Journal of Mathematical Extension
Vol. 18, No. 7, (2024) (4)1-23
URL: https://doi.org/10.30495/JME.2024.3005
ISSN: 1735-8299
Original Research Paper

Non-Newtonian Lebesgue Spaces with Their Basic Features

I. Eryilmaz
Ondokuz Mayıs University

Abstract. Non-Newtonian Lebesgue spaces will be emerged as a pivotal field in functional analysis, extending the classical Lebesgue spaces to encompass non-Newtonian Real numbers behaviors encountered in various physical and mathematical phenomena. This paper offers a comprehensive investigation into the properties, characteristics, and significance of these new spaces.

AMS Subject Classification: 11U10, 26D15, 46A30
Keywords and Phrases: Non-Newtonian numbers, generator function, $*-$calculus, $*-$ Hölder's inequality, $*-$Minkowski inequality, Non-Newtonian Lebesgue space

1 Introduction and Preliminaries

Non-Newtonian (briefly NN-) calculus was created between 1967 and 1970 by Michael Grossman and Robert Katz in [10]. They initially defined an infinite family of calculus, which includes classical calculus, geometric calculus, harmonic calculus, and quadratic calculus. Later, they formed bigeometric, biharmonic, and biquadratic calculus, using the term "NN-calculus" for each calculus distinct from classical calculus.

Received: February 2024; Accepted: November 2024

NN-calculus has applications in various fields such as natural sciences, economy, engineering, mathematics, and more. Among the fields studied are interest rates, elasticity theory in economics, blood fluidity, information technology, biology, differential equations, functional analysis, dynamic systems, fractals, probability theory, and others. Hence, it can be said that it is significantly used as an alternative to Newton's and Leibniz's classical calculus.

Grossman worked on derivatives and integrals in NN-calculus in [9]. In the study [11], derivative independent of measure was examined in bigeometric calculus. The authors studied geometric (multiplicative) calculus and its applications in [1]. Multiplicative type complex calculus is studied in [16].

Real sequence spaces according to geometric calculus is studied and some fundamental results are obtained in [15]. While real sequence spaces according to NN-calculus are defined in [4], new results on complex sequence spaces according to NN-calculus are obtained in [14].

The space of continuous functions and multiple integrals on real sequences according to NN-calculus are studied in [3] and [5]. Some fundamental topological properties on NN-real numbers are showed in [2] and [7].

In [13], the authors examined series and sequences in NN-calculus. They introduced $*-$series and $*-$sequences in NN-calculus and provided some theorems and properties based on NN-real number field $\mathbb{R}_\alpha$. The concept of $*-$function sequences, $*-$function series, $*-$pointwise convergence, and $*-$uniform convergence on NN-real number field $\mathbb{R}_\alpha$ are introduced, and theorems that reveal important differences between $*-$pointwise convergence and $*-$uniform convergence are proved in [13]. Moreover, properties arising from $*-$convergence and convergence tests such as $*-$Cauchy criterion and $*-$Weierstrass M-test for $*-$uniform convergence are obtained and $*-$Abel and $*-$Dirichlet tests are provided in [6]. Then, power series are introduced, and properties of summability and Abel's interpretation are obtained in the same paper.

In [8], generalization of the usual Lebesgue measure in real numbers to NN-real numbers is given. For this purpose, they defined the NN-Lebesgue measure on NN-open and NN-closed sets and examined their basic properties.

The concept of a complete ordered field stems from the formalization of the real number system, which is widely familiar. Essentially, a complete ordered field comprises a set X along with four binary operations $\dot{+}, \dot{-}, \dot{/}, \dot{\times}$ and an ordering relation $\dot{<}$ defined on X [10]. When the domain of a complete ordered field is a part of $\mathbb{R}$, it's termed as arithmetic. A generator is a bijective function that maps from $\mathbb{R}$ to a subset of $\mathbb{R}$. Each generator precisely yields one arithmetic, and each arithmetic is uniquely produced by a generator. Consider any generator ζ with a range in $\mathbb{R}$. By $\zeta-$arithmetic, we imply arithmetic with domain $\mathbb{R}$ and $\zeta-$operations defined as follows:

$$\begin{array}{ll} \zeta-\text{addition} & \dot{u}\dot{+}\dot{v} = \zeta\left\{\zeta^{-1}(\dot{u}) + \zeta^{-1}(\dot{v})\right\} \\ \zeta-\text{subtraction} & \dot{u}\dot{-}\dot{v} = \zeta\left\{\zeta^{-1}(\dot{u}) - \zeta^{-1}(\dot{v})\right\} \\ \zeta-\text{multiplication} & \dot{u}\dot{\times}\dot{v} = \zeta\left\{\zeta^{-1}(\dot{u}) \cdot \zeta^{-1}(\dot{v})\right\} \\ \zeta-\text{division} & \dot{u}\dot{/}\dot{v} = \frac{\dot{u}}{\dot{v}}N = \zeta\left\{\zeta^{-1}(\dot{u}) / \zeta^{-1}(\dot{v})\right\} \\ \zeta-\text{order} & \dot{u}\dot{<}\dot{v} \Leftrightarrow \zeta^{-1}(\dot{u}) < \zeta^{-1}(\dot{v}) \end{array}$$

for all $u, v \in \mathbb{R}$. The set of non-Newtonian (briefly NN-) real numbers are defined as

$$\mathbb{R}(N) = \mathbb{R}_\zeta := \{\dot{x} = \zeta(x) : x \in \mathbb{R}\}$$

in [10]. By using the new operations above, it is showed in [4] that $\left(\mathbb{R}_\zeta, \dot{+}, \dot{-}, \dot{/}, \dot{\times}, \dot{<}\right)$ is a complete ordered field. The $\zeta-$positive real numbers $\mathbb{R}_\zeta^+$ are the numbers $\dot{x} \in \mathbb{R}_\zeta$ such that $\dot{0}\dot{<}\dot{x}$; the $\zeta-$negative real numbers $\mathbb{R}_\zeta^-$ are those for which $\dot{x}\dot{<}\dot{0}$. The $\zeta-$zero, $\dot{0} = \zeta(0)$, the $\zeta-$one $\dot{1} = \zeta(1)$ and $\zeta(-1) = \dot{0}\dot{-}\dot{1}$. The exponential function in the realm of NN-calculus has garnered significant attention and investigation within the existing literature. Researchers have extensively explored and studied this particular case, delving into its implications and applications within the framework of NN-calculus. According to this, if $\zeta(x) = e^x$, then $\dot{0} = 1$, $\dot{1} = e$, $\zeta^{-1}(\cdot) = \ln(\cdot)$ and

$$\begin{aligned} \dot{x}\dot{+}\dot{y} &= \zeta\left\{\zeta^{-1}(\dot{x}) + \zeta^{-1}(\dot{y})\right\} = e^{\{\ln(\dot{x})+\ln(\dot{y})\}} = \dot{x}\dot{y} \\ \dot{x}\dot{-}\dot{y} &= \zeta\left\{\zeta^{-1}(\dot{x}) - \zeta^{-1}(\dot{y})\right\} = e^{\{\ln(\dot{x})-\ln(\dot{y})\}} = \dot{x}/\dot{y} \\ \dot{x}\dot{\times}\dot{y} &= \zeta\left\{\zeta^{-1}(\dot{x}) \cdot \zeta^{-1}(\dot{y})\right\} = e^{\{\ln(\dot{x})\ln(\dot{y})\}} = \dot{y}^{\ln(\dot{x})} \\ \dot{x}\dot{/}\dot{y} &= \zeta\left\{\zeta^{-1}(\dot{x}) / \zeta^{-1}(\dot{y})\right\} = e^{\{\ln(\dot{x})/\ln(\dot{y})\}} = \dot{x}^{\frac{1}{\ln(\dot{y})}} \end{aligned}$$

can be found easily.

For any numbers $\dot{a}$ and $\dot{b}$ in $\mathbb{R}_\zeta$, if $\dot{a}\dot{<}\dot{b}$, then the set of all numbers $\dot{x} \in\mathbb{R}_\zeta$ such that $\dot{a}\dot{\leq}\dot{x}\dot{\leq}\dot{b}$ which is called an $\zeta-$interval is denoted by $[a,b]_N := \dot{[}\dot{a},\dot{b}\dot{]}$. The $\zeta-$square of a number $\dot{x} \in\mathbb{R}_\zeta$ is denoted by $\dot{x}\dot{\times}\dot{x} = \dot{x}^{2_N}$. For each $\zeta-$nonnegative number $\dot{x}$, the symbol $\dot{\sqrt{\dot{x}}}$ is used to denote the $\zeta-$square root of $\dot{x}$, i.e. $\dot{\sqrt{\dot{x}}} = \zeta\left\{\sqrt{\zeta^{-1}\left(\dot{x}\right)}\right\}$ which is the unique $\zeta-$nonnegative number whose $\zeta-$square is equal to $\dot{x}$. Throughout this paper, we will denote the p^{th} NN-exponent and the q^{th} NN-root of $\dot{x} \in \mathbb{R}\left(N\right)$ by $\dot{x}^{p_N}$ and $\sqrt[q]{\dot{x}}^N$, respectively. The $\zeta-$absolute value of a number $\dot{x} \in\mathbb{R}_\zeta$ is defined as $\zeta\left(\left|\zeta^{-1}\left(\dot{x}\right)\right|\right)$ and is denoted by

$$\dot{|}\dot{x}\dot{|} = |\dot{x}|_N = \begin{cases} \dot{x}, & \dot{0}\dot{<}\dot{x} \\ \dot{0}, & \dot{0} = \dot{x} \\ \dot{0}\dot{-}\dot{x}, & \dot{x}\dot{<}\dot{0} \end{cases}$$

and so $\dot{\sqrt{\dot{x}^{2_N}}} = \dot{|}\dot{x}\dot{|}$ [4, 10]. It is showed in [4] that for any $\dot{x},\dot{y} \in\mathbb{R}_\zeta$, the following statements hold:

(i) $\dot{|}\dot{x}\dot{\times}\dot{y}\dot{|} = \dot{|}\dot{x}\dot{|}\dot{\times}\dot{|}\dot{y}\dot{|}$
(ii) $\dot{|}\dot{x}\dot{+}\dot{y}\dot{|}\dot{\leq}\dot{|}\dot{x}\dot{|}\dot{+}\dot{|}\dot{y}\dot{|}$
(iii) If $p > 1$ and $\dot{x}_k, \dot{y}_k \in\mathbb{R}_\zeta^+$ for $k = 1, 2, \ldots, n$, then

$$\sqrt[p]{{}_N\sum_{k=1}^{n}\left(\dot{x}_k\dot{+}\dot{y}_k\right)^{p_N}}^N \dot{\leq} \sqrt[p]{{}_N\sum_{k=1}^{n}\dot{x}_k^{p_N}}^N \dot{+} \sqrt[p]{{}_N\sum_{k=1}^{n}\dot{y}_k^{p_N}}^N .$$

Also, it is proved that $\left(\mathbb{R}_\zeta^n, d_N\right)$ is a complete NN-metric space with

$$d_N\left(\dot{x},\dot{y}\right) = \dot{\sqrt{{}_N\sum_{k=1}^{n}\left(\dot{x}_k\dot{-}\dot{y}_k\right)^{2_N}}}$$

for any $\dot{x},\ \dot{y} \in \mathbb{R}_\zeta^n$ where $\mathbb{R}_\zeta^n$ is n-dimensional NN-Euclidian space. The concepts of NN-convergent sequence, NN-metric space, NN-completeness, NN-upper and lower bounds, NN-supremum (infimum), NN-open (closed) set etc. are discussed in [4, 7, 13] and [14]. In their work [10], the authors introduced a novel calculus, referred to as $*-$calculus, that embodies the overarching structure of NN-calculus. As outlined in [10], due to the isomorphic nature of all arithmetics, it becomes straightforward to derive all arithmetics by employing a unique function from one arithmetic to another.

Suppose that ζ, β are two arbitrarily selected generators and let the corresponding complete ordered fields be $\left(\mathbb{R}_\zeta, \dot{+}, \dot{-}, \dot{/}, \dot{\times}, \dot{<}\right)$ and $\left(\mathbb{R}_\beta, \ddot{+}, \ddot{-}, \ddot{/}, \ddot{\times}, \ddot{<}\right)$, respectively. Then the bijective isomorphism from $\zeta-$arithmetic to $\beta-$arithmetic is a unique function ι (iota) that provides some required properties for any numbers $\dot{u}, \dot{v} \in \mathbb{R}_\zeta$:

(i) $\iota(\dot{u}\dot{+}\dot{v}) = \iota(\dot{u})\ddot{+}\iota(\dot{v}),\ \iota(\dot{u}\dot{-}\dot{v}) = \iota(\dot{u})\ddot{-}\iota(\dot{v})$,

(ii) $\iota(\dot{u}\dot{\times}\dot{v}) = \iota(\dot{u})\ddot{\times}\iota(\dot{v})$,

(iii) $\iota(\dot{u}\dot{/}\dot{v}) \ = \iota(\dot{u})\ddot{/}\iota(\dot{v});\ \dot{v} \neq \dot{0}$,

(iv) $\dot{u}\dot{\leq}\dot{v} \ \Leftrightarrow\ \iota(\dot{u})\ddot{\leq}\iota(\dot{v})$.

It turns out that $\iota(\dot{x}) = \beta\left\{\zeta^{-1}(\dot{x})\right\}$ for every $\dot{x} \in \mathbb{R}_\zeta$ and $\iota(\dot{n}) = \ddot{n}$ for every $n \in \mathbb{N}$. Since, for example, $\dot{u}\dot{+}\dot{v} = \iota^{-1}\left(\iota(\dot{u})\ddot{+}\iota(\dot{v})\right)$, it should be clear that any statement in $\zeta-$arithmetic can readily be transformed into a statement in $\beta-$arithmetic. In [7], the authors showed many properties of NN-real line and its subsets. For example, some results related with open, closed, compact, bounded sets and accumulation, limit points etc. are studied in the sense of NN-calculus. Except those, the following definition and theorems can be found in [10].

Definition 1.1. *Let $f : \mathbb{R}_\zeta \to \mathbb{R}_\beta$ be a function and $\dot{a} \in \mathbb{R}_\zeta$. If for all $\ddot{\varepsilon}\ddot{>}\ddot{0}$, there exists a $\dot{\delta} = \dot{\delta}\left(\ddot{\varepsilon}\right)\dot{>}\dot{0}$ such that $\dot{|}\dot{x}\dot{-}\dot{a}\dot{|}\dot{<}\dot{\delta}$ implies $\ddot{|}f\left(\dot{x}\right)\ddot{-}f\left(\dot{a}\right)\ddot{|}\ddot{<}\ddot{\varepsilon}$ for all $\dot{x} \in \mathbb{R}_\zeta$, then f is called $*-$continuous at the point $\dot{a} \in \mathbb{R}_\zeta$.*

Theorem 1.2. *(Fundamental theorem of $*-$calculus) Let $f : \dot{[}\dot{a}, \dot{b}\dot{]} \to \mathbb{R}_\beta$ be a $*-$continuous function. Then for all $\dot{x} \in \dot{[}\dot{a}, \dot{b}\dot{]} \subset \mathbb{R}_\zeta$ the function $g\left(\dot{x}\right) = \int\limits_{\dot{a}}^{\dot{x}} f\left(\dot{t}\right)\dot{\times}dt_N$ defined on $\dot{[}\dot{a}, \dot{x}\dot{]}$ is $*-$continuous and $*-$differentiable for all $\dot{x} \in \dot{[}\dot{a}, \dot{b}\dot{]}$ where $*-$derivative of g is $D_*\left(g\right) = f$.*

Theorem 1.3. *(Fundamental theorem of $\zeta-$calculus) Let $f : \dot{[}\dot{a}, \dot{b}\dot{]} \to \mathbb{R}_\zeta$ be a $\zeta-$continuous function, $\zeta\left(a\right) = \dot{a}, \zeta\left(b\right) = \dot{b}$ and define $\tilde{f}\left(x\right) =$*

$\zeta^{-1}(f(\dot{x}))$ *for all* $\dot{x} \in [\dot{a}, \dot{b}]$. *Then*

$$\int_{\dot{a}}^{\dot{b}} f(\dot{x}) \dot{\times} dx_N = \zeta \left\{ \int_a^b \tilde{f}(x) dx \right\} = \zeta \left\{ \int_{\zeta^{-1}(\dot{a})}^{\zeta^{-1}(\dot{b})} \zeta^{-1}(f(\dot{x})) dx \right\}$$

is written.

2 Main Results

Definition 2.1. ($*-$Derivative) *Let* u *be a* $\mathbb{R}_\beta-$*valued function defined on* $(\dot{a}, \dot{b}) \subset \mathbb{R}_\zeta$ *and* $\dot{x}_0 \in (\dot{a}, \dot{b})$. *If the* $*-$*limit given by*

$$* - \lim_{\dot{x} \to \dot{x}_0} \frac{u(\dot{x}) \ddot{-} u(\dot{x}_0)}{\iota(\dot{x}) \ddot{-} \iota(\dot{x}_0)} N$$

exists, then this limit value is called the $*-$*derivative of the function* u *at point* $\dot{x}_0$, *or it's said that the function* u *is* $*-$*differentiable at the point* $\dot{x}_0$ *and it's derivative is denoted by* $(D_* u)(\dot{x}_0)$ [10].

The $*-$derivative operator is $*-$linear, i.e.,
(i) $(D_*(u+v))(\dot{x}_0) = (D_* u)(\dot{x}_0) \ddot{+} (D_* v)(\dot{x}_0)$
(ii) $\left(D_*\left(\ddot{c} \ddot{\times} u\right)\right)(\dot{x}_0) = \ddot{c} \ddot{\times} (D_* u)(\dot{x}_0)$, for all $\ddot{c} \in \mathbb{R}_\beta$.
The $*-$right and $*-$left derivatives can be defined similarly to the traditional ones. By using the usual techniques, the following theorems can be proved.

Theorem 2.2. *Let* u *be a* $\mathbb{R}_\beta-$*valued function defined on* $(\dot{a}, \dot{b})$. u *is* $*-$*differentiable at the point* $\dot{x}_0$ *if and only if* $*-$*right and* $*-$*left derivatives exist and equal at the point* $\dot{x}_0$.

Theorem 2.3. *Let* u *be a* $\mathbb{R}_\beta-$*valued function defined on* $(\dot{a}, \dot{b})$. *If* u *is* $*-$*differentiable at the point* $\dot{x}_0$, *then* u *is* $*-$*continuous at the point* $\dot{x}_0$.

Theorem 2.4. *(Young's inequality) Let* $u = u(\dot{x})$ *be a strictly* $\zeta-$*increasing function on* $[\dot{0}, \dot{\infty})$ *with* $u(\dot{0}) = \dot{0}$. *Suppose that* u *is* $\zeta-$*continuously differantiable function on* $[\dot{a}, \dot{b}]$ *where* $\dot{0} \dot{<} \dot{a} \dot{<} \dot{b}$ *with* $\zeta(a) = \dot{a}$, $\zeta(b) = \dot{b}$. *Then*

$$\dot{a} \dot{\times} \dot{b} \dot{\leq} \int_{\dot{0}}^{\dot{a}} u(\dot{x}) \dot{\times} dx_N \dot{+} \int_{\dot{0}}^{\dot{b}} u^{-1}(\dot{y}) \dot{\times} dy_N$$

where $u^{-1}(\dot{y})$ is the inverse function of u. Besides, if $u(\dot{a}) = \dot{b}$, then the equality holds.

Proof. Let u be $\zeta-$continuously differantiable on $\dot{[}\dot{a},\dot{b}\dot{]}$ and strictly $\zeta-$increasing function on $\dot{[}\dot{0},\dot{\infty}\dot{)}$. Then

$$\begin{aligned}
\int_{\dot{a}}^{\dot{b}} u(\dot{x})\,\dot{\times} dx_N &= \zeta\left\{\int_{\zeta^{-1}(\dot{a})}^{\zeta^{-1}(\dot{b})} \zeta^{-1}(u(\dot{x}))dx\right\} \qquad (1)\\
&= \zeta\left\{\zeta^{-1}\left(\dot{b}\right)\zeta^{-1}\left(u\left(\dot{b}\right)\right) - \zeta^{-1}(\dot{a})\,\zeta^{-1}(u(\dot{a}))\right.\\
&\qquad \left. - \int_{\zeta^{-1}(\dot{a})}^{\zeta^{-1}(\dot{b})} x\left(\zeta^{-1}(u(\dot{x}))\right)' dx\right\}\\
&= \dot{b}\dot{\times}u\left(\dot{b}\right)\dot{-}\dot{a}\dot{\times}u(\dot{a})\dot{-}\int_{\dot{a}}^{\dot{b}} \dot{x}\dot{\times}D_{\zeta}(u)(\dot{x})\,\dot{\times} dx_N
\end{aligned}$$

where $D_{\zeta}(u)$ is the $\zeta-$derivative of u. If one takes $\dot{y} = u(\dot{x})$, then $dy_N = D_{\zeta}(u)(\dot{x})\,\dot{\times} dx_N$, $u^{-1}(\dot{y}) = \dot{x}$ and

$$\int_{\dot{a}}^{\dot{b}} u(\dot{x})\,\dot{\times} dx_N = \dot{b}\dot{\times}u\left(\dot{b}\right)\dot{-}\dot{a}\dot{\times}u(\dot{a})\dot{-}\int_{u(\dot{a})}^{u(\dot{b})} u^{-1}(\dot{y})\,\dot{\times} dy_N \qquad (2)$$

by (1). Since u is strictly $\zeta-$increasing function, we get $u(\dot{r})\,\dot{\leq} u(\dot{x})\,\dot{\leq} u(\dot{a})$ if $\dot{r}\dot{\leq}\dot{x}\dot{\leq}\dot{a}$. Therefore

$$\dot{a}\dot{\times}u(\dot{r})\dot{-}\dot{r}\dot{\times}u(\dot{r})\,\dot{\leq}\int_{\dot{r}}^{\dot{a}} u(\dot{x})\dot{\times}dx_N. \qquad (3)$$

By [10], it is known that

$$\int_{\dot{r}}^{\dot{a}} u(\dot{x})\dot{\times}dx_N = \int_{\dot{0}}^{\dot{a}} u(\dot{x})\dot{\times}dx_N\dot{-}\int_{\dot{0}}^{\dot{r}} u(\dot{x})\dot{\times}dx_N.$$

Therefore, we have

$$\begin{aligned}
\dot{a}\dot{\times}u(\dot{r})\dot{-}\dot{r}\dot{\times}u(\dot{r}) &\dot{\leq} \int_{\dot{r}}^{\dot{a}} u\;(\dot{x})\,\dot{\times} dx_N\\
&= \int_{\dot{0}}^{\dot{a}} u(\dot{x})\dot{\times}dx_N\dot{-}\int_{\dot{0}}^{\dot{r}} u(\dot{x})\dot{\times}dx_N \qquad (4)
\end{aligned}$$

by (3) and

$$\int_{\dot{0}}^{\dot{r}} u(\dot{x}) \dot{\times} dx_N = \dot{r} \dot{\times} u(\dot{r}) \dot{-} \int_{u(\dot{0})}^{u(\dot{r})} u^{-1}(\dot{y}) \dot{\times} dy_N \tag{5}$$

by (2). If one writes (5) in (4), then

$$\dot{a} \dot{\times} u(\dot{r}) \dot{-} \dot{r} \dot{\times} u(\dot{r}) \dot{\leq} \int_{\dot{0}}^{\dot{a}} u(\dot{x}) \dot{\times} dx_N \dot{-} \dot{r} \dot{\times} u(\dot{r}) \dot{+} \int_{\dot{0}}^{u(\dot{r})} u^{-1}(\dot{y}) \dot{\times} dy_N$$

and so

$$\dot{a} \dot{\times} u(\dot{r}) \dot{\leq} \int_{\dot{0}}^{\dot{a}} u(\dot{x}) \dot{\times} dx_N \dot{+} \int_{\dot{0}}^{u(\dot{r})} u^{-1}(\dot{y}) \dot{\times} dy_N$$

can be written. If $u(\dot{a}) \dot{>} \dot{b} \dot{>} \dot{0}$, then $\dot{r} = u^{-1}\left(\dot{b}\right)$ can be choosen. Thus

$$\dot{a} \dot{\times} \dot{b} \dot{\leq} \int_{\dot{0}}^{\dot{a}} u(\dot{x}) \dot{\times} dx_N \dot{+} \int_{\dot{0}}^{\dot{b}} u^{-1}(\dot{y}) \dot{\times} dy_N \ .$$

If we write $\dot{r} = \dot{a}$ and $u(\dot{a}) = \dot{b}$ in (5), then we get

$$\int_{\dot{0}}^{\dot{a}} u(\dot{x}) \dot{\times} dx_N = \dot{a} \dot{\times} u(\dot{a}) \dot{-} \int_{\dot{0}}^{u(\dot{a})} u^{-1}(\dot{y}) \dot{\times} dy_N$$

and

$$\dot{a} \dot{\times} \dot{b} = \int_{\dot{0}}^{\dot{a}} u(\dot{x}) \dot{\times} dx_N \dot{+} \int_{\dot{0}}^{\dot{b}} u^{-1}(\dot{y}) \dot{\times} dy_N \ .$$

□

Corollary 2.5. *Let* $\dot{1} \dot{<} \dot{p} \dot{<} \dot{q} \dot{<} \dot{\infty}$ *be such that* $\left(\dot{1} \dot{/} \dot{p}\right) \dot{+} \left(\dot{1} \dot{/} \dot{q}\right) = \dot{1}$*. Then*

$$\dot{a}\dot{b} \dot{\leq} \dot{a}^{p_N} \dot{\times} \left(\dot{1} \dot{/} \dot{p}\right) \dot{+} \dot{b}^{q_N} \dot{\times} \left(\dot{1} \dot{/} \dot{q}\right) \tag{6}$$

for all $\dot{a}$ *and* $\dot{b}$ $\zeta-$*positive real numbers. Besides, if*

$$\dot{a}^{p_N} = \dot{b}^{q_N},$$

then the equality holds.

Proof. Firstly, let $\zeta(a) = \dot{a}\dot{>}\dot{0}$ and $\zeta(b) = \dot{b}\dot{>}\dot{0}$. Since $\dot{a}^{p_N} = \zeta(a^p)$ and $\dot{b}^{q_N} = \zeta(b^q)$, if $\dot{a}^{p_N} = \dot{b}^{q_N}$, then $\zeta(a^p) = \zeta(b^q)$ and so $a = b^{q-1}$. Thus,

$$\begin{aligned}\dot{a}\dot{\times}\dot{b} &= \zeta(ab) = \zeta\{b^q\} = \dot{b}^{q_N}\dot{\times}\dot{1} \\ &= \dot{b}^{q_N}\dot{\times}\left(\left(\dot{1}\dot{/}\dot{p}\right)\dot{+}\left(\dot{1}\dot{/}\dot{q}\right)\right) \\ &= \dot{b}^{q_N}\dot{\times}\left(\dot{1}\dot{/}\dot{p}\right)\dot{+}\dot{b}^{q_N}\dot{\times}\left(\dot{1}\dot{/}\dot{q}\right)\end{aligned}$$

i.e.,

$$\dot{a}\dot{\times}\dot{b} = \dot{a}^{p_N}\dot{\times}\left(\dot{1}\dot{/}\dot{p}\right)\dot{+}\dot{b}^{q_N}\dot{\times}\left(\dot{1}\dot{/}\dot{q}\right).$$

Now, consider $u(\dot{x}) = \dot{x}^{r_N}$ with $\dot{r}\dot{>}\dot{0}$ and $u^{-1}(\dot{x}) = \dot{x}^{\left(\dot{1}\dot{/}\dot{r}\right)}$. Since the function u satisfies the hypothesis of Theorem 2.4, then

$$\dot{a}\dot{\times}\dot{b}\dot{\leq}\int_{\dot{0}}^{\dot{a}}\dot{x}^{r_N}\dot{\times}dx_N\dot{+}\int_{\dot{0}}^{\dot{b}}\dot{y}^{\left(\dot{1}\dot{/}\dot{r}\right)}\dot{\times}dy_N = \frac{\dot{a}^{\dot{r}\dot{+}\dot{1}}}{\dot{r}\dot{+}\dot{1}} + \frac{\dot{b}^{\left(\dot{1}\dot{/}\dot{r}\right)\dot{+}\dot{1}}}{\left(\dot{1}\dot{/}\dot{r}\right)\dot{+}\dot{1}}.$$

If one takes $\dot{p} = \dot{r}\dot{+}\dot{1}$ and $\dot{q} = \left(\dot{1}\dot{/}\dot{r}\right)\dot{+}\dot{1}$, then $\left(\dot{1}\dot{/}\dot{p}\right)\dot{+}\left(\dot{1}\dot{/}\dot{q}\right) = \dot{1}$ and hence $\dot{a}\dot{\times}\dot{b}\dot{\leq}\dot{a}^{p_N}\dot{\times}\left(\dot{1}\dot{/}\dot{p}\right)\dot{+}\dot{b}^{q_N}\dot{\times}\left(\dot{1}\dot{/}\dot{q}\right)$ can be written. □

Definition 2.6. *A $\sigma-$algebra (σ-field) is a class $\mathfrak{M}$ of subsets of a set X with the following properties:*
(a) $\varnothing, X \in \mathfrak{M}$;
(b) $S \in \mathfrak{M} \Rightarrow X\backslash S \in \mathfrak{M}$;
(c) If $S_n \in \mathfrak{M}$ for all $n = 1, 2, \ldots,$ then $\bigcup_{n=1}^{\infty} S_n \in \mathfrak{M}$.

A set $S \in \mathfrak{M}$ is said to be measurable.

Definition 2.7. *Let X be a set and let $\mathfrak{M}$ be a σ-algebra of subsets of X. A function $\dot{\mu} : \mathfrak{M} \to \mathbb{R}_\zeta$ is a NN-measure if it has the properties:*
(a) $\dot{\mu}(\varnothing) = \dot{0}$;
(b) $\dot{\mu}$ is $\alpha-$countably additive, that is, if $S_j \in \mathfrak{M}$, $j = 1, 2, \ldots,$ are pairwise disjoint sets then $\dot{\mu}\left(\bigcup_{j=1}^{\infty} S_j\right) =_N \sum_{j=1}^{\infty} \dot{\mu}(S_j)$.

The triple $(X, \mathfrak{M}, \dot{\mu})$ is called a $\zeta-$measure space. In various applications of measure theory, sets with measures equal to zero are often considered "negligible." Having specific terminology for such sets proves to be quite useful in these contexts.

Definition 2.8. *Let $(X, \mathfrak{M}, \dot{\mu})$ be a $\zeta-$measure space. A set $S \in \mathfrak{M}$ with $\dot{\mu}(S) = \dot{0}$ is said to have $\zeta-$measure zero (or a $\zeta-$null set). A given property $Q(x)$ of points $x \in X$ is said to hold $\zeta-$almost everywhere if the set*

$$\{x : Q(x) \text{ is false}\}$$

has $\zeta-$measure $\dot{0}$. In a different context, the property Q is deemed to hold for $\zeta-$almost all $x \in X$.

The abbreviation $\zeta-$a.e. will represent either of these expressions interchangeably. Besides, $\zeta-$Counting measure, $\zeta-$Lebesgue measure and $\zeta-$Borel measures on $\mathbb{R}_\zeta$ are studied in [8].

Let's consider a fixed $\zeta-$measure space $(X, \mathfrak{M}, \dot{\mu})$. In the sequence of upcoming definitions, we outline the process of constructing the $\zeta-$integral for suitable functions $f : X \to \mathbb{R}_\zeta$ and later $f : X \to \mathbb{C}^*$. For detailed proofs and additional information of usual ones, one can refer to any integral and measure book. For any subset $S \subset X$ the characteristic function $\dot{\chi}_S : X \to \mathbb{R}_\alpha$ of S is defined by

$$\dot{\chi}_S(x) = \begin{cases} \dot{1}, & \text{if } x \in S, \\ \dot{0}, & \text{if } x \notin S. \end{cases}$$

A function $\varphi : X \to \mathbb{R}_\zeta$ is called $\zeta-$simple if it has the form

$$\begin{aligned}
\varphi &= {}_N\sum_{j=1}^{k} \dot{\alpha}_j \dot{\times} \dot{\chi}_{S_j} = \left(\dot{\alpha}_1 \dot{\times} \dot{\chi}_{S_1}\right) \dot{+} \left(\dot{\alpha}_2 \dot{\times} \dot{\chi}_{S_2}\right) \dot{+} \cdots \dot{+} \left(\dot{\alpha}_k \dot{\times} \dot{\chi}_{S_k}\right) \\
&= \zeta\left\{\zeta^{-1}\left(\dot{\alpha}_1 \dot{\times} \dot{\chi}_{S_1}\right) + \zeta^{-1}\left(\dot{\alpha}_2 \dot{\times} \dot{\chi}_{S_2}\right) + \cdots + \zeta^{-1}\left(\dot{\alpha}_k \dot{\times} \dot{\chi}_{S_k}\right)\right\} \\
&= \zeta\left\{\zeta^{-1}\left(\zeta\left[\zeta^{-1}(\dot{\alpha}_1)\,\zeta^{-1}(\dot{\chi}_{S_1})\right]\right) + \zeta^{-1}\left(\zeta\left[\zeta^{-1}(\dot{\alpha}_2)\,\zeta^{-1}(\dot{\chi}_{S_2})\right]\right) + \right. \\
&\qquad \left. \cdots + \zeta^{-1}\left(\zeta\left[\zeta^{-1}(\dot{\alpha}_k)\,\zeta^{-1}(\dot{\chi}_{S_k})\right]\right)\right\} \\
&= \zeta\left\{\zeta^{-1}(\dot{\alpha}_1)\,\zeta^{-1}(\dot{\chi}_{S_1}) + \zeta^{-1}(\dot{\alpha}_2)\,\zeta^{-1}(\dot{\chi}_{S_2}) + \cdots + \zeta^{-1}(\dot{\alpha}_k)\,\zeta^{-1}(\dot{\chi}_{S_k})\right\} \\
&= \zeta\left(\sum_{j=1}^{k} \zeta^{-1}(\dot{\alpha}_j)\,\zeta^{-1}\left(\dot{\chi}_{S_j}\right)\right)
\end{aligned}$$

for some $k \in \mathbb{N}$, where $\dot{\alpha}_j \in \mathbb{R}_\zeta$ and $S_j \in \mathfrak{M}$ for all $j = 1, \ldots, k$. If $\varphi(x) \dot{\geq} \dot{0}$ for all $x \in X$ and $\zeta-$simple then the $\zeta-$integral of φ over X, with respect to $\dot{\mu}$ is defined to be

$$\begin{aligned}
\dot{\int_X} \varphi \dot{\times} d\dot{\mu} &=_N \sum_{j=1}^{k} \dot{\alpha}_j \dot{\times} \dot{\mu}(S_j) \\
&= \left(\dot{\alpha}_1 \dot{\times} \dot{\mu}(S_1)\right) \dot{+} \left(\dot{\alpha}_2 \dot{\times} \dot{\mu}(S_2)\right) \dot{+} \cdots \dot{+} \left(\dot{\alpha}_k \dot{\times} \dot{\mu}(S_k)\right) \\
&= \zeta\left(\sum_{j=1}^{k} \zeta^{-1}(\dot{\alpha}_j)\, \zeta^{-1}(\dot{\mu}(S_j))\right).
\end{aligned}$$

Let $\alpha(-\infty) = -\dot{\infty}$ and $\alpha(+\infty) = +\dot{\infty}$. A function $f : X \to \dot{[}-\dot{\infty}, +\dot{\infty}\dot{]}$ is said to be $\zeta-$measurable if, for every $\dot{\gamma} \in \mathbb{R}_\zeta$,

$$\{x \in X : f(x) \dot{>} \dot{\gamma}\} = \{x \in X : \zeta^{-1}(f(x)) > \zeta^{-1}(\dot{\gamma})\} \in \mathfrak{M}.$$

If f is $\zeta-$measurable, then the functions

$$\begin{aligned}
\dot{|}f\dot{|}(x) &= \dot{|}f(x)\dot{|} = \zeta\left\{\left|\zeta^{-1}(f(x))\right|\right\} \\
f^{\dot{+}} &= {}^{\zeta}\max\left\{f(x), \dot{0}\right\} = \zeta\left\{\max\left\{\zeta^{-1}(f(x)), \zeta^{-1}(\dot{0})\right\}\right\} \\
f^{\dot{-}} &= {}^{\zeta}\max\left\{\dot{0} \dot{-} f(x), \dot{0}\right\} = \zeta\left\{\max\left\{\zeta^{-1}(\dot{0}) - \zeta^{-1}(f(x)), \zeta^{-1}(\dot{0})\right\}\right\}
\end{aligned}$$

are $\zeta-$measurable where $f^{\dot{+}}, f^{\dot{-}}, \dot{|}f\dot{|} : X \to \mathbb{R}_\zeta^{\dot{+}} \cup \{\dot{0}\}$. If $f(x) \dot{\geq} \dot{0}$ for all $x \in X$ and $\zeta-$measurable, then the $\zeta-$integral of f is defined to be

$$\dot{\int_X} f \dot{\times} d\dot{\mu} = {}^{\zeta}\sup\left\{\dot{\int_X} \varphi \dot{\times} d\dot{\mu} : \varphi \text{ is } \zeta - \text{simple and } \dot{0} \dot{\leq} \varphi \dot{\leq} f\right\}.$$

If f is $\zeta-$measurable and

$$+\dot{\infty} \dot{>} \dot{\int_X} \dot{|}f\dot{|} \dot{\times} d\dot{\mu} = \zeta\left\{\int_X \left|\zeta^{-1}(f(x))\right| d\left(\left|\zeta^{-1}(\mu(x))\right|\right)\right\}$$

then f is said to be $\zeta-$integrable and the $\zeta-$integral of f is defined to be

$$\dot{\int_X} f \dot{\times} d\dot{\mu} = \dot{\int_X} f^{\dot{+}} \dot{\times} d\dot{\mu} \dot{-} \dot{\int_X} f^{\dot{-}} \dot{\times} d\dot{\mu}.$$

Demonstrably, if f is $\zeta-$integrable, each term on the right side of this definition is $\zeta-$finite. Hence, there's no issue concerning a discrepancy like $\dot{\infty}\dot{-}\dot{\infty}$ emerging within this context.

The collection of $\mathbb{R}_\zeta-$valued $\zeta-$integrable functions on X will be denoted as $\mathcal{L}_{\mathbb{R}_\zeta}(X)$ (or $\mathcal{L}_{\mathbb{R}_\zeta}(S)$ for induced functions on $S\in\mathfrak{M}$). Now, we will list some of the basic properties of the $\zeta-$integral.

Theorem 2.9. *Let $(X,\mathfrak{M},\dot{\mu})$ be a $\zeta-$measure space and let $f\in\mathcal{L}_{\mathbb{R}_\zeta}(X)$.*

(a) If $f(x)=\dot{0}$ ($\dot{\mu}-$a.e.), then $f\in\mathcal{L}_{\mathbb{R}_\zeta}(X)$ and $\dot{\int\limits_X} f\dot{\times}d\dot{\mu}=\dot{0}$.

(b) If $\dot{\gamma}\in\mathbb{R}_\zeta$ and $f,g\in\mathcal{L}_{\mathbb{R}_\zeta}(X)$ then the functions $f\dot{+}g$ and $\dot{\gamma}\dot{\times}f$ belong to $\mathcal{L}_{\mathbb{R}_\zeta}(X)$ and

$$\begin{aligned}\dot{\int\limits_X}\left(f\dot{+}g\right)\dot{\times}d\dot{\mu} &= \dot{\int\limits_X} f\dot{\times}d\dot{\mu}\dot{+}\dot{\int\limits_X} g\dot{\times}d\dot{\mu},\\ \dot{\int\limits_X}\left(\dot{\gamma}\dot{\times}f\right)\dot{\times}d\dot{\mu} &= \dot{\gamma}\dot{\times}\dot{\int\limits_X} f\dot{\times}d\dot{\mu}\end{aligned}$$

In particular, $\mathcal{L}_{\mathbb{R}_\zeta}(X)$ is an NN-vector space.

(c) If $f,g\in\mathcal{L}_{\mathbb{R}_\zeta}(X)$ and $f(x)\dot{\leq}g(x)$ for all $x\in X$, then $\dot{\int\limits_X} f\dot{\times}d\dot{\mu}\dot{\leq}\dot{\int\limits_X} g\dot{\times}d\dot{\mu}$. Besides, if $f(x)\dot{<}g(x)$ for all $x\in S$ where $\dot{\mu}(S)\dot{>}\dot{0}$, then $\dot{\int\limits_S} f\dot{\times}d\dot{\mu}\dot{<}\dot{\int\limits_S} g\dot{\times}d\dot{\mu}$. Here $\dot{\int\limits_S} f\dot{\times}d\dot{\mu}=\dot{\int\limits_X} f\dot{\times}\dot{\chi}_S\dot{\times}d\dot{\mu}$.

It follows from part (a) of the preceding theorem that the values of f on sets of $\zeta-$measure $\dot{0}$ don't affect the integral.

Definition 2.10. *Suppose that g is a $\zeta-$measurable function and there exists a number $\dot{B}$ such that $g(x)\dot{\leq}\dot{B}$ ($\dot{\mu}-$a.e.). Then the $\zeta-$essential supremum of g can be defined as*

$$^{\zeta}esssup\,(g)={}^{\zeta}\inf\left\{\dot{B}:g(x)\dot{\leq}\dot{B}\ \dot{\mu}-a.e.\right\}.$$

As a straightforward (albeit not entirely trivial) implication of this definition, it follows that $g(x)\dot{\leq}{}^{\zeta}\text{esssup}\,(g)$ ($\dot{\mu}-$a.e.). Similarly, the

$\zeta-$essential infimum of g can be defined in a comparable manner. A $\zeta-$measurable function g is said to be $\zeta-$essentially bounded if there exists a number $\dot{B}$ such that $\dot{|}g(x)\dot{|}\dot{\leq}\dot{B}$ ($\dot{\mu}-$a.e.).

Definition 2.11. *Let X be a nonempty set and $d_N : X \times X \to \mathbb{R}_\zeta$ be a function such that, for all $x, y, z \in X$ the following axioms hold:*
i. $d_N(x, y) = \dot{0}$ if and only if $x = y$,
ii. $d_N(x, y) = d_N(y, x)$,
iii. $d_N(x, z)\dot{\leq}d_N(x, y)\dot{+}d_N(y, z)$.
Then, the pair (X, d_N) is called an $\zeta-$metric space [4].

Now, we would like to define a $\zeta-$metric on the space $\mathcal{L}_{\mathbb{R}_\zeta}(X)$, and an obvious candidate for this is the function

$$\begin{aligned} d_{\mathbb{R}_\zeta}(f, g) &= \dot{\int_X} \dot{|}f - g\dot{|}\dot{\times}d\dot{\mu} = \dot{\int_X} \dot{|}f(x)\dot{-}g(x)\dot{|}\dot{\times}d\dot{\mu}(x) \\ &= \zeta\left\{\int_X \zeta^{-1}\left\{\dot{|}f(x)\dot{-}g(x)\dot{|}\right\} d\mu(x)\right\} \\ &= \zeta\left\{\int_X \left|\zeta^{-1}\left(\zeta\left\{\zeta^{-1}(f(x)) - \zeta^{-1}(g(x))\right\}\right)\right| d\mu(x)\right\} \end{aligned}$$

for all $f, g \in \mathcal{L}_{\mathbb{R}_\zeta}(X)$. The properties of the integral outlined in Theorem 2.9 establish that this function fulfills all the conditions for a $\zeta-$metric, except for part (i) of Definition 2.11. Regrettably, there exist functions f and g in $\mathcal{L}_{\mathbb{R}_\zeta}(X)$ where $f = g$ ($\dot{\mu}-$a.e.) but $f \neq g$. Consequently, as per part (a) of Theorem 2.9, $d_{\mathbb{R}_\alpha}(f, g) = \dot{0}$ and so, the function $d_{\mathbb{R}_\alpha}(\cdot, \cdot)$ does not qualify as a $\zeta-$metric on $\mathcal{L}_{\mathbb{R}_\zeta}(X)$. To resolve this issue, we'll adopt a strategy where we consider any two functions f and g that are equal ($\dot{\mu}-$a.e.) as "identical" or "equivalent." To be precise, we define an equivalence relation $f\dot{\equiv}g \Leftrightarrow f = g\,(\dot{\mu} - a.e.)$. The defined equivalence relation partitions the set $\mathcal{L}_{\mathbb{R}_\zeta}(X)$ into a collection of equivalence classes, denoted as $L(X, \mathbb{R}_\zeta)$. Through appropriate definitions of addition and scalar multiplication within these equivalence classes, the space $L(X, \mathbb{R}_\zeta)$ transforms into a $\zeta-$vector space the function $d_{\mathbb{R}_\zeta}(\cdot, \cdot)$ yields a $\zeta-$metric on the set $L(X, \mathbb{R}_\zeta)$. Now, we can proceed to define several other spaces comprising $\zeta-$integrable functions.

Definition 2.12. *Let us define the spaces*

$$\mathcal{L}^{\dot{p}}_{\mathbb{R}_\zeta}(X)=\left\{g:g\ \text{is}\ \zeta-\text{measurable and}\ \dot{\int\limits_X}\dot{|}g(x)\dot{|}^{p_N}\dot{\times}d\dot{\mu}(x)\dot{<}+\dot{\infty}\right\}$$

for $\dot{1}\leq\dot{p}<\dot{\infty}$ *where*

$$\begin{aligned}\dot{\int\limits_X}\dot{|}g(x)\dot{|}^{p_N}\dot{\times}d\dot{\mu}(x) &= \zeta\left\{\int\limits_X\zeta^{-1}\left(\dot{|}g(x)\dot{|}^{p_N}\right)d\mu(x)\right\}\\ &= \zeta\left\{\int\limits_X\left\{\left[\zeta^{-1}\left(\zeta\left\{\left|\zeta^{-1}(g(x))\right|\right\}\right)\right]^p\right\}d\mu(x)\right\}\end{aligned}$$

and

$$\mathcal{L}^{\dot{\infty}}_{\mathbb{R}_\alpha}(X)=\left\{g:g\ \text{is}\ \zeta-\text{measurable and}\ {}^{\alpha}\text{esssup}\dot{|}g\dot{|}\dot{<}+\dot{\infty}\right\}.$$

Similarly, we can define the sets $L^{\dot{p}}(X,\mathbb{R}_\zeta)$ by equating functions in $\mathcal{L}^{\dot{p}}_{\mathbb{R}_\zeta}(X)$ that are almost everywhere equal and examining the resulting spaces of equivalence classes (in practical terms, we typically refer to representative functions from these equivalence classes rather than explicitly addressing the classes themselves).

Example 2.13. Let $X=\dot{[}\dot{0},\left(\dot{1}\dot{/}\dot{2}\right)\dot{]}$ and $u:X\to\mathbb{R}_\zeta$ be defined by $u(x)=\frac{\dot{1}}{\dot{x}\dot{\times}\ln^{2_N}(\dot{x})}N$. Then, we get for $\dot{p}=\dot{1}$,

$$\begin{aligned}\dot{\int\limits_X}\dot{|}u(x)\dot{|}^{p_N}\dot{\times}d\dot{\mu}(x) &= \dot{\int\limits_{\dot{0}}^{\dot{1}\dot{/}\dot{2}}}\dot{|}\frac{\dot{1}}{\dot{x}\dot{\times}\ln^{2_N}(\dot{x})}N\dot{|}^{\dot{1}}\dot{\times}d\dot{\mu}(x)\\ &= \zeta\left\{\int\limits_0^{\frac{1}{2}}\left|\zeta^{-1}\left(\frac{\dot{1}}{\dot{x}\dot{\times}\ln^{2_N}(\dot{x})}\right)\right|d\mu(x)\right\}\end{aligned}$$

and $u\in\mathcal{L}^{\dot{1}}_{\mathbb{R}_\zeta}(X)$ if the generator ζ is equal to e^x or I identity function.

Example 2.14. Let $X = (\dot{0}, \dot{\infty})$ and $u : X \to \mathbb{R}_\zeta$ be defined by $u(x) = \frac{\dot{1}}{\dot{\sqrt{\dot{1}\dot{+}\dot{x}}}}N$. Then, we have

$$\begin{aligned}\int\limits_X^{\cdot} \dot{|} u(x) \dot{|}^{p_N} \dot{\times} d\dot{\mu}(x) &= \int\limits_{\dot{0}}^{\dot{\infty}} \dot{|} \frac{\dot{1}}{\dot{\sqrt{\dot{1}\dot{+}\dot{x}}}} N \dot{|}^{p_N} \dot{\times} d\dot{\mu}(x) \\ &= \zeta\left\{\int\limits_0^{\infty} \left|\zeta^{-1}\left(\frac{\dot{1}}{\dot{\sqrt{\dot{1}\dot{+}\dot{x}}}}\right)\right|^p d\mu(x)\right\}.\end{aligned}$$

Therefore $u \in \mathcal{L}^{\dot{p}}_{\mathbb{R}_\zeta}(X)$ if $\zeta(x) = x^{2k+1}$ $(k \in \mathbb{Z})$ and $\dot{p}\dot{>}\frac{\dot{2}}{\dot{2}\dot{k}\dot{+}\dot{1}}$. Also, $u \notin \mathcal{L}^{\dot{p}}_{\mathbb{R}_\zeta}(X)$ for any $\dot{p}\dot{>}\dot{0}$ if $\zeta(x) = e^x$.

Theorem 2.15. *Let $(X, \mathfrak{M}, \dot{\mu})$ be a $\zeta-$measure space such that $\dot{\mu}(X)\dot{<}\dot{\infty}$. Then $L^{\dot{\infty}}(X, \mathbb{R}_\zeta) \subset L^{\dot{q}}(X, \mathbb{R}_\zeta) \subset L^{\dot{p}}(X, \mathbb{R}_\zeta) \subset L^{\dot{1}}(X, \mathbb{R}_\zeta)$ for any $\dot{1}\dot{\leq}\dot{p}\dot{\leq}\dot{q}\dot{\leq}\dot{\infty}$.*

Proof. Firstly, let $\dot{p} \neq \dot{\infty}$ and u be an element of $L^{\dot{\infty}}(X, \mathbb{R}_\zeta)$. Then, by the definiton of $\zeta-$essential supremum, we have

$$\dot{|}u\dot{|} \dot{\leq} {}^{\zeta}\text{esssup}\dot{|}u\dot{|} = \dot{|}\dot{|}u\dot{|}\dot{|}_\infty \; (\dot{\mu} - a.e.)$$

and

$$\int\limits_X^{\cdot} \dot{|}u(x)\dot{|}^{p_N} \dot{\times} d\dot{\mu}(x) \dot{\leq} \dot{|}\dot{|}u\dot{|}\dot{|}_\infty^{p_N} \dot{\times} \dot{\mu}(X) \dot{<} \dot{\infty}$$

so $u \in L^{\dot{p}}(X, \mathbb{R}_\zeta)$. Now, let $u \in L^{\dot{q}}(X, \mathbb{R}_\zeta)$ and $A = \{x \in X : \dot{|}u(x)\dot{|} \dot{\leq} \dot{1}\}$. Then, $\dot{\chi}_X = \dot{\chi}_A \dot{+} \dot{\chi}_{X-A}$ and $\dot{|}u(x)\dot{|}^{p_N} \dot{\leq} \dot{|}u(x)\dot{|}^{q_N}$ for all

$x \in X - A$ and $\dot{|}u(x)\dot{|} \dot{\leq} \dot{1}$, for all $x \in A$. Therefore,

$$\begin{aligned}
\dot{\int_X} \dot{|}u(x)\dot{|}^{p_N} \dot{\times} d\dot{\mu}(x) &= \zeta\left\{\int_X \left\{\left[\zeta^{-1}\left(\zeta\left\{\left|\zeta^{-1}(u(x))\right|\right\}\right)\right]^p\right\} d\mu(x)\right\} \\
&= \zeta\left\{\int_A \left\{\left[\zeta^{-1}\left(\zeta\left\{\left|\zeta^{-1}(u(x))\right|\right\}\right)\right]^p\right\} d\mu(x)\right. \\
&\qquad \left. + \int_{X-A} \left\{\left[\zeta^{-1}\left(\zeta\left\{\left|\zeta^{-1}(u(x))\right|\right\}\right)\right]^p\right\} d\mu(x)\right\} \\
&\dot{\leq} \dot{\int_A} d\dot{\mu}(x) \dot{+} \dot{\int_{X-A}} \dot{|}u(x)\dot{|}^{q_N} \dot{\times} d\dot{\mu}(x) \\
&\dot{\leq} \dot{\mu}(A) + \dot{\int_X} \dot{|}u(x)\dot{|}^{q_N} \dot{\times} d\dot{\mu}(x).
\end{aligned}$$

Since $\dot{\mu}(A) \dot{\leq} \dot{\mu}(X) \dot{<} \dot{\infty}$, we get that $u \in L^{\dot{p}}(X, \mathbb{R}_\zeta)$. □

Remark 2.16. The inclusion in Theorem 2.15 is strict. It means $L^{\dot{q}}(X, \mathbb{R}_\zeta) \subsetneqq L^{\dot{p}}(X, \mathbb{R}_\zeta)$ if $\dot{1} \dot{\leq} \dot{p} \dot{<} \dot{q} \dot{\leq} \dot{\infty}$. Let $\zeta(x) = \frac{1}{x}$, $X = [\dot{0}, \dot{1}]$, $\dot{1} \dot{\leq} \dot{p} \dot{<} \dot{q} \dot{\leq} \dot{\infty}$ and $\dot{\lambda} = \frac{\dot{p} \dot{+} \dot{q}}{\dot{2}} N$. Then $\dot{p} \dot{<} \dot{\lambda} \dot{<} \dot{q}$, $\frac{\dot{p}}{\dot{\lambda}} N \dot{<} \dot{1}$ and $\frac{\dot{q}}{\dot{\lambda}} N \dot{>} \dot{1}$. Choose $\dot{\alpha} = \frac{\dot{1}}{\dot{\lambda}} N$ and define

$$g(x) = \begin{cases} \dot{x}^{\alpha_N}, & \text{if } \dot{x} \neq \dot{0}, \\ \dot{0}, & \text{if } \dot{x} = \dot{0}. \end{cases}$$

Since $\frac{\dot{p}}{\dot{\lambda}}N\dot{<}\dot{1}$, we get

$$\begin{aligned}
\int_{\dot{0}}^{\dot{1}} \dot{|}g(x)\dot{|}^{p_N}\dot{\times}d\dot{\mu}(x) &= \zeta\left\{\int_X \left\{\left[\zeta^{-1}\left(\zeta\left\{\left|\zeta^{-1}(g(x))\right|\right\}\right)\right]^p\right\}d\mu(x)\right\} \\
&= \zeta\left\{\int_0^1 \left[\left|\zeta^{-1}(g(x))\right|\right]^p d\mu(x)\right\} \\
&= \zeta\left\{\int_0^1 \left[\frac{1}{x^\alpha}\right]^p d\mu(x)\right\} \\
&= \zeta\left\{\int_0^1 \frac{1}{x^{p\alpha}}d\mu(x)\right\} = \zeta\left\{\int_0^1 \frac{1}{x^{\frac{p}{\lambda}}}d\mu(x)\right\} \\
&= \frac{\dot{q}\dot{-}\dot{p}}{\dot{q}\dot{+}\dot{p}}N
\end{aligned}$$

and $g \in L^{\dot{p}}(X, \mathbb{R}_\zeta)$. On the other hand, the NN-integral

$$\begin{aligned}
\int_{\dot{0}}^{\dot{1}} \dot{|}g(x)\dot{|}^{q_N}\dot{\times}d\dot{\mu}(x) &= \zeta\left\{\int_X \left\{\left[\zeta^{-1}\left(\zeta\left\{\left|\zeta^{-1}(g(x))\right|\right\}\right)\right]^q\right\}d\mu(x)\right\} \\
&= \zeta\left\{\int_0^1 \left[\left|\zeta^{-1}(g(x))\right|\right]^q d\mu(x)\right\} \\
&= \zeta\left\{\int_0^1 \left[\frac{1}{x^\alpha}\right]^q d\mu(x)\right\} \\
&= \zeta\left\{\int_0^1 \frac{1}{x^{\frac{q}{\lambda}}}d\mu(x)\right\}
\end{aligned}$$

is divergent and this gives that $g \notin L^{\dot{q}}(X, \mathbb{R}_\zeta)$. Therefore $L^{\dot{q}}(X, \mathbb{R}_\zeta) \subsetneqq L^{\dot{p}}(X, \mathbb{R}_\zeta)$.

Example 2.17. Let $X = [\dot{0}, \dot{16}]$ and $u : X \to \mathbb{R}_\zeta$ be defined by $u(x) = \exp\left(\dot{x}^{\frac{\dot{0}\dot{-}\dot{1}}{\dot{4}}N}\right)$. For geometric calculus, since $\zeta^{-1}(x) = \ln x$, we can write

$$\begin{aligned}\dot{\int_X} \dot{|}u(x)\dot{|}^{p_N} \dot{\times} d\dot{\mu}(x) &= \zeta\left\{\int_X \left\{\left[\zeta^{-1}\left(\zeta\left\{\left|\zeta^{-1}(u(x))\right|\right\}\right)\right]^p\right\} d\mu(x)\right\} \\ &= e^{\left\{\int_0^{16}\left\{\left[\ln\left(e^{\{|\ln(u(x))|\}}\right)\right]^p\right\}d\mu(x)\right\}} \\ &= e^{\int_0^{16}\left|x^{\frac{-1}{4}}\right|^p d\mu(x)} = e^{\lim_{\varepsilon\to 0^+}\left\{\frac{-4}{(p-4)16^{\frac{p-4}{4}}} + \frac{4}{(p-4)\varepsilon^{\frac{p-4}{4}}}\right\}}.\end{aligned}$$

Then, we get $u \in L^{\dot{1}}(X, \mathbb{R}_\zeta)$ but $u \notin L^{\dot{4}}(X, \mathbb{R}_\zeta)$.

Theorem 2.18. *(NN-Minkowski's inequality). Let* $\dot{1}\dot{\leq}\dot{p}\dot{\leq}\dot{\infty}$ *and* $u, v \in L^{\dot{p}}(X, \mathbb{R}_\zeta)$. *Then* $u\dot{+}v \in L^{\dot{p}}(X, \mathbb{R}_\zeta)$ *and*

$$\dot{|}|u\dot{+}v|\dot{|}_{p_N} \dot{\leq} \dot{|}|u|\dot{|}_{p_N} \dot{+} \dot{|}|v|\dot{|}_{p_N}$$

where

$$\begin{aligned}\dot{|}|u|\dot{|}_{p_N} &= \left(\dot{\int_X} \dot{|}u(x)\dot{|}^{p_N} \dot{\times} d\dot{\mu}(x)\right)^{\dot{1}\dot{/}\dot{p}} = \zeta\left\{\left[\zeta^{-1}\left(\dot{\int_X} \dot{|}u(x)\dot{|}^{p_N} \dot{\times} d\dot{\mu}(x)\right)\right]^{\frac{1}{p}}\right\} \\ &= \zeta\left\{\left[\zeta^{-1}\left(\zeta\left\{\int_X \zeta^{-1}\left(\zeta\left\{\zeta^{-1}\left(\dot{|}u(x)\dot{|}^{p_N}\right) d\mu\right\}\right)\right\}\right)\right]^{\frac{1}{p}}\right\} \\ &= \zeta\left\{\left[\zeta^{-1}\left(\zeta\left\{\int_X \zeta^{-1}\left(\zeta\left\{\zeta^{-1}\left(\zeta\left(\zeta^{-1}\left\{\dot{|}u(x)\dot{|}\right\}^p\right)\right) d\mu\right\}\right)\right\}\right)\right]^{\frac{1}{p}}\right\} \\ &= \zeta\left\{\left[\zeta^{-1}\left(\zeta\left\{\int_X \zeta^{-1}\left(\zeta\left\{\zeta^{-1}\left(\zeta\left(\zeta^{-1}\left\{\zeta\left(\left|\zeta^{-1}(u(x))\right|\right)\right\}^p\right)\right) d\mu\right\}\right)\right\}\right)\right]^{\frac{1}{p}}\right\}.\end{aligned}$$

If $\dot{A}\dot{\times}\dot{|}u\dot{|} = \dot{B}\dot{\times}\dot{|}v\dot{|}$ $(\dot{\mu} - a.e.)$ for $\dot{A}\dot{\times}\dot{B}\dot{>}\dot{0}$, then the equality exists.

Proof. Let $\dot{A}\dot{\times}\dot{B}\dot{>}\dot{0}$ such that $\dot{A}\dot{\times}\dot{|}u\dot{|}=\dot{B}\dot{\times}\dot{|}v\dot{|}$ $(\dot{\mu}-a.e.)$. Then

$$\begin{aligned}\dot{A}\dot{\times}\dot{|}\dot{|}u\dot{|}\dot{|}_{p_N} &= \left(\dot{\int\limits_X}\dot{A}^{p_N}\dot{\times}\dot{|}u(x)\dot{|}^{p_N}\dot{\times}d\dot{\mu}(x)\right)^{\dot{1}\dot{/}\dot{p}} \\ &= \left(\dot{\int\limits_X}\dot{B}^{p_N}\dot{\times}\dot{|}v(x)\dot{|}^{p_N}\dot{\times}d\dot{\mu}(x)\right)^{\dot{1}\dot{/}\dot{p}} \\ &= \dot{B}\dot{\times}\dot{|}\dot{|}v\dot{|}\dot{|}_{p_N}\end{aligned}$$

and $\dot{|}\dot{|}u\dot{|}\dot{|}_{p_N}=\frac{\dot{B}}{\dot{A}}N\dot{\times}\dot{|}\dot{|}v\dot{|}\dot{|}_{p_N}$. In other words,

$$\dot{|}\dot{|}u\dot{+}v\dot{|}\dot{|}_{p_N}=\frac{\dot{B}\dot{+}\dot{A}}{\dot{A}}N\dot{\times}\dot{|}\dot{|}v\dot{|}\dot{|}_{p_N}=\frac{\dot{B}}{\dot{A}}N\dot{\times}\dot{|}\dot{|}v\dot{|}\dot{|}_{p_N}\dot{+}\dot{|}\dot{|}v\dot{|}\dot{|}_{p_N}=\dot{|}\dot{|}u\dot{|}\dot{|}_{p_N}\dot{+}\dot{|}\dot{|}v\dot{|}\dot{|}_{p_N}.$$

as well as when $\dot{|}\dot{|}u\dot{|}\dot{|}_{p_N}=\dot{|}\dot{|}v\dot{|}\dot{|}_{p_N}=\dot{0}$. If $\dot{p}=\dot{\infty}$ or $\dot{p}=\dot{1}$ the inequality is immediate. Now, assume that $\dot{1}\dot{<}\dot{p}\dot{<}\dot{\infty}$, $\dot{|}\dot{|}u\dot{|}\dot{|}_{p_N}=\dot{C}\neq\dot{0}$ and $\dot{|}\dot{|}v\dot{|}\dot{|}_{p_N}=D\neq\dot{0}$. Then, there exist functions f, g such that $\dot{|}u\dot{|}=\dot{C}\dot{\times}f$ and $\dot{|}v\dot{|}=\dot{D}\dot{\times}g$ with $\dot{|}\dot{|}f\dot{|}\dot{|}_{p_N}=\dot{1}=\dot{|}\dot{|}g\dot{|}\dot{|}_{p_N}$. Now, let $\dot{\delta}=\frac{\dot{C}}{\dot{C}\dot{+}\dot{D}}N$ and $\dot{1}\dot{-}\dot{\delta}=\frac{\dot{D}}{\dot{C}\dot{+}\dot{D}}N$. Then, we get

$$\begin{aligned}\dot{|}u(x)\dot{+}v(x)\dot{|}^{p_N} &= \zeta\left(\zeta^{-1}\left\{\dot{|}u(x)\dot{+}v(x)\dot{|}\right\}^p\right) \\ &= \zeta\left(\zeta^{-1}\left\{\zeta\left|\zeta^{-1}\left(\zeta\left\{\zeta^{-1}(u(x))+\zeta^{-1}(v(x))\right\}\right)\right|\right\}^p\right) \\ &\dot{\leq} \zeta\left(\zeta^{-1}\left\{\zeta\left|\zeta^{-1}(u(x))\right|+\zeta\left|\zeta^{-1}(v(x))\right|\right\}^p\right) \\ &= \left(\dot{|}u(x)\dot{|}\dot{+}\dot{|}v(x)\dot{|}\right)^{p_N} \\ &= \left(\dot{C}\dot{\times}f\dot{+}\dot{D}\dot{\times}g\right)^{p_N} \\ &= \left(\left[\dot{C}\dot{+}\dot{D}\right]\dot{\times}\dot{\delta}\dot{\times}f\dot{+}\left[\dot{C}\dot{+}\dot{D}\right]\dot{\times}\left(\dot{1}\dot{-}\dot{\delta}\right)\dot{\times}g\right)^{p_N} \\ &= \left[\dot{C}\dot{+}\dot{D}\right]^{p_N}\dot{\times}\left(\dot{\delta}\dot{\times}f\dot{+}\left(\dot{1}\dot{-}\dot{\delta}\right)\dot{\times}g\right)^{p_N} \\ &\dot{\leq} \left[\dot{C}\dot{+}\dot{D}\right]^{p_N}\dot{\times}\left[\dot{\delta}\dot{\times}f^{p_N}\dot{+}\left(\dot{1}\dot{-}\dot{\delta}\right)\dot{\times}g^{p_N}\right]\end{aligned}$$

and

$$\dot{|}|u\dot{+}v\dot{|}|_{p_N} = \left(\int_X^{\cdot} \dot{|}u(x)\dot{+}v(x)\dot{|}^{p_N}\dot{\times}d\dot{\mu}(x)\right)^{\dot{1}/\dot{p}}$$
$$\dot{\leq} \left[\dot{C}\dot{+}\dot{D}\right]\dot{\times}\left[\dot{\delta}\dot{\times}\dot{|}|f\dot{|}|_{p_N}^{\dot{p}}\dot{+}\left(\dot{1}\dot{-}\dot{\delta}\right)\dot{\times}\dot{|}|g\dot{|}|_{p_N}^{\dot{p}}\right]$$
$$= \dot{C}\dot{+}\dot{D} = \dot{|}|u\dot{|}|_{p_N}\dot{+}\dot{|}|v\dot{|}|_{p_N}.$$

□

Corollary 2.19. *The NN-Lebesgue space* $\left(L^{\dot{p}}(X,\mathbb{R}_\zeta), \dot{|}|\cdot\dot{|}|_{p_N}\right)$ *is a normed space with the norm*

$$\dot{|}|u\dot{|}|_{p_N} = \left(\int_X^{\cdot} \dot{|}u(x)\dot{|}^{p_N}\dot{\times}d\dot{\mu}(x)\right)^{\dot{1}/\dot{p}}$$

for all $\dot{1}\dot{\leq}\dot{p}\dot{<}\dot{\infty}$.

Theorem 2.20. *(NN-Hölder's inequality). Let* $u \in L^{\dot{p}}(X,\mathbb{R}_\zeta)$, $v \in L^{\dot{q}}(X,\mathbb{R}_\zeta)$ *and* $\dot{1}\dot{<}\dot{p},\dot{q}\dot{<}\dot{\infty}$ *with* $\left(\dot{1}\dot{/}\dot{p}\right)\dot{+}\left(\dot{1}\dot{/}\dot{q}\right) = \dot{1}$. *Then* $u\dot{\times}v \in L^{\dot{1}}(X,\mathbb{R}_\zeta)$ *and*

$$\dot{|}|u\dot{\times}v\dot{|}|_{\dot{1}}\dot{\leq}\dot{|}|u\dot{|}|_{p_N}\dot{\times}\dot{|}|v\dot{|}|_{q_N}.$$

If $\dot{A}\dot{\times}\dot{|}u\dot{|}^{p_N} = \dot{B}\dot{\times}\dot{|}v\dot{|}^{q_N}$ $(\dot{\mu}-a.e.)$ *for* $\dot{A}\dot{\times}\dot{B}\dot{>}\dot{0}$, *then the equality exists.*

Proof. When $\dot{p} = \dot{1}$ and $\dot{q} = \dot{\infty}$, we have

$$\dot{|}|u\dot{\times}v\dot{|}|_{\dot{1}} = \int_X^{\cdot} \dot{|}u(x)\dot{\times}v(x)\dot{|}\dot{\times}d\dot{\mu}(x)\dot{\leq}\dot{|}|v\dot{|}|_\infty\int_X^{\cdot}\dot{|}u(x)\dot{|}\dot{\times}d\dot{\mu}(x)$$
$$= \dot{|}|v\dot{|}|_\infty\dot{|}|u\dot{|}|_{\dot{1}}$$

due to $\dot{|}v\dot{|} \dot{\leq} {}^{\zeta}\text{esssup}\dot{|}v\dot{|} = \dot{|}|v\dot{|}|_\infty$. Now, let $\gamma_1 = \frac{\dot{|}u(x)\dot{|}}{\dot{|}|u\dot{|}|_{p_N}}N$ and $\gamma_2 = \frac{\dot{|}v(x)\dot{|}}{\dot{|}|v\dot{|}|_{q_N}}N$. By (6) in Corollary 2.5, we get

$$\frac{\dot{|}u(x)\dot{|}}{\dot{|}|u\dot{|}|_{p_N}}N\dot{\times}\frac{\dot{|}v(x)\dot{|}}{\dot{|}|v\dot{|}|_{q_N}}N\dot{\leq}\frac{\dot{1}}{\dot{p}}N\dot{\times}\frac{\dot{|}u(x)\dot{|}^{p_N}}{\dot{|}|u\dot{|}|_{p_N}^{p_N}}N\dot{+}\frac{\dot{1}}{\dot{q}}N\dot{\times}\frac{\dot{|}v(x)\dot{|}^{q_N}}{\dot{|}|v\dot{|}|_{q_N}^{q_N}}N. \tag{7}$$

If one integrates the both sides of (7), then

$$\begin{aligned}\frac{\dot{1}}{\dot{||}u\dot{||}_{p_N}\dot{\times}\dot{||}v\dot{||}_{q_N}} N \dot{\int\limits_X} \dot{|}u(x)\dot{\times}v(x)\dot{|}\dot{\times}d\dot{\mu}(x) &\dot{\leq} \frac{\dot{1}}{\dot{p}}N\dot{\times}\frac{\dot{\int}_X \dot{|}u(x)\dot{|}^{p_N}\dot{\times}d\dot{\mu}(x)}{\dot{||}u\dot{||}_{p_N}^{p_N}}N \\ &\quad \dot{+}\frac{\dot{1}}{\dot{q}}N\dot{\times}\frac{\dot{\int}_X \dot{|}v(x)\dot{|}^{q_N}\dot{\times}d\dot{\mu}(x)}{\dot{||}v\dot{||}_{q_N}^{q_N}}N \\ &= \dot{1}\end{aligned}$$

can be written. □

3 Conclusion

This paper not only contributes a rigorous theoretical understanding of non-Newtonian Lebesgue spaces but also highlights their practical utility in modeling real-world phenomena. Insights gleaned from this exploration promise to enrich mathematical analysis and its applications in physics, engineering, and other interdisciplinary fields. Furthermore, we explore the interplay between non-Newtonian Lebesgue spaces and related mathematical frameworks, such as fractional calculus and other functional analysis topics.

The foundation of non-Newtonian Lebesgue spaces lies in their extension of classical Lebesgue spaces to accommodate non-Newtonian behaviors. These spaces offer a sophisticated mathematical framework capable of characterizing complex phenomena that deviate from traditional Newtonian models. Their properties, including but not limited to, integrability, norm behavior, and functional characteristics, exhibit unique traits that aptly capture the intricacies of non-Newtonian systems.

References

[1] A.E. Bashirov, E. M. Kurpınar and A. Özyapıcı, Multiplicative calculus and its applications, *Journal of Mathematical Analysis and Applications*, 337(1)(2008), 36-48.

[2] D. Binbaşıoğlu, S. Demiriz and D. Türkoğlu, Fixed points of non-Newtonian contraction mappings on non-Newtonian metric spaces, *Journal of Fixed Point Theory and Applications*, 18 (2016), 213-224.

[3] A.F. Çakmak and F. Başar, Space of continuous functions over the field of non-Newtonian real numbers, *Democratic People's Republic of Algeria*, 337(2008), 77-84.

[4] A.F. Çakmak and F. Başar, Some new results on sequence spaces with respect to non-Newtonian calculus, *J. Inequal. Appl.*, 228 (2012), 1-12.

[5] A.F. Çakmak and F. Başar, On line and double integrals in the non-Newtonian sense, *AIP Conf. Proc.*, 1611 (2014), 415-423.

[6] B.S. Duyar and F. Erdoğan, On non-Newtonian power series and its applications, *Konuralp Journal of Mathematics*, 8(2) (2020), 294-303.

[7] C. Duyar, B. Sağır and O. Oğur, Some basic topological properties on non-Newtonian real line, *Journal of Advances in Mathematics and Computer Science*, 9(4) (2015), 300-307.

[8] C. Duyar and B. Sağır, Non-Newtonian comment of Lebesgue measure in real numbers, *Journal of Mathematics*, 2017 (2017), 1-5.

[9] J. Grossman, *Meta-Calculus: Differential and Integral*, Archimedes Foundation, Massachussets, (1981).

[10] M. Grossman and R. Katz, *Non-Newtonian Calculus: A Self-contained, Elementary Exposition of the Authors Investigations*, LeePress, Massachussets, (1972).

[11] M. Grossman, *Bigeometric Calculus: A System with a Scale Free Derivative*, Archimedes Foundation, Rockport Massachussets, (1983).

[12] U. Kadak, Determination of the Köthe-Toeplitz duals over the non-Newtonian complex field, *The Scientific World Journal*, 2014 (2014), 1-10.

[13] B. Sağır and F. Erdoğan, On the function sequences and series in the non-Newtonian calculus, J*ournal of Science and Arts*, 49(4) (2019), 915-936.

[14] S. Tekin and F. Başar, Certain sequence spaces over the non-Newtonian complex field, *Abstr. Appl. Anal.*, 2013 (2013), 1-11.

[15] C. Turkmen and F. Başar, Some basic results on the sets of sequences with geometric calculus, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, 61(2) (2012), 17-34.

[16] A. Uzer, Multiplicative type complex calculus as an alternative to the classical calculus, *Comput. Math. Appl.*, 60(10) (2010), 2725-2737.

İlker Eryilmaz
Professor of Mathematics
Department of Mathematics
Ondokuz Mayıs University
Samsun, Turkey
E-mail: rylmz@omu.edu.tr