Journal of Mathematical Extension
Vol. 18, No. 3, (2024) (5)1-32
URL: https://doi.org/10.30495/JME.2024.3007
ISSN: 1735-8299
Original Research Paper

The Linear Sequential Fractional Differential System Involving Two Generalized Fractional Orders and Its Application to the Vibration Theory

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Abstract. The main aim of the current paper is to keep developing the theory of conformable fractional calculus and observe its contributions to real-world problems. In this regard, the conformable bivariate Mittag-Leffler function is first proposed. The images of the conformable bivariate Mittag-Leffler functions under the conformable derivatives and the conformable Laplace transforms are calculated. A representation of an explicit solution to the linear sequential fractional differential system involving two generalized fractional orders in the conformable sense is determined based on the conformable bivariate function with the help of the conformable Laplace transform method. Then, it is shown that the obtained solution satisfies the introduced system. The vibration of springs is presented as an application with many simulations and tables for the described system. The effectiveness of the results is shown by discovering a relation between the system's order and its equilibrium position.

AMS Subject Classification: 26A33; 34Axx; 70Jxx
Keywords and Phrases: Conformable fractional derivative, vibration of springs, conformable bivariate Mittag-Leffler function, representation of solutions

Received: February 2024; Accepted: August 2024

1 Introduction

Fractional calculus may be regarded as an extension of integer calculus. This gives fractional calculus various privileges that integer calculus does not have. For instance, it is observed by most of the researchers who study this subject that real-world problems and real-life social issues are more appropriately represented by fractional-order systems rather than integer-order systems. Today, fractional-order systems have been used in almost all areas such as tumor growth[28], microbial propagation[15], viscoelastic models for vibrational analysis[61, 51], nature, biology[53, 12, 31, 13, 14, 60, 11], image and signal possessing, engineering, biophysics, models of neurons, thermodynamics, and mathematical physics; see [44, 19, 27, 58, 20, 33, 59]. When having a look at the literature, we have observed that there are so many definitions of fractional derivatives such as Euler, Fourier, Abel, Liouville, Riemann, Grünwald, Hadamard, Weyl, Erdélyi-Kober, Caputo[42, 40, 48, 63, 49, 50, 46], conformable[16, 2], etc, fractional derivatives. The conformable fractional derivative is of particular importance over the others because one of the most likely reasons is that it satisfies the corresponding quotient rule, the corresponding product rule, the corresponding chain rule, the corresponding mean value theorem, the corresponding Rolle theorem, generally the corresponding semigroup property compared to the classical 1st derivative while the others do not. Also, it begins to attract everyone's attention, whether they are mathematicians or not, because its definition is so simple and so close to the well-known 1st derivative. In addition, the importance of conformable fractional derivatives in the medical world is undeniable based on the available studies[29, 52, 17, 4, 26, 45, 8]. Moreover, Dazhi and Maokang[66] in 2017 first gave the physical and geometrical interpretations of the conformable fractional derivatives.

In recent years, the theory of Laplace transforms, which is also known as Laplace transformation or operational calculus, has emerged as a crucial aspect of the mathematical foundation required for engineers, physicists, mathematicians, and other scientific disciplines. Laplace transform methods are not only of great theoretical interest but also provide a convenient and effective tool for solving problems across different scientific and engineering disciplines[56, 18, 32, 64].

Gösta Mittag-Leffler, the Swedish mathematician, designed the traditional Mittag-Leffler function[43] in 1903

$$\mathbb{E}_{\beta}(t)=\sum_{j=0}^{\infty}\frac{t^{j}}{\Gamma(j\beta+1)},\ Re(\beta)>0,$$

which is a generalization of the well-known exponential function e^t. In the progress of time, it has made efforts to be generalized and extended to several types by modifying coefficients to account for the extra parameters such as the two-parameter Mittag-Leffler function $\mathbb{E}_{\beta,\theta}(t)$ and the three-parameter Mittag-Leffler function $\mathbb{E}^{\delta}_{\beta,\theta}(t)$ discussed in [25, 33], which are depicted by power series similar structure to the classical Mittag-Leffler:

$$\mathbb{E}_{\beta,\theta}(t)=\sum_{j=0}^{\infty}\frac{t^{j}}{\Gamma(j\beta+\theta)},\ Re(\beta)>0,$$

and

$$\mathbb{E}^{\delta}_{\beta,\theta}(t)=\sum_{j=0}^{\infty}\frac{(\delta)_{j}}{\Gamma(j\beta+\theta)}\frac{t^{j}}{j!},\ Re(\beta)>0.$$

In recent times, different kinds of generalizations have been suggested like the bivariate Mittag-Leffler functions[23, 35, 47] and multivariate Mittag-Leffler functions[39, 55] for more details, see [38, 9, 57], which depicted by two power series in two variables and by more than two power series as many as the number of the independent available variables in a number of variables. It is obvious that there is more than one kind of bivariate Mittag-Leffler function which is tantamount to the traditional Mittag-Leffler function. In the reference[22], the bivariate Mittag-Leffler function, which is also another kind of bivariate Mittag-Leffler function different from the above-mentioned ones, is defined and its relation to fractional calculus, especially, Riemann-Liouville and Caputo derivatives is established. The conformable exponential function $e^{\frac{t^{\alpha}}{\alpha}}$ was proposed and discussed in [2]. I wonder how is the bivariate Mittag-Leffler function in the conformable sense and what is the relation to the conformable fractional derivatives. Moreover, differential equations involving two fractional orders were studied and the existence of their solutions, their stabilities[65, 36, 37, 6, 5, 62, 7], etc were investigated.

Inspired by the above explanations and the above-cited studies, we will consider the following linear sequential conformable fractional differential system involving two generalized fractional orders as follows

$$\begin{cases} \mathcal{D}_0^{n\beta} c\left(t\right) - \lambda_1 \mathcal{D}_0^{(n-1)\beta} c\left(t\right) = \lambda_2 c\left(t\right) + \zeta\left(t\right), \quad t \in \left(0, T\right], \\ c\left(0\right) = c_0, \quad \mathcal{D}_0^{k\beta} c\left(0\right) = c_k, \quad k = 1, 2, \ldots, n-1, \end{cases} \tag{1}$$

where the symbols $\mathcal{D}_0^{n\beta}$ and $\mathcal{D}_0^{(n-1)\beta}$ stand for the sequential conformable derivatives of orders $n-1 < n\beta \leq n$ and $n-2 < (n-1)\beta \leq n-1$ with $n \geq 2$. Here,

$$\mathcal{D}_0^{n\beta} = \mathcal{D}_0^{(n-1)\beta} \mathcal{D}_0^{\beta} = \mathcal{D}_0^{\beta} \mathcal{D}_0^{(n-1)\beta} \quad \text{for all} \quad n = 1, 2, \ldots,$$

$\lambda_1, \lambda_2 \in \mathbb{R}$, $\zeta \in C\left([0, T] \times \mathbb{R}, \mathbb{R}\right)$.

Equation 1 can be regarded as a generalization of the classical Langevin formula[41] for a Brownian particle, which is given by

$$m \frac{d^2}{dt^2} x(t) + \gamma \frac{d}{dt} x(t) = f(t, x(t)),$$

here, f is the force on the particle, x stands for the position of the particle, γ is the coefficient of the friction, and m is the particle mass. Many stochastic problems in the presence of a fluctuating medium are characterized by the Langevin equation. However, the standard Langevin equation is not able to express the right description of some sophisticated problems. So, it needs to be generalized to compensate for the lack of the classic one, which enables researchers to describe more physical problems in disordered regions[34]. One of the generalizations of the Langevin equation is acquired by replacing the ordinary derivative with a fractional one, which provides the well-known fractional Langevin equation. The fractional Langevin equation was employed for modeling many physical problems such as diffusion[21], anomalous transport[34], and electrical circuits[10]. In this work, the theory of comfortable fractional calculus is improved and Equation 1, a generalization Langevin equation, is employed for modeling the vibration of springs.

The paper is organized as follows. In Section 1, a brief summary of fractional derivatives and the available Mittag-Leffler functions are given and the tackled system is offered. In Section 2, necessary tools and concepts in the literature, which are available, are remembered. In Section 3, the conformable bivariate Mittag-Leffler function is first proposed, simple calculations with and without the conformable Laplace transform are made, and a representation of a solution to the linear sequential conformable fractional differential system is looked for via two different approaches such as the Laplace transform, and the variation of constants. In Section 4, the vibration of springs is offered as an application with simulations to show the effectiveness of our findings by determining a tie between the fractional order of the tackled system and its equilibrium position. In Section 5, all I obtained are summed up and a couple of possible future works are presented.

2 Preliminaries

The focus of this section is to outline the key tools required for readers to gain a clearer understanding of forthcoming proofs and statements.

For $-\infty < a < b < \infty$, $J = [a, b]$ is the interval of $\mathbb{R}$ that symbolizes the set of all real numbers, and let $C(J, \mathbb{R})$ be the Banach space of all continuous functions from $J \to \mathbb{R}$ endowed with the infinity norm

$$\|\zeta\|_C = \sup_{t\in J} \|\zeta(t)\|,$$

for an arbitrary norm $\|.\|$ on $\mathbb{R}$.

Definition 2.1. [30] The conformable fractional integral of order $0 < \beta \leq 1$ with a function $\zeta : [0, \infty) \to \mathbb{R}$

$$\mathcal{I}_0^{\beta}\zeta(t) = \int_0^t \zeta(s)\, s^{\beta-1} d\tau, \quad t > 0.$$

Definition 2.2. [30] The following fractional expression

$$\mathcal{D}_0^{\beta}\zeta(t) = \lim_{\varepsilon\to 0} \frac{\zeta\left(t + \varepsilon t^{1-\alpha}\right) - \zeta(t)}{\varepsilon}, \quad t > 0, \quad 0 < \beta \leq 1,$$

is said to be the α-ordered conformable fractional derivative of a function $\zeta : [0,\infty) \to \mathbb{R}$. Moreover, if $\zeta(.)$ is differentiable and $\lim_{x\to 0^+} \mathcal{D}_0^{\beta}\zeta(t)$ exists, $\mathcal{D}_0^{\beta}\zeta(0) = \lim_{x\to 0^+} \mathcal{D}_0^{\beta}\zeta(t)$.

Lemma 2.3. *[1] The conformable fractional derivative of order $0 < \beta \leq 1$ of a function $\zeta : [0,\infty) \to \mathbb{R}$ exists iff it is differentiable at a point t, and also*

$$\mathcal{D}_0^{\beta}\zeta(t) = t^{1-\beta}\zeta'(t).$$

Lemma 2.4. *[2] For $0 < \alpha \leq 1$,*

$$\mathcal{I}_0^{\alpha}\mathcal{D}_0^{\alpha}\zeta(t) = \zeta(t) - \zeta(0).$$

Definition 2.5. [2] For each $t \geq 0$, the conformable exponential function is described by

$$E_{\beta}(d,t) = \exp\left(d\frac{t^{\beta}}{\beta}\right) = e^{d\frac{t^{\beta}}{\beta}}, \quad d \in \mathbb{R}, \quad 0 < \beta \leq 1.$$

Definition 2.6. [56] A function $\zeta(t)$ is said to be conformably exponentially bounded if $\zeta(t)$ holds the inequality $\|\zeta(t)\| \leq ME_{\beta}(d,t)$ for all sufficiently large t, where $0 < \beta \leq 1$, $M, d \in \mathbb{R}^+$.

Definition 2.7. [56] The Laplace transform of order $0 < \beta \leq 1$ of a function ζ in the conformable sense is described as noted below

$$\mathcal{L}_{\beta}\{\zeta(t)\}(s) = \int_0^{\infty} E_{\beta}(-s,t)\zeta(t)t^{\beta-1}dt,$$

where the function $\zeta : [0,\infty) \to \mathbb{R}$.

The following lemma expresses the uniqueness of the conformable Laplace transform.

Lemma 2.8. *[64] Let $g(t)$ and $f(t)$ be conformably exponentially bounded. If*

$$\mathcal{L}_{\beta}\{g(t)\}(s) = \mathcal{L}_{\beta}\{f(t)\}(s), \quad s > a,$$

then

$$g(t) = f(t), \ \forall t \geq 0,$$

where both functions are continuous. Furthermore,

$$g(t) = \mathcal{L}_{\beta}^{-1}\{G_{\beta}(t)\}(s) \Leftrightarrow \mathcal{L}_{\beta}\{g(t)\}(s) = G_{\beta}(t). \tag{2}$$

Theorem 2.9. *[18] Let $\zeta : [0, \infty) \to \mathbb{R}$ be differentiable and $0 < \beta \leq 1$, then for an arbitrary integer number n we have*

$$\mathcal{L}_\beta \left\{ \mathcal{D}_0^{n\beta} \zeta(t) \right\}(s) = s^n \mathcal{L}_\beta \{ \zeta(t) \}(s) - \sum_{k=0}^{n-1} s^{n-k-1} \mathcal{D}_0^{k\beta} \zeta(0),$$

where $\mathcal{D}_0^{n\beta} = \mathcal{D}_0^{(n-1)\beta} \mathcal{D}_0^{\beta} = \mathcal{D}_0^{\beta} \mathcal{D}_0^{(n-1)\beta}$ for all $n = 1, 2, \ldots$.

Theorem 2.10. *[32] Assume that $\zeta, \eta : [0, \infty) \to \mathbb{R}$ and $0 < \beta \leq 1$. The conformable convolution of ζ and η is given by*

$$(\zeta * \eta)(t) = \int_0^t \zeta\left(\frac{t^\beta}{\beta} - \frac{s^\beta}{\beta} \right) \eta\left(\frac{s^\beta}{\beta} \right) d\frac{s^\beta}{\beta},$$

and

$$\mathcal{L}_\beta \{ (\zeta * \eta)(t) \}(s) = \mathcal{L}_\beta \{ \zeta(t) \}(s) \, \mathcal{L}_\beta \{ \eta(t) \}(s),$$

if the conformable Laplace transforms of both ζ and η exist for $s \geq 0$.

Proposition 2.11. *[54] The conformable gamma function $\Gamma_\beta(\alpha)$ satisfies the below equations:*

1. $\Gamma_\beta(\alpha + 1) = (\alpha + \beta - 1)\Gamma_\beta(\alpha)$,
2. $\Gamma_\beta(\alpha) = \beta^{\frac{\alpha+\beta-1}{\beta}} \Gamma\left(\frac{\alpha+\beta-1}{\beta} \right)$,
3. $\Gamma_\beta(1) = \beta$,

here, Γ denotes the gamma function, which is well-established in mathematical literature.

Theorem 2.12. *[54] Given $p > 0$ and $0 < \beta \leq 1$, the Laplace transforms of 1, t, and t^p can be expressed as:*

1. $\mathcal{L}_\beta \{1\}(s) = s^{-1}\Gamma_\beta(1)$,
2. $\mathcal{L}_\beta \{t\}(s) = s^{-\frac{1+\beta}{\beta}}\Gamma_\beta(2)$,
3. $\mathcal{L}_\beta \{t^p\}(s) = s^{-\frac{p+\beta}{\beta}}\Gamma_\beta(p+1) = \frac{1}{s^{1+\frac{p}{\beta}}}\Gamma_\beta(p+1)$.

Our novel contributions are presented starting from Section 3 onwards.

3 An Explicit Solution to the Considered System

In this section, we will expend energy to obtain an explicit solution to non-homogeneous sequential linear conformable fractional differential systems through two different approaches.

Definition 3.1. The conformable bivariate Mittag-Leffler function with three parameters $E_{\alpha,\beta,\gamma}(u,v) : [0,\infty)\times[0,\infty)\to\mathbb{R}$ is given by

$$E^{\gamma+1}_{\beta,\alpha}(u,v) := \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{i+j}{j}\frac{u^i v^j}{\Gamma_\beta(i\beta+j\alpha+\gamma+1)},$$

where $\beta,\alpha,\gamma>0$.

To make the proofs easy, we make some preparations.

Corollary 3.2. *For $0<\beta\le 1$ and $\alpha>0$, we have*

$$\Gamma_\beta(\alpha+1)=\alpha\Gamma_\beta(\alpha-\beta+1).$$

Proof. It is an immediate result of items (1) and (2) in Proposition 2.11. ☐

Lemma 3.3. *For $\alpha,\beta,\gamma>0$, $u=\lambda_2 t^\beta$, $b=\lambda_1 t^\alpha$, $t\in[0,\infty)$, we have*

$$\mathcal{D}^\beta_0\left(t^\gamma E^{\gamma+1}_{\beta,\alpha}\left(\lambda_2 t^\beta,\lambda_1 t^\alpha\right)\right)=t^{\gamma-\beta}E^{\gamma-\beta+1}_{\beta,\alpha}\left(\lambda_2 t^\beta,\lambda_1 t^\alpha\right).$$

Proof. If Lemma 2.3 and Corollary 3.2 are exploited, one can easily acquire

$$\begin{aligned}
&\mathcal{D}^\beta_0\left(t^\gamma E^{\gamma+1}_{\beta,\alpha}\left(\lambda_2 t^\beta,\lambda_1 t^\alpha\right)\right)\\
&=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{i+j}{j}\lambda_2^i\lambda_1^j\mathcal{D}^\beta_0\left(\frac{t^{i\beta+j\alpha+\gamma}}{\Gamma_\beta(i\beta+j\alpha+\gamma+1)}\right)\\
&=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{i+j}{j}\lambda_2^i\lambda_1^j\left(\frac{t^{i\beta+j\alpha+\gamma-\beta}}{\Gamma_\beta(i\beta+j\alpha+\gamma-\beta+1)}\right)\\
&=t^{\gamma-\beta}E^{\gamma-\beta+1}_{\beta,\alpha}\left(\lambda_2 t^\beta,\lambda_1 t^\alpha\right).
\end{aligned}$$

□

I would like to give an explicit proof of the following lemma which exists in the literature.

Lemma 3.4. *The following equation holds true:*

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{i+j}{j}a_{ij} = a_{00} + \sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\binom{i+j-1}{j-1}a_{ij} + \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\binom{i+j-1}{j}a_{ij},$$

where $a_{ij} \in \mathbb{R}$.

Proof. It is known that $\binom{0}{0} = 1$, $\binom{i}{j} = 0$, for $j > i$, and $\binom{i}{j} = \binom{i-1}{j} + \binom{i-1}{j-1}$, for $0 < j < i$. Consider

$$\begin{aligned}
\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{i+j}{j}a_{ij} &= \sum_{j=0}^{\infty}\binom{j}{j}a_{0j} + \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\binom{i+j}{j}a_{ij} \\
&= \sum_{j=0}^{\infty}\binom{j}{j}a_{0j} + \sum_{i=1}^{\infty}\binom{i}{0}a_{i0} + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\binom{i+j}{j}a_{ij} \\
&= \binom{0}{0}a_{00} + \sum_{j=1}^{\infty}\binom{j}{j}a_{0j} \\
&+ \sum_{i=1}^{\infty}\binom{i}{0}a_{i0} + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\binom{i+j}{j}a_{ij} \\
&= a_{00} + \sum_{j=1}^{\infty}\binom{j-1}{j-1}a_{0j} + \sum_{i=1}^{\infty}\binom{i-1}{0}a_{i0} \\
&+ \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\binom{i+j-1}{j}a_{ij} + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}+\binom{i+j-1}{j-1}a_{ij} \\
&= a_{00} + \sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\binom{i+j-1}{j-1}a_{ij} \\
&+ \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\binom{i+j-1}{j}a_{ij}.
\end{aligned}$$

□

Lemma 3.5. *The following expressions hold true.*

$$\frac{1}{\left(s^n-\lambda_1 s^{n-1}\right)^{l+1}}=\sum_{k=0}^{\infty}\binom{l+k}{k}\frac{\lambda_1^k}{s^{n(l+1)+k}},$$

and

$$\frac{s^d}{s^n-\lambda_1 s^{n-1}-\lambda_2}=\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\binom{l+k}{k}\frac{\lambda_2^l\lambda_1^k}{s^{nl+k+n-d}},$$

where $\left|\frac{\lambda_1}{s}\right|<1,\ \left|\frac{\lambda_2}{s^n-\lambda_1 s^{n-1}}\right|<1\ \lambda_1,\lambda_2\in\mathbb{R},\ s>0.$

Proof. By employing the binomial series, one easily demonstrates the former

$$\begin{aligned}\frac{1}{\left(s^n-\lambda_1 s^{n-1}\right)^{l+1}}&=\frac{1}{s^{n(l+1)}}\frac{1}{\left(1-\frac{\lambda_1}{s}\right)^{l+1}}\\&=\frac{1}{s^{n(l+1)}}\sum_{k=0}^{\infty}\binom{l+k}{k}\left(\frac{\lambda_1}{s}\right)^k,\quad\left|\frac{\lambda_1}{s}\right|<1,\\&=\sum_{k=0}^{\infty}\binom{l+k}{k}\frac{\lambda_1^k}{s^{n(l+1)+k}}.\end{aligned}$$

In the light of geometric series and the first item, we get

$$\begin{aligned}\frac{s^d}{s^n-\lambda_1 s^{n-1}-\lambda_2}&=\frac{s^d}{s^n-\lambda_1 s^{n-1}}\frac{1}{1-\frac{\lambda_2}{s^n-\lambda_1 s^{n-1}}}\\&=\frac{s^d}{s^n-\lambda_1 s^{n-1}}\sum_{l=0}^{\infty}\frac{\lambda_2^l}{\left(s^n-\lambda_1 s^{n-1}\right)^l},\quad\left|\frac{\lambda_2}{s^n-\lambda_1 s^{n-1}}\right|<1,\\&=\sum_{l=0}^{\infty}\frac{\lambda_2^l s^d}{\left(s^n-\lambda_1 s^{n-1}\right)^{l+1}},\\&=\sum_{l=0}^{\infty}\lambda_2^l s^d\sum_{k=0}^{\infty}\binom{l+k}{k}\frac{\lambda_1^k}{s^{n(l+1)+k}},\quad\left|\frac{\lambda_1}{s}\right|<1,\\&=\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\binom{l+k}{k}\frac{\lambda_2^l\lambda_1^k}{s^{nl+k+n-d}}.\end{aligned}$$

□

Lemma 3.6. *We have the following equalities, for* $\lambda_1, \lambda_2 \in \mathbb{R}$, $\frac{n-1}{n} < \beta \leq 1$, *and* $s > 0$,

$$\mathcal{L}_\beta^{-1}\left\{\frac{s^d}{s^n - \lambda_1 s^{n-1} - \lambda_2}\right\}(t) = t^{(n-1)\beta - d\beta} E_{n\beta,\beta}^{(n-1)\beta - d\beta + 1}\left(\lambda_2 t^{n\beta}, \lambda_1 t^\beta\right).$$

Proof. Under the choices of $\lambda_1, \lambda_2 \in \mathbb{R}$, $0 < \beta \leq 1$, and $s > 0$, we take the former into consideration

$$\begin{aligned}
&\mathcal{L}_\beta^{-1}\left\{\frac{s^d}{s^n - \lambda_1 s^{n-1} - \lambda_2}\right\}(t) \\
&= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\binom{l+k}{k}\lambda_2^l \lambda_1^k \mathcal{L}_\beta^{-1}\left\{\frac{1}{s^{nl+k+n-d}}\right\}(t) \\
&= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\binom{l+k}{k}\lambda_2^l \lambda_1^k \mathcal{L}_\beta^{-1}\left\{\frac{1}{s^{1+\frac{nl\beta + k\beta + n\beta - \beta - d\beta}{\beta}}}\right\}(t) \\
&= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\binom{l+k}{k}\lambda_2^l \lambda_1^k \frac{t^{nl\beta + k\beta + n\beta - \beta - d\beta}}{\Gamma_\beta\left(nl\beta + k\beta + n\beta - \beta - d\beta + 1\right)} \\
&= t^{(n-1)\beta - d\beta} E_{n\beta,\beta}^{(n-1)\beta - d\beta + 1}\left(\lambda_2 t^{n\beta}, \lambda_1 t^\beta\right).
\end{aligned}$$

□

Theorem 3.7. *The following expression*

$$\begin{aligned}
c(t) = \sum_{k=0}^{n-2}\frac{t^{k\beta}}{\Gamma_\beta(k\beta+1)}c_k + \lambda_2\sum_{k=0}^{n-2} t^{(n+k)\beta} E_{n\beta,\beta}^{(n+k)\beta+1}\left(\lambda_2 t^{n\beta}, \lambda_1 t^\beta\right) c_k \\
+ t^{(n-1)\beta} E_{n\beta,\beta}^{(n-1)\beta+1}\left(\lambda_2 t^{n\beta}, \lambda_1 t^\beta\right) c_{n-1}, \qquad t \geq 0, \qquad (3)
\end{aligned}$$

is a solution to the homogeneous linear generalized sequential conformable fractional differential equations.

Proof. Our principle idea is to show that $c(t)$ in (3) satisfies the homogeneous generalized sequential conformable fractional differential equations. Thus, taking the conformable fractional derivative of order $0<\beta<1$ of $c(t)$ in the light of Lemma 3.3, one gets $\mathcal{D}_0^{\beta}c(t)$:

$$= \sum_{k=1}^{n-2} \frac{t^{k\beta-\beta}}{\Gamma_\beta(k\beta-\beta+1)} c_k + t^{(n-1)\beta-\beta} E_{n\beta,\beta}^{(n-1)\beta-\beta+1}\left(\lambda_2 t^{n\beta}, \lambda_1 t^{\beta}\right) c_{n-1}$$
$$+ \lambda_2 \sum_{k=0}^{n-2} t^{(n+k)\beta-\beta} E_{n\beta,\beta}^{(n+k)\beta-\beta+1}\left(\lambda_2 t^{n\beta}, \lambda_1 t^{\beta}\right) c_k,$$

if it is derived in the conformable sense once again, one gets $\mathcal{D}_0^{2\beta}c(t)$:

$$= \sum_{k=2}^{n-2} \frac{t^{k\beta-2\beta}}{\Gamma_\beta(k\beta-2\beta+1)} c_k + t^{(n-1)\beta-2\beta} E_{n\beta,\beta}^{(n-1)\beta-2\beta+1}\left(\lambda_2 t^{n\beta}, \lambda_1 t^{\beta}\right) c_{n-1}$$
$$+ \lambda_2 \sum_{k=0}^{n-2} t^{(n+k)\beta-2\beta} E_{n\beta,\beta}^{(n+k)\beta-2\beta+1}\left(\lambda_2 t^{n\beta}, \lambda_1 t^{\beta}\right) c_k.$$

If it is kept going in similar procedures, one can acquire

$$\mathcal{D}_0^{(n-2)\beta}c(t) = \frac{1}{\Gamma_\beta(1)} c_{n-2} + t^{\beta} E_{n\beta,\beta}^{\beta+1}\left(\lambda_2 t^{n\beta}, \lambda_1 t^{\beta}\right) c_{n-1}$$
$$+ \lambda_2 \sum_{k=0}^{n-2} t^{(k+2)\beta} E_{n\beta,\beta}^{(k+2)\beta+1}\left(\lambda_2 t^{n\beta}, \lambda_1 t^{\beta}\right) c_k.$$

If it is derived in the conformable sense once again, one gets $\mathcal{D}_0^{(n-1)\beta}c(t)$:

$$= E_{n\beta,\beta}^{1}\left(\lambda_2 t^{n\beta}, \lambda_1 t^{\beta}\right) c_{n-1} + \lambda_2 \sum_{k=0}^{n-2} t^{(k+1)\beta} E_{n\beta,\beta}^{(k+1)\beta+1}\left(\lambda_2 t^{n\beta}, \lambda_1 t^{\beta}\right) c_k.$$

Now we use double series expansions of the conformable bivariate Mittag-Leffler functions

$$\mathcal{D}_0^{(n-1)\beta}c(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{i+j}{j} \frac{\lambda_2^i \lambda_1^j \quad t^{n\beta i+\beta j}}{\Gamma_\beta(n\beta i+\beta j+1)} c_{n-1}$$
$$+ \lambda_2 \sum_{k=0}^{n-2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{i+j}{j} \frac{\lambda_2^i \lambda_1^j \quad t^{n\beta i+\beta j+(k+1)\beta}}{\Gamma_\beta(n\beta i+\beta j+(k+1)\beta+1)} c_k.$$

Lemma 3.4 gives $\mathcal{D}_0^{(n-1)\beta} c(t)$ as follows

$$
\begin{aligned}
&= \frac{1}{\Gamma_\beta(1)} c_{n-1} + \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} \binom{i+j-1}{j-1} \frac{\lambda_2^i \lambda_1^j \ \ t^{n\beta i+\beta j}}{\Gamma_\beta(n\beta i+\beta j+1)} c_{n-1} \\
&+ \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \binom{i+j-1}{j} \frac{\lambda_2^i \lambda_1^j \ \ t^{n\beta i+\beta j}}{\Gamma_\beta(n\beta i+\beta j+1)} c_{n-1} \\
&+ \lambda_2 \sum_{k=0}^{n-2} \frac{t^{(k+1)\beta}}{\Gamma_\beta((k+1)\beta+1)} c_k \\
&+ \lambda_2 \sum_{k=0}^{n-2}\sum_{i=0}^{\infty}\sum_{j=1}^{\infty} \binom{i+j-1}{j-1} \frac{\lambda_2^i \lambda_1^j \ \ t^{n\beta i+\beta j+(k+1)\beta}}{\Gamma_\beta(n\beta i+\beta j+(k+1)\beta+1)} c_k \\
&+ \lambda_2 \sum_{k=0}^{n-2}\sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \binom{i+j-1}{j} \frac{\lambda_2^i \lambda_1^j \ \ t^{n\beta i+\beta j+(k+1)\beta}}{\Gamma_\beta(n\beta i+\beta j+(k+1)\beta+1)} c_k,
\end{aligned}
$$

rearranging the indices and using the definition of the conformable bivariate Mittag-Leffler functions

$$
\begin{aligned}
\mathcal{D}_0^{(n-1)\beta} c(t) = &\frac{1}{\Gamma_\beta(1)} c_{n-1} + \lambda_1 t^\beta E_{n\beta,\beta}^{\beta+1}\left(\lambda_2 t^{n\beta}, \lambda_1 t^\beta\right) c_{n-1} \\
&+ \lambda_2 t^{n\beta} E_{n\beta,\beta}^{n\beta+1}\left(\lambda_2 t^{n\beta}, \lambda_1 t^\beta\right) c_{n-1} \\
&+ \lambda_2 \sum_{k=0}^{n-2} \frac{t^{(k+1)\beta}}{\Gamma_\beta((k+1)\beta+1)} c_k \\
&+ \lambda_1\lambda_2 \sum_{k=0}^{n-2} t^{(k+2)\beta} E_{n\beta,\beta}^{(k+2)\beta+1}\left(\lambda_2 t^{n\beta}, \lambda_1 t^\beta\right) c_k \\
&+ \lambda_2^2 \sum_{k=0}^{n-2} t^{(k+1)\beta+n\beta} E_{n\beta,\beta}^{(k+1)\beta+n\beta+1}\left(\lambda_2 t^{n\beta}, \lambda_1 t^\beta\right) c_k.
\end{aligned}
$$

If the conformable fractional derivative is applied to the just-above equation, one can get $\mathcal{D}_0^{n\beta}c\,(t)$:

$$= \lambda_1 E^1_{n\beta,\beta}\left(\lambda_2 t^{n\beta}, \lambda_1 t^{\beta}\right) c_{n-1} + \lambda_2 t^{(n-1)\beta} E^{(n-1)\beta+1}_{n\beta,\beta}\left(\lambda_2 t^{n\beta}, \lambda_1 t^{\beta}\right) c_{n-1}$$
$$+ \lambda_2 \sum_{k=0}^{n-2} \frac{t^{k\beta}}{\Gamma_\beta\,(k\beta+1)} c_k + \lambda_1\lambda_2 \sum_{k=0}^{n-2} t^{(k+1)\beta} E^{(k+1)\beta+1}_{n\beta,\beta}\left(\lambda_2 t^{n\beta}, \lambda_1 t^{\beta}\right) c_k$$
$$+ \lambda_2^2 \sum_{k=0}^{n-2} t^{k\beta+n\beta} E^{k\beta+n\beta+1}_{n\beta,\beta}\left(\lambda_2 t^{n\beta}, \lambda_1 t^{\beta}\right) c_k. \tag{4}$$

We also have,

$$-\lambda_1 \mathcal{D}_0^{(n-1)\beta} c\,(t) = -\lambda_1 E^1_{n\beta,\beta}\left(\lambda_2 t^{n\beta}, \lambda_1 t^{\beta}\right) c_{n-1}$$
$$- \lambda_1\lambda_2 \sum_{k=0}^{n-2} t^{(k+1)\beta} E^{(k+1)\beta+1}_{n\beta,\beta}\left(\lambda_2 t^{n\beta}, \lambda_1 t^{\beta}\right) c_k, \tag{5}$$

and $-\lambda_2 c\,(t)$:

$$= -\lambda_2 \sum_{k=0}^{n-2} \frac{t^{k\beta}}{\Gamma_\beta\,(k\beta+1)} c_k - \lambda_2\lambda_2 \sum_{k=0}^{n-2} t^{(n+k)\beta} E^{(n+k)\beta+1}_{n\beta,\beta}\left(\lambda_2 t^{n\beta}, \lambda_1 t^{\beta}\right) c_k$$
$$- \lambda_2 t^{(n-1)\beta} E^{(n-1)\beta+1}_{n\beta,\beta}\left(\lambda_2 t^{n\beta}, \lambda_1 t^{\beta}\right) c_{n-1}. \tag{6}$$

The summation of equalities (4), (5), and (6) provides the desired result, that is,

$$\mathcal{D}_0^{n\beta} c\,(t) - \lambda_1 \mathcal{D}_0^{(n-1)\beta} c\,(t) - \lambda_2 c\,(t) = 0.$$

□

Theorem 3.8. *The following convolution function*

$$c\,(t) = t^{(n-1)\beta} E^{(n-1)\beta+1}_{n\beta,\beta}\left(\lambda_2 t^{n\beta}, \lambda_1 t^{\beta}\right) * \zeta(t) \quad t \geq 0,$$

is a solution to the inhomogeneous generalized sequential conformable fractional differential equations.

Proof. This proof can be done in the same approaches as Theorem 3.7. However, we use the constant variation technique to simplify the proof.

Then an arbitrary solution $c(t)$ of the inhomogeneous generalized sequential conformable fractional differential equations should be fulfilled in the following structure

$$c\left(t\right)=t^{(n-1)\beta}E_{n\beta,\beta}^{(n-1)\beta+1}\left(\lambda_2 t^{n\beta},\lambda_1 t^{\beta}\right)\ast \urcorner(t),$$

where $\urcorner(t)$ is an unknown continuous function. Keeping Theorems 2.9 and 2.10 in mind and applying the conformable Laplace transform to the inhomogeneous system,

$$\begin{aligned}&\mathcal{L}_{\beta}\left\{\zeta(t)\right\}\\&=\left(s^{n}-\lambda_1 s^{n-1}-\lambda_2\right)\mathcal{L}_{\beta}\left\{t^{(n-1)\beta}E_{n\beta,\beta}^{(n-1)\beta+1}\left(\lambda_2 t^{n\beta},\lambda_1 t^{\beta}\right)\right\}\mathcal{L}_{\beta}\left\{\urcorner(t)\right\}.\end{aligned}$$

Lemma 3.6 provides

$$\mathcal{L}_{\beta}\left\{\urcorner(t)\right\}=\mathcal{L}_{\beta}\left\{\zeta(t)\right\}.$$

Based on the uniqueness of the conformable Laplace transform, we get

$$\urcorner(t)=\zeta(t),\quad t\geq 0,$$

which completes the proof. □

Theorem 3.9. *An explicit solution to the linear generalized sequential conformable fractional differential equations is given by*

$$\begin{aligned}c\left(t\right)=&\sum_{k=0}^{n-2}\frac{t^{k\beta}}{\Gamma_{\beta}\left(k\beta+1\right)}c_k+\lambda_2\sum_{k=0}^{n-2}t^{(n+k)\beta}E_{n\beta,\beta}^{(n+k)\beta+1}\left(\lambda_2 t^{n\beta},\lambda_1 t^{\beta}\right)c_k\\&+t^{(n-1)\beta}E_{n\beta,\beta}^{(n-1)\beta+1}\left(\lambda_2 t^{n\beta},\lambda_1 t^{\beta}\right)c_{n-1}\\&+t^{(n-1)\beta}E_{n\beta,\beta}^{(n-1)\beta+1}\left(\lambda_2 t^{n\beta},\lambda_1 t^{\beta}\right)\ast\zeta\left(t\right).\end{aligned}$$

Proof. The proof of this theorem is obvious. Hence, we will omit it. □

We can verify the explicit solution to the equation (1) as given in Theorem 3.9 by using the conformable Laplace transformation technique.

Theorem 3.10. *The following integral equation*

$$\begin{aligned} c(t) &= \sum_{k=0}^{n-2} \frac{t^{k\beta}}{\Gamma_\beta(k\beta+1)} c_k + \lambda_2 \sum_{k=0}^{n-2} t^{(n+k)\beta} E_{n\beta,\beta}^{(n+k)\beta+1}\left(\lambda_2 t^{n\beta}, \lambda_1 t^{\beta}\right) c_k \\ &+ t^{(n-1)\beta} E_{n\beta,\beta}^{(n-1)\beta+1}\left(\lambda_2 t^{n\beta}, \lambda_1 t^{\beta}\right) c_{n-1} \\ &+ t^{(n-1)\beta} E_{n\beta,\beta}^{(n-1)\beta+1}\left(\lambda_2 t^{n\beta}, \lambda_1 t^{\beta}\right) * \zeta(t), \end{aligned}$$

is a representation of a solution to the system (1).

Proof. Applying the conformable Laplace integral transformation to the system (1), we get

$$\begin{aligned} &\mathcal{L}_\beta\left\{\mathcal{D}_0^{n\beta} c(t)\right\}(s) - \lambda_1 \mathcal{L}_\beta\left\{\mathcal{D}_0^{(n-1)\beta} c(t)\right\}(s) \\ &= \lambda_2 \mathcal{L}_\beta\{c(t)\}(s) + \mathcal{L}_\beta\{\zeta(t)\}(s). \end{aligned}$$

Implementing Theorem 2.9 to the just above equation, we get

$$\begin{aligned} &s^n \mathcal{L}_\beta\{c(t)\}(s) - \sum_{i=0}^{n-1} s^{n-i-1} \mathcal{D}_0^{i\beta} c(0) \\ &- \lambda_1 \left(s^{n-1} \mathcal{L}_\beta\{c(t)\}(s) - \sum_{i=0}^{n-2} s^{n-i-2} \mathcal{D}_0^{i\beta} c(0) \right) \\ &= \lambda_2 \mathcal{L}_\beta\{c(t)\}(s) + \mathcal{L}_\beta\{\zeta(t)\}(s), \end{aligned}$$

and

$$\begin{aligned} &\Rightarrow \left(s^n - \lambda_1 s^{n-1} - \lambda_2\right) \mathcal{L}_\beta\{c(t)\}(s) \\ &= \sum_{i=0}^{n-2} \left(s^{n-i-1} - \lambda_1 s^{n-i-2}\right) c_i + c_{n-1} + \mathcal{L}_\beta\{\zeta(t)\}(s) \\ &= \sum_{i=0}^{n-2} s^{-i-1} \left(s^n - \lambda_1 s^{n-1}\right) c_i + c_{n-1} + \mathcal{L}_\beta\{\zeta(t)\}(s). \end{aligned}$$

$$
\begin{aligned}
\mathcal{L}_\beta \left\{c\left(t\right)\right\}\left(s\right) &= \sum_{i=0}^{n-2} s^{-i-1}\frac{s^n-\lambda_1 s^{n-1}}{s^n-\lambda_1 s^{n-1}-\lambda_2}c_i + \frac{1}{s^n-\lambda_1 s^{n-1}-\lambda_2}c_{n-1} \\
&+ \frac{1}{s^n-\lambda_1 s^{n-1}-\lambda_2}\mathcal{L}_\beta\left\{\zeta\left(t\right)\right\}\left(s\right) \\
&= \sum_{i=0}^{n-2} s^{-i-1}\left(1+\frac{\lambda_2}{s^n-\lambda_1 s^{n-1}-\lambda_2}\right)c_i + \frac{1}{s^n-\lambda_1 s^{n-1}-\lambda_2}c_{n-1} \\
&+ \frac{1}{s^n-\lambda_1 s^{n-1}-\lambda_2}\mathcal{L}_\beta\left\{\zeta\left(t\right)\right\}\left(s\right) \\
&= \sum_{i=0}^{n-2} s^{-i-1}c_i + \lambda_2\sum_{i=0}^{n-2}\frac{s^{-i-1}}{s^n-\lambda_1 s^{n-1}-\lambda_2}c_i \\
&+ \frac{1}{s^n-\lambda_1 s^{n-1}-\lambda_2}c_{n-1} + \frac{1}{s^n-\lambda_1 s^{n-1}-\lambda_2}\mathcal{L}_\beta\left\{\zeta\left(t\right)\right\}\left(s\right).
\end{aligned}
$$

The Laplace inverse transform is taken on both sides of the aforementioned equation, using (2), one acquires

$$
\begin{aligned}
c\left(t\right) &= \sum_{i=0}^{n-2}\mathcal{L}_\beta^{-1}\left\{s^{-i-1}\right\}\left(t\right)c_i + \lambda_2\sum_{i=0}^{n-2}\mathcal{L}_\beta^{-1}\left\{\frac{s^{-i-1}}{s^n-\lambda_1 s^{n-1}-\lambda_2}\right\}\left(t\right)c_i \\
&+ \mathcal{L}_\beta^{-1}\left\{\frac{1}{s^n-\lambda_1 s^{n-1}-\lambda_2}\right\}\left(t\right)c_{n-1} \\
&+ \mathcal{L}_\beta^{-1}\left\{\frac{1}{s^n-\lambda_1 s^{n-1}-\lambda_2}\mathcal{L}_\beta\left\{\zeta\left(t\right)\right\}\left(s\right)\right\}\left(t\right).
\end{aligned}
$$

By utilizing Lemma 3.6, we can achieve the intended outcome in the following manner.

$$
\begin{aligned}
c\left(t\right) &= \sum_{k=0}^{n-2}\frac{t^{k\beta}}{\Gamma_\beta\left(k\beta+1\right)}c_k + \lambda_2\sum_{k=0}^{n-2}t^{(n+k)\beta}E_{n\beta,\beta}^{(n+k)\beta+1}\left(\lambda_2 t^{n\beta},\lambda_1 t^\beta\right)c_k \\
&+ t^{(n-1)\beta}E_{n\beta,\beta}^{(n-1)\beta+1}\left(\lambda_2 t^{n\beta},\lambda_1 t^\beta\right)c_{n-1} \\
&+ t^{(n-1)\beta}E_{n\beta,\beta}^{(n-1)\beta+1}\left(\lambda_2 t^{n\beta},\lambda_1 t^\beta\right)*\zeta\left(t\right).
\end{aligned}
$$

□

Remark 3.11. Although, in Theorem 3.10, we have supposed that the conformable Laplace transforms of $c(t)$, $\zeta(t)$, and $\mathcal{D}_0^{n\beta}c(t)$ for $n = 1, 2, \ldots$ exist, Theorem 3.9 has shown that these conditions could be removed.

The following corollary expresses the special case of our problem for $n = 2$, which corresponds to that of [10].

Corollary 3.12. *The below continuous function*

$$c(t) = \frac{1}{\Gamma_\beta(1)} c_0 + \lambda_2 t^{2\beta} E_{2\beta,\beta}^{2\beta+1}\left(\lambda_2 t^{2\beta}, \lambda_1 t^{\beta}\right) c_0 + t^{\beta} E_{2\beta,\beta}^{\beta+1}\left(\lambda_2 t^{2\beta}, \lambda_1 t^{\beta}\right) c_1 + t^{\beta} E_{2\beta,\beta}^{\beta+1}\left(\lambda_2 t^{2\beta}, \lambda_1 t^{\beta}\right) * \zeta(t),$$

is a solution to the following linear system

$$\begin{cases} \mathcal{D}_0^{2\beta} c(t) - \lambda_1 \mathcal{D}_0^{\beta} c(t) = \lambda_2 c(t) + \zeta(t), & t \in (0, T], \\ c(0) = c_0, \quad \mathcal{D}_0^{\beta} c(0) = c_1, \end{cases}$$

where the symbols of $\mathcal{D}_0^{2\beta}$ and $\mathcal{D}_0^{\beta}$ stand for the sequential conformable derivatives of orders $1 < 2\beta \leq 2$ and $0 < \beta \leq 1$, respectively. Here, $\mathcal{D}_0^{2\beta} = \mathcal{D}_0^{\beta}\mathcal{D}_0^{\beta}$.

4 An Application to the Vibration of Springs

Inhomogeneous linear ordinary differential equations with second-order appear in the works of electrical circuits and the vibration of springs[3, 24].

The vibration theory of springs is mostly exploited in the primary suspension system of road vehicles such as trucks, buses, cars, etc; and heavy vehicles such as wagons, railway coaches, etc. It is also employed in the suspension system of machine beds which are exposed to vibrations and damping systems to absorb shocks.

We will investigate the motion of an object with mass m at the end of a spring that is exposed to a frictional force F_f in the case of a horizontal spring as in Figure 1 or a damping force F_d in the case of the fact that a vertical spring moves through a fluid as in Figure 2 in addition to the spring affected by external force $F(t)$.

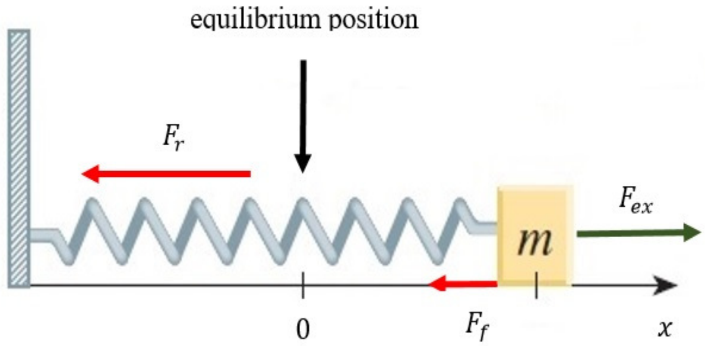

Figure 1: The horizontal system

According to Hooke's Law, when the spring is compressed or stretched, it extends c units from its natural length. So it spends a force that is proportional to c, that is,

$$r = \text{restoring force} = -kc, \quad k > 0 \text{ (spring constant)},$$

can be verified via experimental data. We suppose that a force of friction or damping force is proportional to the velocity of the mass and it is in the opposite direction of the motion which has been verified approximately by some physical experiments, i.e.

$$F_d = \text{damping force} = -e\frac{dc}{dt} := \text{frictional force} := F_r,$$

where $e > 0$ is a frictional or damping force. We also assume that the motion of the spring is affected by an external force $F_{ex} = F(t)$.

According to Newton's second law which expresses that the sum of all forces in a motion is equal to mass times acceleration, we have

$$\sum F = m.a,$$

where $a = \frac{dv}{dt} = \frac{d^2}{dt^2}c$. So we acquire

$$F_r + F_f + F_{ex} = m\frac{d^2c}{dt^2},$$

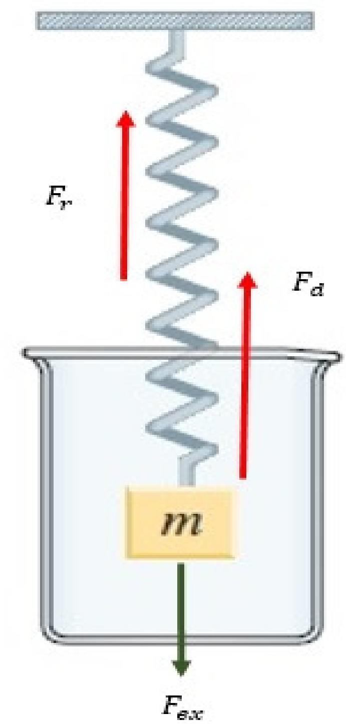

Figure 2: The vertical system

and

$$-kc - e\frac{dc}{dt} + F(t) = m\frac{d^2c}{dt^2}.$$

One can rewrite an IVP consisting of ordinary derivatives as noted below

$$\begin{cases} m\frac{d^2}{dt^2}c(t) + e\frac{d}{dt}c(t) + kc(t) = F(t), \\ c\left(0\right) = c_0, \quad c^{'}\left(0\right) = c_1. \end{cases}$$

As done in the reference[24], one can reformulate the initial value problem for the linear conformable fractional equations by switching integer-order derivatives with fractional-order ones $0 < \beta \leq 1$, $1 < 2\beta \leq 2$. In this regard, the desired initial value problem could be remodeled as stated below

$$\begin{cases} m\mathcal{D}_0^{2\beta} c\left(t\right) + e\mathcal{D}_0^{\beta} c\left(t\right) + kc\left(t\right) = F\left(t\right), \\ \quad c\left(0\right) = c_0, \quad \mathcal{D}_0^{\beta} c\left(0\right) = c_1, \end{cases} \tag{7}$$

where

$$\lim_{\beta \to 1} \mathcal{D}_0^{\beta} c\left(t\right) = \frac{d}{dt} c(t), \quad \lim_{\beta \to 1} \mathcal{D}_0^{2\beta} c\left(t\right) = \mathcal{D}_0^{2\beta} c\left(t\right).$$

Based on Corollary 3.12, one gets an exact analytical solution to system (7) as follows:

$$\begin{aligned} c\left(t\right) &= \frac{1}{\Gamma_{\beta}\left(1\right)} c_0 - \frac{k}{m} t^{2\beta} E_{2\beta,\beta}^{2\beta+1} \left(-\frac{k}{m} t^{2\beta}, -\frac{e}{m} t^{\beta} \right) c_0 \\ &\quad + t^{\beta} E_{2\beta,\beta}^{\beta+1} \left(-\frac{k}{m} t^{2\beta}, -\frac{e}{m} t^{\beta} \right) c_1 + t^{\beta} E_{2\beta,\beta}^{\beta+1} \left(-\frac{k}{m} t^{2\beta}, -\frac{e}{m} t^{\beta} \right) * F\left(t\right). \end{aligned}$$

The graphs of the position $c(t)$ and the velocity $v(t)$ for $k = 128$, $e = 30$, $m = 5$, $\beta = 0.55$, $c_0 = 20$, $c_1 = 52$, and $F(t) = 20t^2 + 5t + 2$ are given in Figure 3. As seen in Figure 1, the zero point is accepted as the equilibrium position. Thus, Figure 3 expresses the position and velocity at an arbitrary time of a mass that starts at the equilibrium position and is exposed to some mentioned forces, again reaching the equilibrium position.

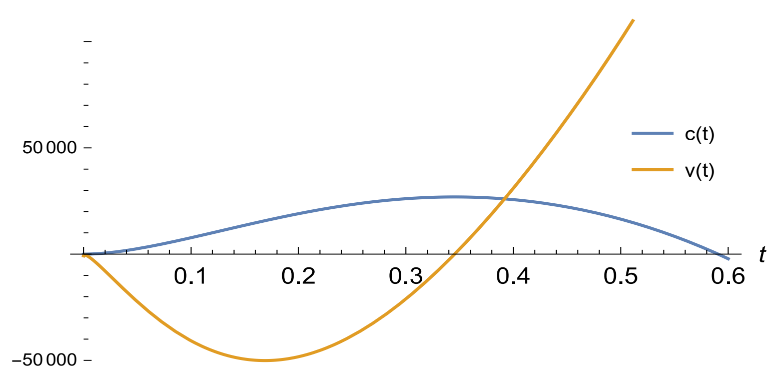

Figure 3

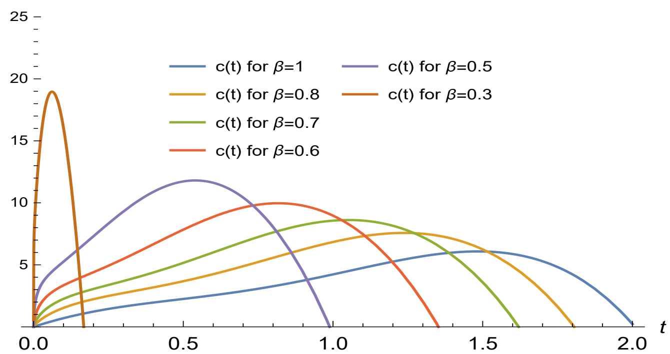

Figure 4

Table 1: The amplitude (amp) change

	$\beta = 1$	$\beta = 0.8$	$\beta = 0.7$	$\beta = 0.6$	$\beta = 0.5$	$\beta = 0.3$
Amp.	6.08361	7.57481	8.61478	9.96791	11.8069	18.9655

The graphs of the positions $c(t)$ for each of $\beta = 1$, $\beta = 0.8$, $\beta = 0.7$, $\beta = 0.6$, $\beta = 0.5$, $\beta = 0.3$ with the common parameters $k = 7.8$, $e = 9$, $m = 3$, $c_0 = 0$, $c_1 = 8$, and $F(t) = t$ are given in Figure 4. One can infer from Figure 4 that a mass that starts at the equilibrium position again reaches the equilibrium position earlier as the values of the beta parameters decrease.

Note that the amplitude (of vibration of springs) refers to the maximum displacement of a vibrating body from its equilibrium position. In Table 1, amplitudes obtained from the position functions, which depend on the distinct beta parameters, presented in Figure 4 are shared. For example, when $\beta = 1$, the corresponding amplitude of vibration of springs is equal to $6.08361cm$. In the case of $\beta = 0.6$, the corresponding one is $9.96791cm$. For the last one, it is $18.9655cm$ It is easily observed from Table 1 that the values of amplitudes increase as the values of the beta parameters decrease.

It should be emphasized that system (7) transforms to the ordinary second-order linear system when $\beta = 1$. So the results of the fractional linear system with $0 < \beta < 1$ and the ordinary second order linear

Table 2: Positions of the mass at some time (t seconds) for $\beta = 0.7$

0.1	0.3	0.4	0.5	0.7	0.8	0.9	1	Amp.
1.93	5.87	9.14	12.81	19.31	21.08	21.26	19.44	21.4142

system with $\beta = 1$ are compared in Figure 4 and Table 1.

In addition, let's draw the graphs of the position functions $c(t)$ of the mass at any time for each of $\beta = 0.6$, $\beta = 0.7$ with the new parameters $k = 10$, $e = 8$, $m = 2$, and $F(t) = t$ are given in Figure 4 when it starts from the equilibrium position and is given a push to start it with an initial velocity of $8cm/sec$. It is drawn in Figure 5. In Table 2 and 3, the positions and amplitudes of the mass at some time in the cases of $\beta = 0.6$ and $\beta = 0.6$ are respectively presented. It is observed from Table 2, Table 3, and Figure 5 that the mass moves away from the equilibrium position rapidly and comes to the equilibrium position even faster when the case $\beta = 0.6$ is compared to the case $\beta = 0.7$. This also shows that there is a strong relationship between the system's order β and the velocity of the mass. The similar situation is also true for their amplitudes, that is, the amplitude for $\beta = 0.6$ is greater than that for $\beta = 0.7$.

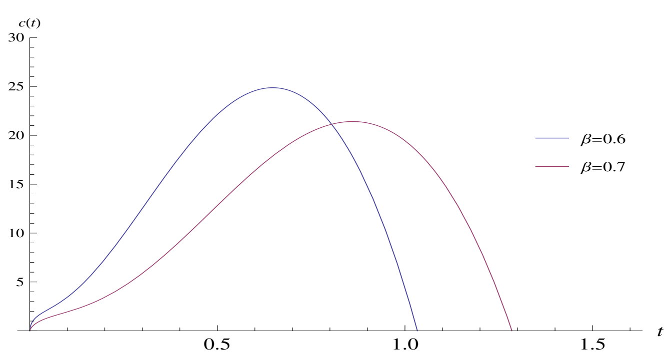

Figure 5

Table 3: Positions of the mass at some time (t seconds) for $\beta = 0.6$

0.1	0.3	0.4	0.5	0.7	0.8	0.9	1	Amp.
3.43	12.52	17.82	22.13	24.47	21.35	14.80	4.41	24.8752

5 Conclusion and Future Work

This study primarily focuses on the introduction of a linear sequential conformable fractional differential system with two generalized fractional orders and bivariate conformable Mittag-Leffler functions. An explicit solution that incorporates the bivariate conformable Mittag-Leffler functions for the system is derived. Additionally, an analysis on the vibration of springs is conducted to solidify the theoretical results.

As possible future work, one can explore finite-time stability, asymptotic stability, Lyapunov-type stability, and exponential stability of the tackled system. As another possible future work, one can investigate whether it is iteratively learning controllable or relatively and approximately controllable.

6 Declarations

The conflict of interest statement:
This work does not have any conflicts of interest.

The funding:
There are no funders to report for this submission.

7 Acknowledgement

I would like to present sincere pleasure to the editor and the anonymous reviewers very much for their positive contributions.

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