Journal of Mathematical Extension
Journal Pre-proof
ISSN: 1735-8299
URL: http://www.ijmex.com
Original Research Paper

On Generating Matrices of the Bidimensional Balancing, Lucas-Balancing, Lucas-Cobalancing and Cobalancing Numbers

J. Chimpanzo*
Higher Polytechnic Institute of Soyo

P. Catarino
University of Trás-os-Montes e Alto Douro

M. V. Otero-Espinar
University of Santiago de Compostela

Abstract. In this article, we define parameter-dependent tridiagonal matrices from which we will find the bidimensional balancing, Lucas-balancing, Lucas-cobalancing numbers and cobalancing numbers.

AMS Subject Classification: 11B39; 11B83
Keywords and Phrases: balancing numbers, Lucas-balancing numbers, Lucas-cobalancing numbers, cobalancing numbers, generating matrices

1 Introduction

The concept of balancing numbers was introduced by Behera and Panda in 1999, [1]. In particular, $n \in \mathbb{Z}_+$ is a balancing number with balancer $r \in \mathbb{Z}_+$ if it is a solution of this Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r)\,,$$

Received: March 2024; Accepted: September 2024
*Corresponding Author

and if n is a balancing number, $8n^2+1$ is a perfect square and its square root is called a Lucas-balancing number.

Panda in [10] introduced the Lucas-balancing numbers C_n as $C_n = \sqrt{8B_n^2+1}$, where B_n is called the balancing number of order n. The recurrence relation of the balancing numbers is

$$B_{n+2} = 6B_{n+1} - B_n,$$

with initial conditions $B_0 = 0$ and $B_1 = 1$. In the case of the Lucas-balancing numbers $\{C_n\}_{n=0}^{\infty}$, the recurrence relation is the same as that of the balancing numbers, differing in their initial conditions, being, in this case, $C_0 = 1$ and $C_1 = 3$.

In [12], Panda and Ray introduced the sequence $\{c_n\}_{n\geq 1}$ of Lucas-cobalancing numbers which satisfy the following recurrence relation

$$c_{n+2} = 6c_{n+1} - c_n,$$

with initial conditions $c_1 = 1$ and $c_2 = 7$.

Also in [12], Panda and Ray introduced the sequence $\{b_n\}_{n\geq 1}$ of cobalancing numbers that satisfies the recurrence relation

$$b_{n+2} = 6b_{n+1} - b_n + 2,$$

with initial conditions $b_1 = 0$ and $b_2 = 2$.

The first four sequences defined above are in *The On-Line Encyclopedia of Integer Sequences*® (OEIS®) [15] and Table 1 gives us their first elements.

Table 1: Some first elements of the sequences $\{B_n\}_{n\geq 0}$, $\{C_n\}_{n\geq 0}$, $\{c_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$

n	0	1	2	3	4	5	6	7	8	9
B_n	0	1	6	35	204	1189	6930	40391	235416	1372105
C_n	1	3	17	99	577	3363	19601	114243	665857	3880899
c_n	-	1	7	41	239	1393	8119	47321	275807	1607521
b_n	-	0	2	14	84	492	2870	16730	97512	568344

Remark 1.1. *For the sequences of Lucas-cobalancing and cobalancing numbers in Table* 1*, we consider the value* c_1 *and* b_1 *as the first element of these sequences, respectively.*

Some detailed studies on balancing, Lucas-balancing, Lucas-cobalancing and cobalancing numbers can also be found in [7, 8, 9, 11, 13, 14].

In [3], the bidimensional version of the balancing and Lucas-balancing numerical sequences were introduced. In particular, in this work, the authors defined, respectively, the bidimensional recurrence relations of these two numerical sequences, as follows:

- The bidimensional balancing numerical sequence $B_{(n,m)}$ satisfies the following recurrence conditions, where n and m are non-negative integers:

$$\begin{cases} B_{(n+1,m)} = 6B_{(n,m)} - B_{(n-1,m)}, \\ B_{(n,m+1)} = 6B_{(n,m)} - B_{(n,m-1)}, \end{cases} \tag{1}$$

with the initial conditions $B_{(0,\ 0)} = 0$, $B_{(1,\ 0)} = 1$, $B_{(0,\ 1)} = i$, $B_{(1,\ 1)} = 1 + i$, where $i^2 = -1$.

- The bidimensional Lucas-balancing numerical sequence $C_{(n,m)}$ satisfies the following recurrence conditions, where n and m are non-negative integers:

$$\begin{cases} C_{(n+1,m)} = 6C_{(n,m)} - C_{(n-1,m)}, \\ C_{(n,\ m+1)} = 6C_{(n,m)} - C_{(n,m-1)}, \end{cases} \tag{2}$$

with the initial conditions $C_{(0,\ 0)} = 1$, $C_{(1,\ 0)} = 3$, $C_{(0,\ 1)} = 1+i$, $C_{(1,1)} = 3 + i$, where $i^2 = -1$.

On the other hand, the bidimensional version of the Lucas-cobalancing and cobalancing numbers was introduced in [4]. In this paper, in particular, the authors defined, respectively, the bidimensional recurrence relations of these two numerical sequences, as follows:

- The bidimensional Lucas-cobalancing numbers $c_{(m,n)}$ satisfy the following recurrence relations, where n and m are non-negative integers:

$$\left\{ \begin{array}{rcl} c_{(n+1,m)} & = & 6c_{(n,m)} - c_{(n-1,m)}, \\ c_{(n,m+1)} & = & 6c_{(n,m)} - c_{(n,m-1)}, \end{array} \right. \tag{3}$$

with the initial conditions $c_{(0,0)} = 1$, $c_{(1,0)} = 7$, $c_{(0,1)} = 1+i$, $c_{(1,1)} = 7+i$ and $i^2 = -1$.

- The bidimensional cobalancing numbers $b_{(m,n)}$, $\forall n, m \in \mathbb{N}_0$, are defined by:

$$b_{(n,m)} = \frac{1}{8}c_{(n+1,m)} - \frac{3}{8}c_{(n,m)} - \frac{1}{2}, \tag{4}$$

with the initial conditions $b_{(0,0)} = 0$, $b_{(1,0)} = 2$, $b_{(0,1)} = -\frac{1}{4}i$, $b_{(1,1)} = 2 + \frac{1}{4}i$ and $i^2 = -1$.

In this article, our purpose is to find balancing, Lucas-balancing, Lucas-cobalancing and cobalancing numbers using the determinants of some tridiagonal matrices. For this, we closely follow some of the work done in [2] and [5] for the sequences k-Pell, k-Pell-Lucas and Modified k-Pell and k-Fibonacci, respectively.

2 Tridiagonal Matrices and the Bidimensional Balancing and Lucas-Balancing Numbers

In this section, we will use the matrices defined in [6] and apply them to the bidimensional balancing and Lucas-balancing numerical sequences. For this, we will consider tridiagonal matrices similar to the one expressed in [5] and calculate its determinant by Laplace expansion. In Linear Algebra, Laplace expansion, named after Pierre-Simon Laplace (1749-1827), also known as cofactorial expansion, is an expression of the determinant of an $n \times n$-matrix A as a weighted sum of minors, which are the determinants of some $(n-1) \times (n-1)$-submatrices of A. In particular, for each i, the Laplace expansion along the i-th row is the equality

$$|A| = \sum_{j}^{n} a_{i,j} A_{i,j}$$

where, $a_{i,j}$ is the element in the i-th row and the j-th column of A, and, $A_{i,j} = (-1)^{i+j} D_{i,j}$ is the cofactor in which $D_{i,j}$ is the determinant of the matrix resulting from the elimination of the i-th row and the j-th column from A. Similarly, the Laplace expansion along the j-th column is the equality

$$\left|A\right| = \sum_{i}^{n} a_{i,j} A_{i,j}.$$

2.1 The determinant of a special model of tridiagonal matrices

In this Subsection, we will present the definition of the tridiagonal matrices given in [5], as mentioned before.

Let us consider the n-order tridiagonal matrices, denoted by M_n,

$$M_n = \begin{bmatrix} a & b & 0 & 0 & \cdots & 0 & 0 & 0 \\ c & d & e & 0 & \cdots & 0 & 0 & 0 \\ 0 & c & d & e & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & c & d & e \\ 0 & 0 & 0 & 0 & \cdots & 0 & c & d \end{bmatrix},$$

where a, b, c, d and e are real numbers.

Solving the sequence of determinants of the tridiagonal matrices M_n, we get

$$\begin{aligned} \left|M_1\right| &= a \\ \left|M_2\right| &= d\left|M_1\right| - bc \\ \left|M_3\right| &= d\left|M_2\right| - ce\left|M_1\right| \\ \left|M_4\right| &= d\left|M_3\right| - ce\left|M_2\right| \\ &\vdots \end{aligned}$$

and, in general,

$$\left|M_{n+1}\right| = d\left|M_n\right| - ce\left|M_{n-1}\right|. \tag{5}$$

In the next subsections, we are going to use these determinants in order to obtain the balancing, Lucas-balancing, Lucas-cobalancing and cobalancing numbers.

2.2 The case of bidimensional balancing numbers

We will use the tridiagonal matrix presented in Subsection 2.1 as a basis and apply it to the bidimensional balancing numbers. Now, we adapt the matrix M_n considering that a, b, c, d and e could be complex numbers.

So if $a = B_{(1,m)}$, $b = B_{(m,0)}$, $c = i$, $d = 6$ and $e = -i$, the matrix M_n expressed in Subsection 2.1 is transformed into a tridiagonal matrix,

$$R_n = \begin{bmatrix} B_{(1,m)} & B_{(m,0)} & 0 & 0 & \cdots & 0 & 0 & 0 \\ i & 6 & -i & 0 & \cdots & 0 & 0 & 0 \\ 0 & i & 6 & -i & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & i & 6 & -i \\ 0 & 0 & 0 & 0 & \cdots & 0 & i & 6 \end{bmatrix},$$

with $n, m \in \mathbb{N}$, m fixed.

Solving the sequence of determinants of the tridiagonal matrix R_n, we obtain

$$\begin{aligned} |R_1| &= |B_{(1,m)}| = B_{(1,m)} \\ |R_2| &= 6B_{(1,m)} - B_{(0,m)}i \\ &= 6B_{(1,m)} - B_{(0,m)} = B_{(2,m)} \ \text{(by the first recurrence relation described in (1))} \\ |R_3| &= 6B_{(2,m)} - i(-i)B_{(1,m)} \ \text{(by Laplace expansion along the third column)} \\ &= 6B_{(2,m)} - B_{(1,m)} = B_{(3,m)} \ \text{(once again, by the first recurrence relation in (1))} \\ &\vdots \end{aligned}$$

so that (5) is given by,

$$|R_{n+1}| = 6|R_n| - |R_{n-1}| = 6B_{(n,m)} - B_{(n-1,m)} = B_{(n+1,m)}.$$

The result below gives us the bidimensional balancing numbers of order n as the determinant of a tridiagonal matrix given in the next result.

Theorem 2.1. *Let us consider the square $n \times n$ matrix for $n, m \in \mathbb{N}$, m fixed:*

$$R_n = \begin{bmatrix} B_{(1,m)} & B_{(m,0)} & 0 & 0 & \cdots & 0 & 0 & 0 \\ i & 6 & -i & 0 & \cdots & 0 & 0 & 0 \\ 0 & i & 6 & -i & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & i & 6 & -i \\ 0 & 0 & 0 & 0 & \cdots & 0 & i & 6 \end{bmatrix}.$$

Then

$$\left|R_n\right| = B_{(n,m)}.$$

Proof. The proof is done by induction on n:

For $n = 1$ we have $\left|R_1\right| = B_{(1,m)}$, by the determinant of the matrix R_1, calculated before.

For $n = 2$ we have $\left|R_2\right| = B_{(2,m)}$, by the determinant of the matrix R_2, calculated before.

Suppose the statement of the theorem is true for all $k \leq n$ and we show that it remains true for $n + 1$. Then we have

$$\left|R_{n+1}\right| = \begin{vmatrix} B_{(1,m)} & B_{(m,0)} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ i & 6 & -i & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & i & 6 & -i & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & i & 6 & -i & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & i & 6 & -i \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & i & 6 \end{vmatrix}.$$

Thus, by Laplace's rule (expansion along $(n+1)$-th row) and by the induction hypothesis, we have

$$
\begin{aligned}
\left|R_{n+1}\right| &= iA_{n+1,n}+6A_{n+1,n+1} \\
&= i\,(-1)^{2n+1}\begin{vmatrix} B_{(1,m)} & B_{(m,0)} & 0 & 0 & \cdots & 0 & 0 & 0 \\ i & 6 & -i & 0 & \cdots & 0 & 0 & 0 \\ 0 & i & 6 & -i & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & i & 6 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & i & -i \end{vmatrix} + 6\left|R_n\right| \\
&= (-i)\begin{vmatrix} B_{(1,m)} & B_{(m,0)} & 0 & 0 & \cdots & 0 & 0 & 0 \\ i & 6 & -i & 0 & \cdots & 0 & 0 & 0 \\ 0 & i & 6 & -i & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & i & 6 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & i & -i \end{vmatrix} + 6\left|R_n\right| \\
&= (-i)\,(-i)\,A_{n,n}+6\left|R_n\right| \\
&= i^2\,(-1)^{2n}\begin{vmatrix} B_{(1,m)} & B_{(m,0)} & 0 & 0 & 0 & \cdots & 0 & 0 \\ i & 6 & -i & 0 & 0 & \cdots & 0 & 0 \\ 0 & i & 6 & -i & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & i & 6 \end{vmatrix} + 6\left|R_n\right| \\
&= -\left|R_{n-1}\right|+6\left|R_n\right| \\
&= 6B_{(n,m)}-B_{(n-1,m)}=B_{(n+1,m)} \quad \text{(by the expression (1))},
\end{aligned}
$$

as we wanted to prove. □

Now if $a=B_{(n,1)}$, $b=B_{(n,0)}$, $c=1$, $d=6$ and $e=1$, the matrix M_m is transformed into the following tridiagonal matrix,

$$
\widetilde{R}_m=\begin{bmatrix} B_{(n,1)} & B_{(n,0)} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 6 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 6 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 6 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 6 \end{bmatrix},
$$

with n fixed.

Solving the sequence of determinants of the tridiagonal matrices $\widetilde{R}_m$, we get

$$\begin{aligned}
\left|\widetilde{R}_1\right| &= \left|B_{(n,1)}\right| = B_{(n,1)} \\
\left|\widetilde{R}_2\right| &= 6B_{(n,1)} - B_{(n,0)} = B_{(n,2)} \quad \text{(by the second recurrence relation in (1))} \\
\left|\widetilde{R}_3\right| &= 6B_{(n,2)} - B_{(n,1)} \quad \text{(by Laplace expansion along column three)} \\
&= B_{(n,3)} \quad \text{(again, by the second recurrence relation in (1))} \\
&\vdots
\end{aligned}$$

so that (5) is given by,

$$\left|\widetilde{R}_{m+1}\right| = 6\left|\widetilde{R}_m\right| - \left|\widetilde{R}_{m-1}\right| = 6B_{(n,m)} - B_{(n,m-1)} = B_{(n,m+1)}.$$

The following result gives us the bidimensional balancing numerical sequence of order m as the determinant of a tridiagonal matrix:

Theorem 2.2. *Let us consider the $m \times m$ matrix for a fixed n natural number and $m \in \mathbb{N}$:*

$$\widetilde{R}_m = \begin{bmatrix} B_{(n,1)} & B_{(n,0)} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 6 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 6 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 6 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 6 \end{bmatrix}.$$

Then

$$\left|\widetilde{R}_m\right| = B_{(n,m)}.$$

Proof. We prove by induction on m:

For $m = 1$ we have $\left|\widetilde{R}_1\right| = B_{(n,1)}$, by the determinant of the matrix $\widetilde{R}_1$, as calculated before.

For $m=2$ we have $\left|\widetilde{R}_2\right| = B_{(n,2)}$, by the determinant of the matrix $\widetilde{R}_2$, as calculated before.

Suppose the statment of the theorem is true for any integer less than or equal to m. Let us show that $\left|\widetilde{R}_{m+1}\right| = B_{(n,m+1)}$. Hence

$$\left|\widetilde{R}_{m+1}\right| = \begin{vmatrix} B_{(n,1)} & B_{(n,0)} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 6 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 6 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 6 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 6 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 6 \end{vmatrix}$$

Thus, according to Laplace's rule (expansion along the $(n+1)$-th row) and the induction hypothesis, we have

$$\begin{aligned} \left|\widetilde{R}_{m+1}\right| &= 1A_{m+1,m} + 6A_{m+1,m+1} \\ &= (-1)^{2m+1} \begin{vmatrix} B_{(n,1)} & B_{(n,0)} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 6 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 6 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{vmatrix} + (-1)^{2m+2}\, 6 \left|\widetilde{R}_m\right| \\ &= (-1)\, A_{m,m} + 6\left|\widetilde{R}_m\right| \quad \text{(by expansion along last column)} \\ &= 6\left|\widetilde{R}_m\right| - \left|\widetilde{R}_{m-1}\right| \\ &= 6B_{(n,m)} - B_{(n,m-1)} = B_{(n,m+1)} \quad \text{(by expression (1))}, \end{aligned}$$

as we wanted to prove. □

2.3 The case of bidimensional Lucas-balancing numbers

As presented in Subsection 2.2, we will use the tridiagonal matrices defined in Subsection 2.1 for the Lucas-balancing numbers.

Hence, if $a = C_{(1,m)}$, $b = C_{(0,m)}$, $c = 1$, $d = 6$ and $e = 1$, the matrix M_n is converted into a tridiagonal matrix given by

$$S_n = \begin{bmatrix} C_{(1,m)} & C_{(0,m)} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 6 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 6 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 6 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 6 \end{bmatrix},$$

with m fixed.

Considering the determinants of the tridiagonal matrix S_n, we get

$$\begin{aligned} |S_1| &= |C_{(1,m)}| = C_{(1,m)} \\ |S_2| &= 6C_{(1,m)} - C_{(0,m)} = C_{(2,m)} \text{ (by the first recurrence relation in (2))} \\ |S_3| &= 6C_{(2,m)} - C_{(1,m)} \text{ (by Laplace expansion along the third column)} \\ &= C_{(3,m)} \text{ (once again, by the first recurrence relation in (2))} \\ &\vdots \end{aligned}$$

and (5) is given by,

$$|S_{n+1}| = 6|S_n| - |S_{n-1}| = 6C_{(n,m)} - C_{(n-1,m)} = C_{(n+1,m)}.$$

Below is the generic result that gives us the bidimensional balancing numbers of order n in terms of the determinant of a tridiagonal matrix:

Theorem 2.3. *Let the following matrix be given for $n, m \in \mathbb{N}$, m fixed:*

$$S_n = \begin{bmatrix} C_{(1,m)} & C_{(0,m)} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 6 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 6 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 6 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 6 \end{bmatrix}.$$

Then

$$|S_n| = C_{(n,m)}.$$

Proof. The proof is done by induction on n:

For $n = 1$ we have $|S_1| = C_{(1,m)}$, by the definition of the determinant of the matrix S_1, what was calculated previously.

For $n = 2$ we have $|S_2| = C_{(2,m)}$, once again, by the definition of the determinant of the matrix S_2 and take into account what was calculated before.

Suppose the statment of the theorem is valid for any integer less than or equal to n and let us prove that it remains valid for $n + 1$.

$$|S_{n+1}| = \begin{vmatrix} C_{(1,m)} & C_{(0,m)} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 6 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 6 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 6 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 6 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 6 \end{vmatrix}.$$

By the calculation of $|S_{n+1}|$ using Laplace's rule (expansion along the $(n+1)$-th row) and by the induction hypothesis, we get

$$\begin{aligned} |S_{n+1}| &= 1A_{n+1,n} + 6A_{n+1,n+1} \\ &= 1\,(-1)^{2n+1} \begin{vmatrix} C_{(1,m)} & C_{(0,m)} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 6 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 6 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{vmatrix} + 6\,|S_n| \\ &= - \begin{vmatrix} C_{(1,m)} & C_{(0,m)} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 6 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 6 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{vmatrix} + 6\,|S_n| \\ &= (-1)\,A_{n,n} + 6\,|S_n| \quad \text{(by Laplace expansion in last column)} \\ &= -\,|S_{n-1}| + 6\,|S_n| \\ &= 6C_{(n,m)} - C_{(n-1,m)} = C_{(n+1,m)}, \end{aligned}$$

as we wanted to prove. $\square$

Now if $a = C_{(n,1)}$, $b = C_{(n,0)}$, $c = 1$, $d = 6$ and $e = 1$, then the matrix of the form M_m defined in Subsection 2.1 is transformed into a tridiagonal matrix,

$$\widetilde{S}_m = \begin{bmatrix} C_{(n,1)} & C_{(n,0)} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 6 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 6 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 6 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 6 \end{bmatrix},$$

with n fixed.

Solving the sequence of determinants of the matrix $\widetilde{S}_m$, in similar way that was done before, we get

$$\left|\widetilde{S}_{m+1}\right| = 6\left|\widetilde{S}_m\right| - \left|\widetilde{S}_{m-1}\right|.$$

The following result, which proof is omitted, gives us the Lucas-balancing numbers in terms of the determinant of a tridiagonal matrix:

Theorem 2.4. *Let us consider the matrix for $n, m \in \mathbb{N}$, n fixed:*

$$\widetilde{S}_m = \begin{bmatrix} C_{(n,1)} & C_{(n,0)} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 6 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 6 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 6 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 6 \end{bmatrix},$$

Then

$$\left|\widetilde{S}_m\right| = C_{(n,m)}. \quad \square$$

2.4 The case of bidimensional Lucas-cobalancing and cobalancing numbers

In this Subsection, we will present the definition of the tridiagonal matrices for the bidimensional Lucas-cobalancing numbers based on the definition given in Subsection 2.1. Also as a consequence of this tridiagonal matrix and taking into account the relation between Lucas-cobalancing and cobalancing numbers, we will obtain expressions for the cobalancing numbers $b_{(n,m)}$ in terms of sums of special tridiagonal matrices determinants.

Then, if $a = c_{(1,m)}$, $b = c_{(0,m)}$, $c = 1$, $d = 6$ and $e = 1$, the tridiagonal matrix presented in Subsection 2.1 is transformed into a tridiagonal matrix,

$$T_n = \begin{bmatrix} c_{(1,m)} & c_{(0,m)} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 6 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 6 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 6 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 6 \end{bmatrix},$$

with m a natural number that is fixed.

Solving the sequence of determinants of the tridiagonal matrix S_n, we obtain

$$\begin{aligned} |T_1| &= |c_{(1,m)}| = c_{(1,m)} \\ |T_2| &= 6c_{(1,m)} - c_{(0,m)} = c_{(2,m)} \text{ (by the first recurrence relation described in (3))} \\ |T_3| &= 6c_{(2,m)} - c_{(1,m)} \text{ (by Laplace expansion along the third column)} \\ &= c_{(3,m)} \text{ (once again, by the first recurrence relation in (3))} \\ &\vdots \end{aligned}$$

so that (5) is given by,

$$|T_{n+1}| = 6\,|T_n| - |T_{n-1}|\,.$$

The following result gives us the bidimensional Lucas-cobalancing numbers in terms of the determinant of a tridiagonal matrix:

Theorem 2.5. *Let the following triangular matrix be given for* $n, m \in$

$\mathbb{N}$, *m fixed:*

$$T_n = \begin{bmatrix} c_{(1,m)} & c_{(0,m)} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 6 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 6 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 6 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 6 \end{bmatrix}.$$

Then

$$\left|T_n\right| = c_{(n,m)}.$$

Proof. We proceed to the proof by induction on n:

For $n = 1$ we have $\left|T_1\right| = c_{(1,m)}$, by the definition of the determinant of the matrix T_1, what was previously calculated.

For $n = 2$ we have $\left|T_2\right| = c_{(2,m)}$, once again, by the definition of the determinant of the matrix T_2, was calculated before.

Suppose the statment of the theorem is true for all integers less than or equal to n and prove that it remains true for $n+1$. Hence we obtain the following:

$$\left|T_{n+1}\right| = \begin{vmatrix} c_{(1,m)} & c_{(0,m)} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 6 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 6 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 6 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 6 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 6 \end{vmatrix}.$$

Thus, taking into account Laplace's rule (expansion along the $(n+1)$-th row) and the induction hypothesis, we obtain

$$
\begin{aligned}
\left|T_{n+1}\right| &= 1A_{n+1,n}+6A_{n+1,n+1}\\
&= 1\,(-1)^{2n+1}\begin{vmatrix} c_{(1,m)} & c_{(m,0)} & 0 & 0 & \cdots & 0 & 0 & 0\\ 1 & 6 & 1 & 0 & \cdots & 0 & 0 & 0\\ 0 & 1 & 6 & 1 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 1 & 6 & 0\\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{vmatrix}+6\left|T_n\right|\\
&= -\begin{vmatrix} c_{(1,m)} & c_{(0,m)} & 0 & 0 & \cdots & 0 & 0 & 0\\ 1 & 6 & 1 & 0 & \cdots & 0 & 0 & 0\\ 0 & 1 & 6 & 1 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 1 & 6 & 0\\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{vmatrix}+6\left|T_n\right|\\
&= (-1)\,A_{(n,n)}+6\left|T_n\right| \quad \text{(by Laplace expansion in last column)}\\
&= -(-1)^{2n}\begin{vmatrix} c_{(1,m)} & c_{(0,m)} & 0 & 0 & \cdots & 0 & 0\\ 1 & 6 & 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & 6 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 1 & 6 \end{vmatrix}+6\left|T_n\right|\\
&= 6\left|T_n\right|-\left|T_{n-1}\right|\\
&= 6c_{(n,m)}-c_{(n-1,m)}=c_{(n+1,m)},
\end{aligned}
$$

as we wanted to prove. □

Now if $a=c_{(n,1)}$, $b=c_{(n,0)}$, $c=1$, $d=6$ and $e=1$, the matrix of the form M_m as defined in Subsection 2.1 is transformed into a tridiagonal matrix,

$$
\widetilde{T}_m=\begin{bmatrix} c_{(n,1)} & c_{(n,0)} & 0 & 0 & \cdots & 0 & 0 & 0\\ 1 & 6 & 1 & 0 & \cdots & 0 & 0 & 0\\ 0 & 1 & 6 & 1 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 1 & 6 & 1\\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 6 \end{bmatrix},
$$

n fixed.

In a similar way to the previous subsections, we get

$$\left|\widetilde{T}_{m+1}\right| = 6\left|\widetilde{T}_{m}\right| - \left|\widetilde{T}_{m-1}\right|.$$

The result that follows gives us the bidimensional cobalancing numbers in terms of the determinant of a tridiagonal matrix, which proof is similar to those presented previously:

Theorem 2.6. *Let us consider the matrix for a fixed natural number n:*

$$\widetilde{T}_m = \begin{bmatrix} c_{(n,1)} & c_{(n,0)} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 6 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 6 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 6 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 6 \end{bmatrix},$$

Then

$$\left|\widetilde{T}_m\right| = c_{(n,m)}. \qquad \square$$

According to the recurrence relation (4), the following results follow:

Corollary 1. *For any non-negative integers n and m, the bidimensional cobalancing numbers satisfy*

$$b_{(n,m)} = \frac{1}{8}\left|T_{n+1}\right| - \frac{3}{8}\left|T_n\right| - \frac{1}{2}.$$

3 Conclusion

This article presents four types of bidimensional numerical sequences in terms of the determinant of tridiagonal matrices. Some results related to these sequences are presented and we consider that this article constitutes a contribution to the field of mathematics and gives an opportunity to researchers interested in this topic of numerical sequences.

Acknowledgements

The first two authors are members of the CMAT Research Center -

CMAT - UTAD. The last author is a member of the Mathematics Research Group of the University of Santiago de Compostela - USC GI 2136 and also of the Homology, Homotopy and Categorical Invariants in Groups and Non-Associative Algebras Project - PID2020 115155GB I00, within the scope of the Dynamical Systems, Genetic Algebras and Non-Stochastic Quadratic Operators objective. This research was partially funded by FCT - Foundation for Science and Technology, under the projects UIDB/00013/2020 and UIDP/00013/2020 and this research was also partially funded by the Ministry of Science and Innovation, State Research Agency (Spain), under the grant PID2020-115155GB-I0 and by the Council of Education, University and Vocational Training (Council of Galicia), grant ED431C 2023/3.

References

[1] A. Behera, G. K. Panda, On the square roots of triangular numbers, *Fibonacci Quart.* 37 (2) (1999), 98-105.

[2] P. Catarino, On generating matrices of the k-Pell, k-Pell-Lucas and modified k-Pell sequences, *Pure Mathematical Sciences* 3 (2) (2014), 71-77.

[3] J. Chimpanzo, P. Catarino, P. Vasco and A. Borges, Bidimensional extensions of balancing and Lucas-balancing numbers. *J. Discrete Math. Sci. Cryptogr.* 27 (1) (2024), 95-115.

[4] J. Chimpanzo, M. V. Otero-Espinar, A. Borges, P. Vasco and P. Catarino, Bidimensional extensions of cobalancing and Lucas-cobalancing numbers, *Annales Mathematicae Silesianae* (2023), DOI: https://doi.org/10.2478/amsil-2023-0022.

[5] S. Falcon, On the generating matrices of the k-Fibonacci numbers, *Proyecciones Journal of Mathematics* 32 (4) (2013), 347-357.

[6] J. Feng, Fibonacci identities via the determinant of a tridiagonal matrix, *Applied Mathematics and Computation* 217 (12) (2011), 5978-5981.

[7] K. Liptai, Lucas balancing numbers, *Acta Math. Univ. Ostrav.* 14 (1) (2006), 43-47.

[8] P. Olajos, Properties of balancing, cobalancing and generalized cobalancing numbers, *Ann. Math. Inform.* 37 (2010), 125-138.

[9] G. K. Panda, Sequence balancing and cobalancing numbers, *Fibonacci Quart.* 45 (3) (2007), 265-271.

[10] G. K. Panda, Some fascinating properties of balancing numbers, in Proc. Eleventh Internat. *Conference on Fibonacci Numbers and Their Applications, Cong. Numerantium* 194 (2009), 185-189.

[11] G. K. Panda, T. Komatsu and R. K. Davala, Reciprocal sums of sequences involving balancing and Lucas-balancing numbers, *Mathematical Reports* 20 (2) (2018), 201-214.

[12] G. K. Panda and P. K. Ray, Some links of balancing and cobalancing numbers with Pell and associated Pell numbers. *Bul. of Inst. of Math. Acad. Sinica* 6 (1) (2011), 41-72.

[13] B. K. Patel, N. Irmak and P. K. Ray, Incomplete balancing and Lucas-balancing numbers, *Mathematical Reports* 20 (1) (2018), 59-72.

[14] P. K. Ray, *Balancing and Cobalancing Numbers*, Doctoral Thesis, National Institute of Technology Rourkela, India, 2009.

[15] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, 2023, Available at *https://oeis.org*.

José António Lelo Chimpanzo
Department of Exact Sciences
Assistant Professor of Mathematics
Higher Polytechnic Institute of Soyo
Soyo, Angola
E-mail: jalchimpanzo@gmail.com

Paula Maria Machado Cruz Catarino
Department of Mathematics

PhD in Mathematics
University of Trás-os-Montes e Alto Douro
Vila Real, Portugal
E-mail: pcatarin@utad.pt

María Victoria Otero Espinar
Department of Statistics, Mathematical Analysis and Optimization
PhD in Mathematics
University of Santiago de Compostela
Santiago de Compostela, Spain
E-mail: mvictoria.otero@usc.es