, \dots, n, d = 1, \dots, l, r = 1, \dots, s.
\end{aligned} \tag{5}
$$

Definition 3.2. DMU_o is overall DEA-inefficient in the two-stage network whenever $NE^*_{overall} = 0$.

3.2 Non-radial Two-stage Network in DEA-R

In this section, in accordance with the output-oriented additive model in DEA-R, we first assume the output-to-input ratios in the first network stage, i.e. z_{dj}/x_{ij}, are defined for $\forall i$ and $\forall d$ also zb_{dj}/xb_{ij}, are indexes of undesirable vectors. Then, we propose the following inefficiency model while considering the presence of undesirable outputs in the N_1 subscript set.

$$\begin{aligned}
&\max \left\{ \sum_{i=1}^{m} \sum_{d=1}^{D} a_{id} \right\} \\
&s.t.\ \sum_{j=1}^{n} \mu_j^1 (z_{dj}/x_{ij}) - a_{id} = (z_{do}/x_{io}),\ \ i = 1, \ldots, m,\ \ d = 1, \ldots, l, \\
&\sum_{j=1}^{n} \mu_j^1 (z_{b2j}/x_{b1j}) = (z_{b2o}/x_{b1o}),\quad b_1 = 1, \ldots, m_2,\quad b_2 = 1, \ldots, l_2, \\
&\sum_{j=1}^{n} \mu_j^1 = 1, \mu_j^1 \geq 0, j = 1, \ldots, n, \\
&a_{id} \geq 0,\quad i = 1, \ldots, m, d = 1, \ldots, l.
\end{aligned} \tag{6}$$

Model (5) is based on the Data Envelopment Analysis (DEA) structure, while model (6) follows the framework of Fractional Analysis models. Both structures initially involve multiple objectives; however, model (6) is transformed into a linear programming problem through the aggregation of objective functions. Using both models for city evaluation is definitely recommended, as each model calculates inefficiency scales.

Similarly, we propose the following inefficiency model for the second network stage assuming that $(r, d) \in N_2$ and that the y_{rj}/z_{dj} ratios are defined. Also y_{b3j}/z_{b2j} are indexes of undesirable vectors which are related with undesirable vectors in the second stage.

$$
\begin{aligned}
&\max \left\{ \sum_{r=1}^{s} \sum_{d=1}^{l} b_{dr} \right\} \\
&s.t. \ \sum_{j=1}^{n} \mu_j^2 (y_{rj}/z_{dj}) - b_{dr} = (y_{ro}/z_{do}), \ d = 1, \ldots, l, \ r = 1, \ldots, s, \\
&\sum_{j=1}^{n} \mu_j^1 (y_{b3j}/z_{b2j}) = (y_{b3o}/z_{b2o}), \quad b_3 = 1, \ldots, s_2, \ b_2 = 1, \ldots, l_2, \\
&\sum_{j=1}^{n} \mu_j^2 = 1, \mu_j^2 \geq 0, j = 1, \ldots, n, \\
&b_{rd} \geq 0, r = 1, \ldots, s, d = 1, \ldots, l. \qquad (7)
\end{aligned}
$$

The model (7) is a linear programming problem, in which the variable μ_j^2 corresponds to the second network stage. In this stage, the slack variables play a crucial role in the evaluation of DMU_o.

Theorem 3.3. *The model (7) is always feasible.*

Proof. To establish the feasibility of model (7), we proceed with the following steps:

I. **Consideration of Model (7):** We begin by analyzing model (2). Let $DMU_o = (Y_o/Z_o)$, where $o \in \{1, ..., n\}$. Here, Y_o and Z_o represent the output and input vectors of the decision-making unit DMU_o's output and input vectors, respectively.

II. **Assumption of Variables:** For model (7), assume the variables are defined as follows:

$$\mu_o^2 = e_o, b = 0.$$

III. **Verification of Constraints:** We need to verify that the constraints of model (7) are satisfied. Specifically, the constraint is:

$$
\begin{aligned}
&y_{ro}/z_{do} - 0 = y_{ro}/z_{do}, \ d = 1, \ldots, l, \ r = 1, \ldots, s, \\
&y_{b3o}/z_{b2o} - 0 = y_{ro}/z_{do}, \ b_3 = 1, \ldots, s_2, \ b_2 = 1, \ldots, l_2.
\end{aligned}
$$

Since this equality holds by definition, the constraints are satisfied, which implies that model (7) is feasible. □

We propose the model (8) below to calculate the overall inefficiency in the two-stage DEA-R network, assuming that the z_{dj}/x_{ij} , y_{rj}/z_{dj} ratios are defined and considering the μ_j^1 and μ_j^1 variables corresponding to the first and second network stages, respectively.

$$
\begin{aligned}
&\max \left\{ \sum_{i=1}^{m}\sum_{d=1}^{l} a_{id}, \sum_{r=1}^{s}\sum_{d=1}^{l} b_{rd} \right\} \\
&s.t. \quad \sum_{j=1}^{n} \mu_j^1 (z_{dj}/x_{ij}) - a_{id} = (z_{do}/x_{io}),\ i=1,\dots,m,\ d=1,\dots,l, \\
&\sum_{j=1}^{n} \mu_j^1 (z_{b2j}/x_{b1j}) = (z_{b2o}/x_{b1o}),\ b_1=1,\dots,m_2, \quad b_2=1,\dots,l_2, \\
&\sum_{j=1}^{n} \mu_j^2 (y_{rj}/z_{dj}) - b_{dr} = (y_{ro}/z_{do}), \quad d=1,\dots,l, \qquad r=1,\dots,s, \\
&\sum_{j=1}^{n} \mu_j^1 (y_{b3j}/z_{b2j}) = (y_{b3o}/z_{b2o}), \quad b_3=1,\dots,s_2, \qquad b_2=1,\dots,l_2, \\
&\sum_{j=1}^{n} \mu_j^1 = 1, \quad \mu_j^1 \geq 0,\ j=1,\dots,n, \\
&\sum_{j=1}^{n} \mu_j^2 = 1,\ \mu_j^2 \geq 0,\ j=1,\dots,n, \\
&a_{id} \geq 0, \quad i=1,\dots,m, \quad d=1,\dots,l, \\
&b_{rd} \geq 0, \quad r=1,\dots,s, \quad d=1,\dots,l.
\end{aligned}
\tag{8}
$$

In evaluating organizational performance, specific ratios, such as the liquidity and leverage ratios, play a crucial role. In this paper, DEA-R models incorporate ratio analysis structures to consider cost-to-income or cost-to-manager satisfaction ratios. This approach allows for evaluating cities from a different perspective. Given that the structure of the model (8) is based on slack variables and dependent on the subscripts N_1 and N_2. This model, a two-objective linear programming problem, calculates the network's overall inefficiency.

Theorem 3.4. *Model (8) is Feasible.*

Proof. To establish the feasibility of Model (8), we follow these steps:

I. **Consideration of Model (8):** We start by analyzing Model (8). Let $DMU_o = (Z_o/X_o, Y_o/Z_o)$, where $o \in \{1, ..., n\}$. Here, Y_o and Z_o represent the decision-making unit (DMU_o) 's output and input vectors, respectively.

II. **Assumption of Variables:** For model (8), assume the variables are defined as follows:

$$(\mu_o^1 = e_o, \mu_o^2 = e_o, a = 0).$$

III. **Verification of Constraints:** We need to verify that the constraints of model (8) are satisfied. Specifically, the constraints are:

$$\sum_{j=1}^{n} \mu_j^1 (z_{dj}/x_{ij}) - 0 = (z_{do}/x_{io}), \quad i = 1, \ldots, m, \quad d = 1, \ldots, l,$$

$$\sum_{j=1}^{n} \mu_j^1 (z_{b2j}/x_{b1j}) = (z_{b2o}/x_{b1o}), \; b_1 = 1, \ldots, m_2, \; b_2 = 1, \ldots, l_2,$$

$$\sum_{j=1}^{n} \mu_j^2 (y_{rj}/z_{dj}) - 0 = (y_{ro}/z_{do}), \quad d = 1, \ldots, l, \qquad r = 1, \ldots, s,$$

$$\sum_{j=1}^{n} \mu_j^1 (y_{b3j}/z_{b2j}) = (y_{b3o}/z_{b2o}), \; b_3 = 1, \ldots, s_2, \; b_2 = 1, \ldots, l_2,$$

$$\sum_{j=1}^{n} \mu_j^1 = 1, \sum_{j=1}^{n} \mu_j^2 = 1.$$

Therefor, $(\mu_o^1 = e_o, \mu_o^2 = e_o, a = 0)$

$$\begin{aligned}
&(z_{do}/x_{io}) - 0 = (z_{do}/x_{io}), \quad i = 1, \ldots, m, \quad d = 1, \ldots, l \\
&(z_{b2o}/x_{b1o}) = (z_{b2o}/x_{b1o}), \quad b_1 = 1, \ldots, m_2, \quad b_2 = 1, \ldots, l_2, \\
&(y_{ro}/z_{do}) - 0 = (y_{ro}/z_{do}), \quad d = 1, \ldots, l, \qquad r = 1, \ldots, s, \\
&(y_{b3o}/z_{b2o}) = (y_{b3o}/z_{b2o}), \quad b_3 = 1, \ldots, s_2, \qquad b_2 = 1, \ldots, l_2,
\end{aligned}$$

Since these equalities hold by definition, the constraints are satisfied, which implies that model (8) is feasible. $\square$

4 Applied Study

In this section, 26 storage centers for electric power supply equipment are evaluated in a two-stage network. In the first network stage, human workforce is considered as the input, and the output is the value of the items. The inputs of the second stage include costs related to loading and unloading and network expansion, as well as the overall costs. Further, the outputs of the second stage, which are undesirable outputs, include scrap and dead indexes and Inverse execution time.

The Fars Regional Electric Company is located in Fars Province, Iran. The company's headquarters is typically in Shiraz, the capital of Fars Province. This company is responsible for managing and distributing electricity across the region, including various cities and areas within Fars Province. Its duties include overseeing the power grid, developing energy infrastructure, and ensuring a stable electricity supply for consumers in this region. The data in Section 4 relates to the following link, where data from 26 storage centers for electric power supply equipment in Fars Province are evaluated in a two-stage network. https://www.farsedc.ir/site/en/default.aspx

Tables 1 and 2 present the input and output data and the intermediate vectors, respectively.

Table 3 shows the inefficiency scores of electric power equipment storage centers in Fars province. The second column provides the inefficiency scores produced by the model (3), where undesirable outputs are considered in CRS technology. According to this column, the Lar branch is non-radially efficient, and the Bavanat, Bakhtegan, and Sarchehan have shown significant inefficiency.

On the other hand, the third column of Table 3, which is related to the second network stage of the model (4), shows the non-radial inefficiency scores while considering undesirable outputs. According to the column, the Gerash, Bavanat, and Khafr branches are non-radially efficient in the second stage. On the other hand, the Darab, Lamerd, and Mohr branches are inefficient. Generally, for the inefficient units in models (3) and (4), the following equations are proposed for calculating targets.

As can be observed in the last column of Table 3, overall inefficiency

Table 1: Inputs and outputs in two-stage network.

DMUs	Human resources cost I1(stage 1)	Value of items I1(stage 2)	The cost of loading and s unloading good I2 (stage 2)	Network development cost I3 (stage 2)	The sum of the planned costs per urban subscriber delivered and transformed (Rials) I4 (stage 2)
Kazerun	1891977792	234227839949	726000000	48735661664	66852759
Nurabad	1678007304	152294807233	687550000	23392858340	109312422
Abadeh	1895599056	75360271909	256450000	10224065720	63503514
Eqlid	1844778264	160399847350	729459975	14205003477	151117058
Khorrambid	1864684572	31800397468	125079994	4910730617	106755013
Darab	2016361680	265699579014	2848165451	17055090076	116021021
Estahban	2010121032	65633561197	283300000	5236344951	93506160
Fasa	1928178336	137741355657	414109975	19056509490	91179471
Neyriz	2067931128	81237965471	244930000	19251930988	84810269
Jahrom	2073820440	132090134313	1410607982	26669727956	57477862
Lar	2131653408	325659603799	1376919966	60620364696	69839130
Firuzabad	2010121032	90942119965	472370000	14462323997	90389525
Ghir	2098701108	301165313987	1032659999	12302355481	96112152
Evaz	1864684572	71244423219	846090000	11911731739	130898151
Gerash	2069360232	92991559633	1061400501	7863963052	34043130
Khonj	1716387696	86676984468	1756828895	12755928815	98122529
Rostam	1997261124	101631805316	896520000	11037718661	132984562
Zarrin Dasht	1630080132	85596781558	1222606350	15746677990	111678567
Mohr	1958817984	131138139698	2352500000	31581995594	140991052
Farashband	1864684572	112284095680	2093213350	8735343122	92929182
Bavanat	2230525812	22048321989	46699994	3401514181	80988433
Lamerd	1974355704	219723875613	1583209999	47878360801	96334730
Khafr	2062943184	50519516844	142800004	10767668534	46815950
Bakhtegan	2234257500	30548625290	217890000	2468969675	129945772
Sarchehan	1864684572	24697807117	433660200	3913752880	134956996
Kuhchenar	2234257500	88505791551	626500000	4264231240	125418566

refers to a combination of the first and second network stages in DEA. According to this column, the Gerash, Bavanat, and Khafr branches are non-radially efficient. At the same time, the other units are non-radially inefficient, for which the following equations provide targets on the efficiency frontier.

Since DEA-R models are referred to as hybrid models, according to Table 4, using model (6) in the first network stage, the Lar branch is non-radially efficient. In contrast, the Bakhtegan, Sarchehan, Bavanat, and Khorambid branches are non-radially inefficient in DEA-R. Meanwhile, regarding the second network stage in DEA-R, it can be observed in the third column of Table 4 that the Gerash and Bavanat branches are non-radially efficient. The final column of Table 4 reflects the overall inefficiency scores derived from model (8) in DEA-R under the CRS assumption. This column does not display non-radially efficient units due to the inherent structure of DEA-R. In DEA-R, a similar inefficiency in the first and second stages does not necessarily indicate overall

Table 2: Two-stage network outputs table.

DMUs	index of scrap (out)	dead index (out)	Inverse execution time	Manager's satisfaction
Kazerun	28.1	0.84	0.0939	90
Nurabad	14.8	0.37	0.0561	70
Abadeh	5.7	0.37	0.2188	90
Eqlid	40.6	0.66	0.0899	30
Khorrambid	12.7	1.29	0.2169	70
Darab	16.4	2.59	0.0871	70
Estahban	47	1.42	0.1337	85
Fasa	2.6	0.17	0.0982	85
Neyriz	25	3.55	0.3584	90
Jahrom	44.5	0.29	0.0779	100
Lar	41	2.07	0.0806	90
Firuzabad	34.7	2.53	0.084	85
Ghir	31.4	4.2	0.4545	85
Evaz	24.5	0.63	0.1264	46
Gerash	40.9	0.24	0.1855	100
Khonj	37.5	0.24	0.1107	85
Rostam	20.1	1.74	0.3165	46
Zarrin Dasht	59.1	1.63	0.2801	70
Mohr	21.1	6.98	0.0308	40
Farashband	40.5	1.27	0.4329	85
Bavanat	6.6	1.18	0.3817	90
Lamerd	15.9	0.23	0.0421	85
Khafr	5.2	2.36	0.5495	100
Bakhtegan	13.6	18.46	0.1082	46
Sarchehan	12.3	18.06	0.303	46
Kuhchenar	52.5	0.15	0.1383	46

inefficiency. This discrepancy arises because DEA and DEA-R use different benchmarks for DMUs, resulting in divergent non-radial inefficiency scores across the overall network in these models.

Based on the pie chart in Figure 2, the cities of Mohr, Lamerd, and Nurabad have the highest average inefficiency. The ratios used in DEA-R models allow for the consideration of even managerial satisfaction, though first-stage costs play a significant role. In other words, the

Table 3: provides the results produced by the two-stage network DEA models in the first, second, and overall stages according to the non-radial models (3), (4), and (5).

DMUs	Inefficiency in stage 1 (Model 3)	Inefficiency in stage 2 (Model 4)	Overall inefficiency (Model 5)
Kazerun	0.234	1.4934	2.5703
Nurabad	0.6833	2.2578	3.9259
Abadeh	2.8428	0.4848	0.6118
Eqlid	0.7571	2.7202	0.4074
Khorrambid	7.9582	1.5808	0.3434
Darab	0.1594	2.2892	2.284
Estahban	3.6789	1.4128	1.7607
Fasa	1.1386	1.886	2.5889
Neyriz	2.8889	1.6777	0.0556
Jahrom	1.3985	1.3166	2.591
Lar	0	1.7504	3.1715
Firuzabad	2.3768	1.8383	3.4861
Ghir	0.0646	1.6212	0.2539
Evaz	2.9985	2.5096	0.494
Gerash	2.3997	0	0
Khonj	2.0252	1.9435	2.2626
Rostam	2.0023	2.1838	0.6476
Zarrin Dasht	1.9094	2.2382	0.0556
Mohr	1.282	2.688	4.6296
Farashband	1.5371	1.5479	0.1981
Bavanat	14.4554	0	0
Lamerd	0.3728	2.1111	7.9722
Khafr	5.2384	0	0
Bakhtegan	10.1735	1.8677	0.7657
Sarchehan	10.5344	1.7551	0.5446
Kuhchenar	2.8566	2.2241	0.3873

proposed models can play an important role in evaluating cities. Since service time and costs play a significant role in power distribution companies, it is even more crucial to equip each of the warehouses in the cities of Fars Province. This is because, in the event of electrical supply issues, managers might have to incur several times higher costs. There-

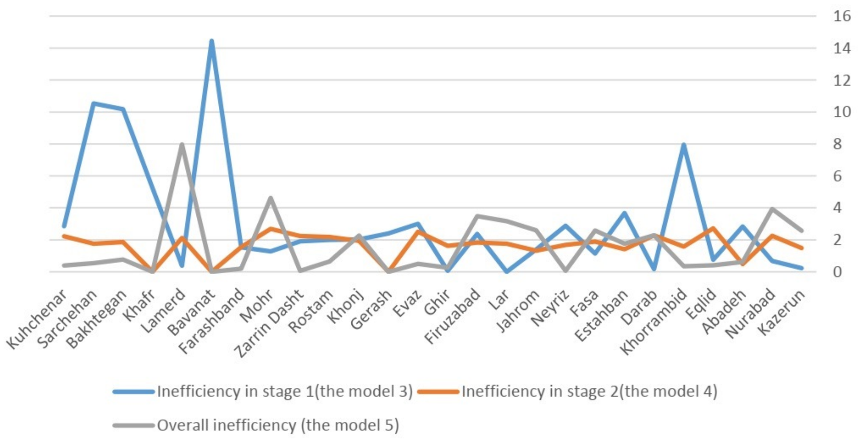

Figure 1: shows a comparison between the inefficiency scores in the first, second, and overall network stages.

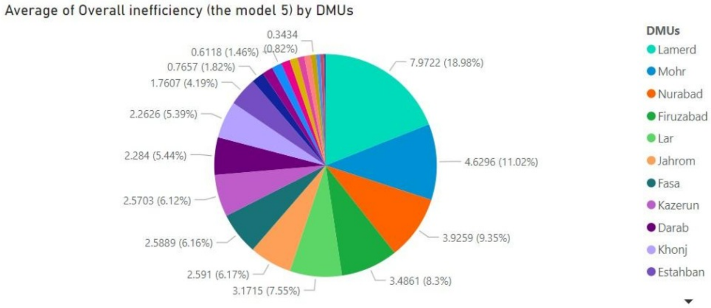

Figure 2: shows the inefficiency scores in the overall network stage.

fore, it is logical that a more accurate evaluation in multi-stage networks can help prevent higher costs.

5 Conclusion

This study introduces two-stage DEA and DEA-R models tailored to evaluate decision-making units (DMUs) with both desirable and unde-

Table 4: Inefficiencies in Stages 1, 2, and Overall Inefficiency.

DMUs	Inefficiency in stage 1 (Model 6)	Inefficiency in stage 2 (Model 7)	Overall inefficiency (Model 8)
Kazerun	0.234	4.6165	25.9536
Nurabad	0.6833	4.4178	28.8257
Abadeh	2.8428	1.0469	6.0616
Eqlid	0.7571	10.3311	25.8318
Khorrambid	7.9582	0.6658	9.6901
Darab	0.1594	13.9504	61.0641
Estahban	3.6789	1.0459	8.4927
Fasa	1.1386	2.2522	12.768
Neyriz	2.8889	1.492	5.9835
Jahrom	1.3985	5.3134	36.8938
Lar	0	7.528	45.1613
Firuzabad	2.3768	2.2323	16.3663
Ghir	0.0646	4.2499	7.6027
Evaz	2.9985	7.0769	20.8954
Gerash	2.3997	0	2.3997
Khonj	2.0252	7.0143	33.0559
Rostam	2.0023	7.3536	13.6226
Zarrin Dasht	1.9094	6.3975	15.0875
Mohr	1.282	22.5251	152.93
Farashband	1.5371	8.0655	16.2516
Bavanat	14.4554	0	14.4554
Lamerd	0.3728	8.1765	85.8967
Khafr	5.2384	0.0472	5.3921
Bakhtegan	10.1735	1.7113	15.4829
Sarchehan	10.5344	3.4466	16.0361
Kuhchenar	2.8566	4.7884	14.4816

sirable outputs. These models offer a practical framework for organizations aiming to assess inefficiencies in their performance. We have proposed methodologies for calculating inefficiency within DEA and DEA-R frameworks and provided a comparative analysis of inefficiency scores through an applied study involving undesirable outputs. For future research, we recommend exploring models that accommodate fuzzy data, as these could enhance the robustness and flexibility of efficiency evaluations. A notable limitation of our study was the lack of access to comprehensive online data from the regional electricity company of Fars province. Such data would have enabled a more precise assessment of the DMUs. Future research could also focus on evaluating inefficiencies in two-stage networks using random data and investigating efficiency within variable returns to scale technology. These areas offer promising avenues for advancing the application of DEA and DEA-R models in diverse and dynamic contexts.

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