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Original Research Paper

Bayesian k-record Analysis For The Lomax Distribution Using Objective Priors

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Abstract. In this paper, we present new perspectives of the parameters of a Lomax model that measure the relevance of different priors on the posteriors using upper kth records. The importance of this analysis is shown through the establishment of convenient rankings based on objective priors. Among several possible priors, such as Jeffrey's, reference and maximal data information priors, we identify those priors that satisfy specific convergence concepts. For illustration purposes, we measure the relevance of the priors within simulated data and real medical data of cancer patients.

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1 Introduction and Motivation

The Lomax distribution has a substantial impact within heavy-tailed fitting models for data sets from various humanitarian, economic or industrial fields. We say that a random variable X has the Lomax distribution [**?**] if its density function is

$$f(x;\alpha,\beta)=\frac{\alpha}{\beta}\Big(1+\frac{x}{\beta}\Big)^{-(\alpha+1)},\ x>0, \tag{1}$$

with the corresponding distribution function as

$$F(x;\alpha,\beta)=1-\Big(1+\frac{x}{\beta}\Big)^{-\alpha},\ x>0, \tag{2}$$

where $\alpha>0$ is the shape parameter and $\beta>0$ is the scale parameter. Equation (1) is a special form of the Pearson type VI distribution. Its ability to quantify the relevance of heavy-tailed phenomena in data, while at the same time holds similar properties as the exponential distribution, is a determinant in the design of a successful modeling strategy. This is especially the case with data related to business finances [**? ? ?**], medical and biological sciences [**?**], different lifetime data groups [**?**], etc. Interesting inference examples where the Lomax distribution has found its place as a basis may be found in e.g. [**? ? ? ? ?**].

When dealing with large data, it is often convenient to deal only with those observations that overpass other observations, for instance, in magnitude. Within such situations it is interesting to focus at records. The concept of record values were introduced in [**?**] as observations that overpass the previous ones in a sequence of independent and identically distributed (iid) random variables. Thus, it seems easier to collect and memorize only record values because they rarely occur instead of the whole sample. This implies that the probability of record values in most cases is skewed and asymmetric.

This paper deals with so called kth record values. The definition of upper kth record values and upper kth record times is as follows ([**?**]): Let $T_{1,k}=k$, $R_{1(k)}=X_{1:k}$ and for $n\geq 2$, let $T_{n,k}=\min\{j: j>T_{n-1,k},\ X_j>X_{T_{n-1,k}-k+1:T_{n-1,k}}\}$, where $X_{i:m}$ denotes the ith order statistics in a sample of size m from the underlying iid sequence $\{X_i,\ i\geq 1\}$. Then, the sequence $\{T_{n,k},\ k\geq 1\}$ is denoted as sequence of kth upper record times, while the sequence $\{R_{n(k)} = X_{T_{n-1,k}-k+1:T_{n-1,k}},\ n \geq 1\}$ is denoted as sequence of upper kth record values. Case when $n = 1$ is a trivial case, which we do not observe in detail in this paper, due to limited significance.

Records provide a basis for various inferential procedures, such as characterization problems [**?** **?**], goodness of fit tests [**?**], predictions [**?** **?** **?** **?**], information theory [**?**], reliability analysis [**?** **?**], etc. Within this concept, some inference results on records for the Lomax distribution have been addressed in e.g. [**?** **?** **?**].

The log-likelihood function for $(\alpha,\ \beta)$ based on the first n upper kth realizations of record values $r_{1(k)}, r_{2(k)} \ldots, r_{n(k)}$, such that $r_{1(k)} < r_{2(k)} < \cdots < r_{n(k)}$, of random variables $R_{1(k)}, \ldots, R_{n(k)}$ from the cdf (2), has the form

$$l(\alpha, \beta) = n \ln k - n \ln \beta + n \ln \alpha - k\alpha \ln \left(1 + \frac{r_{n(k)}}{\beta}\right) - \sum_{i=1}^{n} \ln \left(1 + \frac{r_{i(k)}}{\beta}\right). \tag{3}$$

It may be easily seen, based on normal equations of the likelihood for the two-parameter Lomax model (1), that tractable form of the maximum likelihood estimate (MLEs) for $\beta(\widehat{\beta})$ based on kth records can not be obtainable. Clearly, $\widehat{\alpha} = \alpha(\beta) = \frac{n}{k}\left(\ln\left(1 + \frac{r_{n(k)}}{\beta}\right)\right)^{-1}$, which is neglected for $\widehat{\beta}$ since it can not be presented in a similar form. Furthermore, it can be observed that the MLE for β has no unique value (see Appendix). We may point out to [**?**] for some additional details on this matter.

Basically, it is found hard to make reliable inference on the parameters in cases where small samples are used with high level of skewness and with obvious asymmetry behaviour as in the case of record values. In such circumstances, Bayesian procedures are used to provide concise and reliable estimates of unknown parameters. However, Bayesian inference produces posteriors that highly depend on the choice of the priors. It is on the researcher to determine this level of impact. That is the baseline for the idea of properly evaluating the impact of the information about the parameters of interest through the priors.

One approach to address this problem is to view prior information

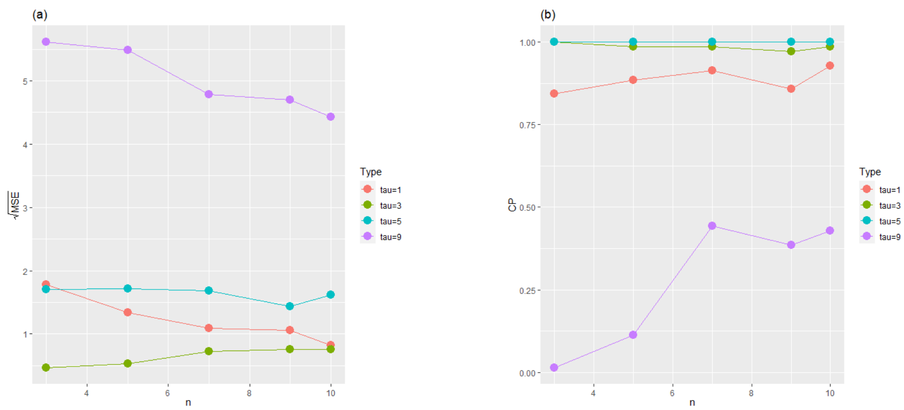

Figure 1: Square root of mean squared error and coverage probability of the Bayesian estimation of α based on prior (4) with $\tau = 1, 3, 5$ and 9. Left graph (a) is for $\sqrt{MSE}$ and right graph (b) is for CP.

of parameters from the perspective of (subjective) prior distribution

$$\pi(\alpha, \beta) \propto \frac{1}{\beta}\alpha^{\tau-1}e^{-\alpha}, \tag{4}$$

where $\tau > 0$ is a hyperparameter. The model description prior (4) may be seen as the product of marginal gamma prior for α and Pareto prior for β. For illustration purposes, we conducted a brief simulation analysis on the Bayesian estimators for parameter α for comparison purposes with respect to the square root of the mean square error ($\sqrt{MSE}$) and coverage probability (CP). Sample sizes of records are selected to be $n = 3, 5, 7, 9$ and 10, while the value of τ is set as 1, 3, 5 and 9. Results are presented in Figure 1. As expected, the performance of $\sqrt{MSE}$ and CP based on the prior (4) highly depends on the value of τ. This yields the necessity for avoiding the ambiguity on the selection of τ in practice. Thus, one should devote special attention on priors based on scientific principles.

The main issue is to find the most relevant posterior with minor deviations with the amount of information found in the data and in the posteriors. In order to do so, some well-known objective priors are used. These kind of priors cover situations where the posteriors behave similarly as the likelihood function but they differ by definition and construction procedures. As so, we will distinguish several cases of objective priors such as probability matching priors, the maximal data information (MDI) prior, Jeffreys prior and reference priors. Comprehensive studies following this framework were done for the Weibull, Gamma and Lomax distribution under the iid concept and they can be found in papers [**? ? ? ? ? ?**]. However, this was not done for cases where the iid structure is weakened. One such example is the case of record values. Hence, such an analysis is challenging and attractive at the same time. Therefore, using records global popularity, the Bayesian inference under record values and objective priors will have various applications in different life aspects thus imposing the practical worth of this paper. One such application can be found in e.g. [**?**].

In this paper, we examine the scenario in which record values are considered to be the basic sample scheme and the Lomax distribution is the selected probability model for summarizing the behaviour of objective priors that preserve and emphasis, as much as possible, the information found in the sample scheme.

The rest of the paper is organized as follows. Section 2 presents objective priors which we will consider and analyse their relevant properties. Our main results are also given in this section. In Section 3, we conduct in-depth review of the posteriors in the context of prior distributions for which it is reasonable to make such inference. Section 4 illustrates the application of such inference to the case of cancer patient data. Finally, we present some concluding remarks in Section 5.

2 Noninformative Priors

This section is devoted to incorporating different objective priors for the parameters (α, β) from the Lomax distribution (1) applied within a record scheme. Objective priors such as probability matching priors, the MDPI prior and reference priors are considered.

2.1 Second-order matching prior

Probability matching priors have been characterized in [**? ?**] as a set of procedures that tend to construct credible intervals that preserve the quantitive relevance of the coverage probability within Bayesian and frequentist context. Thus, given a prior $\pi(\cdot)$ for the parameters (ϕ, ξ), and suppose that ϕ is the parameter of intereset, $\phi^{(1-\gamma)}(\pi(\cdot), \mathbf{X})$ is the $(1-\gamma)$th percentile of the marginal posterior distribution of ϕ. Then, $\pi(\cdot)$ is called a second-order probability matching prior if

$$P\{\phi \leq \phi^{(1-\gamma)}(\pi(\cdot), \mathbf{X})\} = 1 - \gamma + o(n^{-1}),$$

holds for all $\gamma \in (0, 1)$.

For the parameters α and β of the Lomax distribution, we have obtained the following probability matching priors.

Theorem 1. *(a) When α is the parameter of interest and β is the nuisance parameter, the second-order probability matching prior has the form of*

$$\pi_{M_1}(\alpha, \beta) \propto F_1(\alpha) \cdot G_1(\beta), \tag{5}$$

where

$$F_1(\alpha) \propto \alpha^{n/2-2}(2+\alpha)^{-n/2} e^{(1+c_1)H_1(\alpha)},$$

and

$$H_1(\alpha) = \frac{\alpha^{-n}}{n} {}_2F_1(-n, -n, 1-n, -\alpha)$$

with $G_1(\beta) \propto \beta^{c_1}$, and c_1 as an arbitrary constant.

(b) When β is the parameter of interest and α is the nuisance parameter, the second-order probability matching prior has the form of

$$\pi_{M_2}(\alpha, \beta) \propto F_2(\alpha) \cdot G_2(\beta), \tag{6}$$

where

$$F_2(\alpha) \propto (1+\alpha)^{-n}\alpha^{n-2}e^{-(1+c_2)H_2(\alpha)}$$

and

$$\begin{aligned} H_2(\alpha) =& \frac{(1+\alpha)^{-n}(2+\alpha)^n}{n} {}_2F_1\left(1, -n, 1-n, \frac{2(1+\alpha)}{2+\alpha}\right) \\ &+ \frac{(1+\alpha)^{-n}}{n} {}_2F_1\left(-n, -n, 1-n, -1-a\right) \end{aligned}$$

with $G_2(\beta) \propto \beta^{c_2}$*, and* c_2 *as an arbitrary constant.*

(c) For the case when $c_1 = c_2 = -1$*, the posterior distribution under* π_{M_1} *and* π_{M_2} *is proper for* $n \geq 2$*.*

2.2 MDI prior

Another perspective of a non-informative prior is offered in [**?**], which accounts the maximization of the information found in the data with respect to priors. Such prior is called Maximal data information(MDI) prior. For parameters (α, β) of the Lomax distribution model (1), we have derived the MDI prior and other results.

Theorem 2. *(a) The MDI prior for the parameters* (α, β) *is given by*

$$\pi_M(\alpha, \beta) \propto \frac{\alpha^n}{\beta e^{1/\alpha}}.$$

(b) For any $n \geq 2$*, the posterior distribution under* $\pi_M(\alpha, \beta)$ *is improper.*

(c) The MDI prior is not a second-order probability matching prior for $n \geq 2$*.*

2.3 Reference prior

Reference prior is based on the idea to maximize the expected Kullback-Leibler divergence of the posterior distribution relative to the prior. One of its main features is a different treatment for interest and nuisance parameters which is established by the order of occurrence in a parameterization. Reference priors were first introduced in [**?**] after which a detailed inference was reported in [**?**].
Eventually, For the parameters α and β of the Lomax distribution, we obtain the following reference priors.

Theorem 3. *(a) If α is the parameter of interest and β is the nuisance parameter, the reference prior is of the form*

$$\pi_{R1}(\alpha,\beta) \propto \frac{1}{\alpha\beta(\alpha+1)^n},$$

and, if β is the parameter of interest and α is the nuisance parameter, the reference prior for (α,β) is

$$\pi_{R2}(\alpha,\beta) \propto \frac{1}{\alpha\beta}.$$

(b) The posterior distribution under the reference prior π_{R1} is proper for $n \geq 2$, but under reference prior π_{R2} is improper.

(c) Neither one of the priors π_{R1} or π_{R2} is a second-order probability matching prior for $n \geq 2$.

Remark 4. When $n = 1$ (trivial case), the prior π_{R1} is a second-order probability matching prior according to [**?**].

2.4 Jeffrey's prior

The most popular and influential objective prior in Bayesian analysis is the Jeffrey's prior [**?**], which is defined as

$$\pi_J(\alpha,\beta) \propto |I(\alpha,\beta)|^{1/2}, \tag{7}$$

where I denotes the Fisher information(FI) matrix, which form is given by

$$I(\alpha,\beta) \propto \begin{pmatrix} \frac{\alpha^n}{(\alpha+2)^n\beta^2} & -\frac{\alpha^{n-1}}{(\alpha+1)^n\beta} \\ -\frac{\alpha^{n-1}}{(\alpha+1)^n\beta} & \frac{1}{\alpha^2} \end{pmatrix}. \tag{8}$$

For the parameters α and β of the Lomax distribution, we have derived the following result.

Theorem 5. *(a) The Jeffrey's prior for* (α, β) *is*

$$\pi_J(\alpha,\beta) \propto \alpha^{3n/2-2}(\alpha+1)^{-n}(\alpha+2)^{-n/2}\beta^{-1}.$$

(b) The posterior distribution under $\pi_J(\alpha,\beta)$ *is proper for* $n \geq 2$.

(c) The Jeffrey's prior is not a second-order probability matching prior whenever α *is the parameter of interest or* β *is the parameter of interest.*

The proofs of the Theorems 2.1-2.5 and the Proposition 2.3. are given in the Appendix. They refer to priors π_J, π_{R_1}, π_{M_1} and π_{M_2} as those that develop proper posteriors and can be considered for use in practice. Priors π_J and π_{R_1} are not a second-order matching priors, and so it is interesting to perform a simulation study in order to gain additional information on the performances of all mentioned priors. This will be discussed in more details in the following section.

3 Simulation Study

This section represents frequentist influence indices via a comparison study that can be applied to quantify the relevance of prior distributions on the posterior. This is achieved by implementing the random-walk Metropolis algorithm [**?**] within the package *MHadaptive* in software R. A sample of 100 000 random variates is generated from which the initial 40 000 was discarded as a burn-in sample. Under this setting, an acceptance rate in the range of 10-40% is used as a condition that need to be fulfilled in order to accept the generated sample as representative. Such condition was suggested in [**?**]. The sample was then tinned by a factor of 10 in order to yield a low mutually autocorrelations and use those remaining observations to estimate the posterior density functions. This led to a final sample of 4001 values for each parameter. In order to provide a reliable performance, the median was chosen as a Bayesian estimator. This intuitively makes sense since the median provides more optimal and robust estimation compared to the sample mean. We selected the highest posterior density intervals (HPDs) as the appropriate estimate of Bayesian credible intervals (CIs) due to non-symmetrical form of marginal posteriors of parameters α and β, see [**?**]. We compare performance of Bayesian estimators using mean-square errors (MSEs) and the coverage probabilities (CPs). The MSEs and CPs are computed based on replications of 500 times for record samples with size $n = 5, 10, 15$ and 20, from the Lomax model (1) using different true values of the parameters α and β. We present results for two different values of k: $k = 1$ and $k = 2$. Table 1 and 2 list the values of MSEs and CPs for 95% CIs, with respect to value of k.

From the reported values, we can derive the following conclusions:

- The MSEs appear to decrease for all estimators when the sample size n increases. This becomes quite obvious for the parameter α under the cases $(\alpha, \beta) = (1.5, 5)$ and $(\alpha, \beta) = (3, 10)$. Also, it is noted that in case of small n the CPs tend to be wider in application thus yielding higher performances for all true values. This is directly invoked by the low precision of the record values statistics in general. For instance, a similar problem is encounted in [**?**].

- According to MSEs, it seems evident that Bayes estimators under prior π_{R_2} outperform those estimators under priors π_J, π_{M_1} and π_{M_2}.

- Bayes estimators under priors π_{M_1} and π_{M_2} appear as the dominant ones in terms of the highest CPs. This is determined by their construction principle.

Table 1: Empirical MSEs (CPs in parentheses) of Bayesian estimators based on priors π_J, π_{R_2}, π_{M_1} and π_{M_2} for $k = 1$.

	n	5	10	15	20
$\alpha = 3, \beta = 1$					
π_J	α	138.8332(1)	243.5515(0.92)	664.5009(0.875)	12.993(0.675)
	β	405.2481(1)	2417.639(0.97)	8227.3445(0.95)	577.3507(0.985)
π_{R_2}	α	2.3599(0.825)	2.94(0.24)	2.8998(0.05)	2.9412(0.005)
	β	0.6988(0.84)	0.6591(0.67)	0.5441(0.635)	0.5793(0.62)
π_{M_1}	α	1354.055(0.99)	321.7536(0.985)	119.6123(0.965)	2.1908(0.945)
	β	6019.744(0.995)	492.1408(0.98)	53.9371(0.985)	17.6279(0.985)
π_{M_2}	α	2030.846(0.985)	168.5836(0.985)	7.8034(0.98)	2.1874(0.95)
	β	1968.177(0.99)	486.1223(0.98)	259.1341(0.965)	11.0607(0.99)
$\alpha = 2, \beta = 1.5$					
π_J	α	1378.114(0.958)	430.6451(0.83)	8.1009(0.655)	2.5323(0.43)
	β	19450.008(0.975)	50599.4603(0.95)	2450.7514(0.97)	1723.61(0.965)
π_{R_2}	α	0.5084(0.955)	0.8578(0.545)	0.8941(0.265)	0.9137(0.115)
	β	130.0291(0.925)	5.3118(0.86)	1.5942(0.78)	2.2015(0.765)
π_{M_1}	α	923.048(0.985)	8.7507(0.98)	2.4011(0.945)	0.6912(0.92)
	β	5339.168(0.995)	576.1558(0.99)	1244.1646(0.985)	89.2187(0.97)
π_{M_2}	α	53.3098(1)	287.0818(0.98)	3.238(0.91)	1.2326(0.815)
	β	3794.1295(0.99)	26513.4378(0.98)	274.5186(0.975)	343.8095(0.965)
$\alpha = 1.5, \beta = 2$					
π_J	α	1379.091(0.955)	19.5894(0.73)	3.1212(0.475)	1.2742(0.34)
	β	83419.935(0.940)	21068.623(0.945)	2215.2924(0.975)	11750.9045(0.96)
π_{R_2}	α	0.2835(0.95)	0.3434(0.695)	0.3845(0.42)	0.398(0.27)
	β	739.7006(0.94)	14.2715(0.885)	8699.5963(0.88)	3975.5209(0.81)
π_{M_1}	α	74.6011(0.985)	5.8945(0.98)	0.7696(0.885)	0.6067(0.845)
	β	30865.1036(0.985)	12104.1438(0.97)	1464.5083(0.98)	2433.6369(0.975)
π_{M_2}	α	1130.537(0.975)	55.4598(0.94)	122.42(0.795)	0.5964(0.75)
	β	56110.513(0.965)	8013.1739(0.98)	10102.063(0.975)	488.5763(0.975)
$\alpha = 1.5, \beta = 5$					
π_J	α	106.0308(0.945)	12.3409(0.705)	3.0501(0.465)	1.568(0.31)
	β	179622.0320(0.93)	42964.6794(0.92)	8365.4809(0.97)	25025.7632(0.945)
π_{R_2}	α	0.2179(0.97)	0.3567(0.655)	0.3807(0.495)	0.4071(0.23)
	β	2166.2408(0.965)	1241.4228(0.885)	41.6347(0.84)	222.9069(0.855)
π_{M_1}	α	25.5421(0.99)	3.4941(0.955)	0.8523(0.915)	0.4183(0.88)
	β	20193.264(0.975)	32973.9764(0.945)	2751.1703(0.965)	6704.4385(0.98)
π_{M_2}	α	57.9922(0.965)	3.7121(0.915)	1.0763(0.835)	0.7065(0.73)
	β	81076.5816(0.97)	50515.4055(0.94)	18106.9258(0.96)	34062.08(0.955)
$\alpha = 3, \beta = 10$					
π_J	α	697.7192(0.965)	516.0249(0.86)	84.5236(0.78)	7.0229(0.65)
	β	87921.2806(0.96)	15508.7203(0.91)	36983.0463(0.945)	12148.6283(0.925)
π_{R_2}	α	2.1646(0.905)	2.8376(0.295)	2.8592(0.07)	2.9655(0.005)
	β	125.062(0.89)	58.7198(0.705)	61.0706(0.61)	59.162(0.615)
π_{M_1}	α	14.8206(0.99)	38.5733(0.995)	13.0889(0.965)	7.5047(0.925)
	β	5869.9024(0.995)	18711.8209(0.96)	16977.1403(0.965)	5507.1673(0.97)
π_{M_2}	α	317.9774(0.98)	63.645(0.97)	11.2376(0.955)	4.3357(0.9)
	β	21735.7015(0.985)	18506.9769(0.975)	7157.5734(0.965)	17085.7888(0.95)

Table 2: Empirical MSEs (CPs in parentheses) of Bayesian estimators based on priors π_J, π_{R_2}, π_{M_1} and π_{M_2} for $k = 2$.

	n	5	10	15	20
$\alpha = 3, \beta = 1$					
π_J	α	267.1435(1)	4115.769(0.85)	2029.087(0.66)	1341.282(0.65)
	β	146.7821(1)	3981.325(0.94)	15406.714(0.885)	6394.559(0.85)
π_{R_2}	α	3.029(0.88)	3.796(0.04)	3.6737(0)	3.6087(0)
	β	0.4918(0.835)	0.6235(0.465)	0.6237(0.365)	0.6432(0.315)
π_{M_1}	α	5.9509(0.995)	1767.0618(0.985)	1023.99(0.96)	275.8564(0.94)
	β	9.55(0.995)	508.3372(0.99)	2079.137(0.99)	975.4477(0.98)
π_{M_2}	α	194.0302(1)	236.2958(0.995)	1061.065(0.945)	98.3138(0.92)
	β	66.2948(1)	118.3721(0.99)	1826.01(0.975)	165.7086(0.975)
$\alpha = 2, \beta = 1.5$					
π_J	α	1091.498(0.975)	706.3238(0.715)	780.3347(0.545)	62.4794(0.455)
	β	10474.34(0.985)	16568.5705(0.895)	66191.0306(0.81)	5052.5587(0.855)
π_{R_2}	α	0.7997(0.95)	1.1042(0.445)	1.1246(0.13)	1.1018(0.025)
	β	1.9853(0.905)	1.2235(0.72)	1.2652(0.665)	1.3673(0.535)
π_{M_1}	α	379.9688(1)	1463.727(0.99)	26.6312(0.92)	5.603(0.935)
	β	523.1862(1)	13044.425(0.995)	897.3921(0.975)	152.0279(0.975)
π_{M_2}	α	509.669(0.98)	605.5415(0.985)	114.8227(0.87)	36.4965(0.88)
	β	41183.460(0.98)	14232.1206(0.985)	8279.2307(0.96)	965.5869(0.965)
$\alpha = 1.5, \beta = 2$					
π_J	α	862.8587(0.98)	401.3709(0.625)	87.1428(0.495)	7.8489(0.3)
	β	45678.4355(0.96)	41085.5187(0.825)	39538.8619(0.855)	4442.0313(0.94)
π_{R_2}	α	0.2386(0.97)	0.4173(0.69)	0.4693(0.32)	0.4679(0.17)
	β	6.54(0.97)	2.042(0.785)	2.3431(0.77)	2.3228(0.725)
π_{M_1}	α	106.2368(0.99)	156.6144(0.98)	3.8284(0.97)	1.6466(0.85)
	β	17157.2159(0.99)	12617.344(0.99)	342.4811(0.98)	470.3085(0.955)
π_{M_2}	α	207.848(1)	233.4819(0.935)	424.0057(0.83)	1.9682(0.765)
	β	6291.798(1)	10148.5016(0.97)	7025.7925(0.955)	868.1283(0.98)
$\alpha = 1.5, \beta = 5$					
π_J	α	565.4356(0.965)	238.1688(0.495)	60.7445(0.445)	10.5562(0.27)
	β	72671.5348(0.95)	127216.466(0.745)	90755.9156(0.83)	27884.6593(0.835)
π_{R_2}	α	0.3021(0.945)	0.4473(0.64)	0.4375(0.40)	0.4565(0.185)
	β	45.3277(0.90)	34.4974(0.80)	21.7274(0.78)	14.3331(0.705)
π_{M_1}	α	267.6252(0.99)	87.2102(0.955)	83.8766(0.91)	2.9484(0.88)
	β	12817.8388(0.99)	69736.1034(0.95)	10629.0952(0.97)	13078.5456(0.975)
π_{M_2}	α	212.7938(0.985)	54.7583(0.915)	15.3453(0.87)	1.5014(0.75)
	β	21857.3257(0.985)	24890.3064(0.965)	25505.5475(0.94)	1916.4231(0.97)
$\alpha = 3, \beta = 10$					
π_J	α	432.2631(0.99)	1151.653(0.825)	1451.127(0.53)	185.1263(0.555)
	β	36692.6873(0.985)	140350.369(0.85)	188910.68(0.705)	79023.9296(0.74)
π_{R_2}	α	3.126(0.865)	3.7978(0.06)	3.7126(0)	3.49(0)
	β	54.8589(0.885)	61.9585(0.475)	63.4184(0.39)	61.4751(0.355)
π_{M_1}	α	126.4224(0.995)	425.5736(0.995)	91.6406(0.975)	34.9405(0.935)
	β	1232.6373(0.995)	35324.7054(0.995)	21323.24137(0.945)	19805.3631(0.96)
π_{M_2}	α	588.2681(0.98)	726.1167(0.985)	1325.288(0.96)	48.2131(0.905)
	β	39315.9712(0.98)	45657.9298(0.96)	65112.76(0.945)	13977.9524(0.96)

4 Real Data Analysis

In this section, we use a real life data to illustrate the significance of the proposed methods and verify how our estimates work in practice. The data set consists on remission times (in months) taken from a random sample of 128 bladder cancer patients and presented as a model study in [**?**]. According to [**?**] and [**?**], the two-parameter Lomax distribution can be used to describe these data. The maximum likelihood estimates of the Lomax distribution parameters for this data are $\widehat{\alpha} = 13.9384$ and $\widehat{\beta} = 121.0225$. For the purpose of our analysis, we consider those estimates of α and β as the true values. For clearlity, we extracted kth record samples from the data and listed them in Table 3.

Table 3: Extracted kth records for a data set.

$k = 1$	0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	25.74	25.82
	26.31	32.15	34.26	36.66	43.01	46.12	79.05			
$k = 2$	0.08	2.09	3.48	4.87	6.94	8.66	13.11	13.29	13.80	23.63
	25.74	25.82	26.31	32.15	34.26	36.66	43.01	46.12		

In order to attribute more information to ones in previous section, we compare Bayesian estimators based on priors π_J, π_{R_2}, π_{M_1} and π_{M_2}, with respect to different values of k, on a real data sample. We briefly summarize the values of estimators, their corresponding standard deviation(SDs) and HPDs in Table 4. These results 4 indicate that Bayes estimates under Jeffrey's prior for the parameters of Lomax distribution are preffered than others when precision of point estimates is taken but with deficiency of large SDs. On contrary, Bayes estimates under reference prior outperform all other Bayesian estimates in terms of SDs and HPDs, but lack of precision.

A similar behaviour was observed for Bayesian estimates under matching priors, which outperform Bayesian estimates under Jeffrey's prior taking into consideration the length of the HPDs and the SDs, but provides results with high deviations from true values.

Finally, the overall conclusion is that Bayesian estimates under priors π_J, π_{M_1} or π_{M_2} can be selected as adequate depending on the researchers needs and preferences. As a last note, we can highlight that the k value had no influence on the final conclusion.

Table 4: Summary of the Bayesian estimates.

	Prior	Parameter	Median	SD	95% HPD
	π_J	α	12.4991	14.3042	(3.1895, 46.6744)
		β	28.0167	67.2049	(0.358, 185.4571)
	π_{R_2}	α	1.5735	0.5893	(0.6969, 2.8151)
$k = 1$		β	0.4247	1.0128	(0.0006, 2.7286)
	π_{M_1}	α	8.7085	5.6408	(2.9647, 20.5452)
		β	11.9815	20.5556	(0.1095, 57.2737)
	π_{M_2}	α	8.7489	13.9356	(2.6924, 28.3414)
		β	12.1409	64.4685	(0.0654, 92.8945)
	π_J	α	29.0884	33.9457	(4.2656, 103.8848)
		β	128.3906	188.5059	(3.5477, 565.0499)
	π_{R_2}	α	1.2376	0.4455	(0.5766, 2.1962)
$k = 2$		β	0.9309	1.5522	(0.0004, 4.203437)
	π_{M_1}	α	16.0583	42.6722	(2.0717, 118.7862)
		β	62.4734	238.968	(0.7698, 656.4567)
	π_{M_2}	α	17.1652	44.3993	(2.5136, 132.1943)
		β	67.4939	239.4859	(0.7772, 651.2028)

5 Summary and Conclusion

In cases where MLEs of the model parameters do not exist, Bayesian procedures have an overwhelming impact. This is quite natural in case of absence some frequentist details. In this paper, we have dealt with the issue of quantifying the relevance of objective priors on the properness of the posteriors within record values from the Lomax distribution. This comes naturally, since it is known that Lomax distribution fits heavy-tailed data and is therefore oftenly faced with various extremes and records. Thus, we analyzed families of objective priors and their impact on the posteriors for the parameters of the Lomax distribution based on records. We considered families of second-order probability matching priors, reference priors, MDI priors and Jeffrey's priors. Inference analysis indicated that neither reference priors, MDI priors or Jeffrey's priors belong to the family of second-order probability matching priors. We also compared their performances through a simulation study. Overall, it was indicated that researchers needs may take a dominating role in the process of prior selection. Therefore, results presented in this paper may have a tremendous impact in various further research.

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Conflict of interest

Authors declare no conflict of interest.

6 Appendix

6.1 Proof for the non-existence of MLEs α and β

By using standard notation, the profile log-likelihood function (3), is given by

$$\begin{aligned} l_p(\beta) = \sup_{\beta} l(\widehat{\alpha},\beta) = \sup_{\beta} l(\alpha(\beta),\beta) &= n\ln k - n - n\ln\beta + n\ln(\widehat{\alpha}) \\ &- \sum_{i=1}^{n} \ln\left(1+\frac{r_{i(k)}}{\beta}\right). \end{aligned}$$

Let us denote the function $g(\beta)$ as

$$g(\beta) = n\ln k - n - n\ln\beta + n\ln(\widehat{\alpha}) - n\ln\left(1+\frac{r_{n(k)}}{\beta}\right).$$

It is evident that $l_p(\beta) \geq g(\beta)$ for all $\beta > 0$. Further, we can represent $g(\beta)$ as $g(\beta) = nh(\beta)$, where

$$h(\beta) = \ln k - 1 - \ln\beta + \ln(\widehat{\alpha}) - \ln\left(1+\frac{r_{n(k)}}{\beta}\right). \tag{9}$$

We can write

$$h'(\beta) = -\frac{1}{\beta}\frac{\ln\left(1+\frac{r_{n(k)}}{\beta}\right) - \frac{r_{n(k)}}{\beta+r_{n(k)}}}{\ln\left(1+\frac{r_{n(k)}}{\beta}\right)} + \frac{r_{n(k)}}{\beta(\beta+r_{n(k)})}, \tag{10}$$

which gives us the information that $\lim_{\beta\to\infty} -\beta^2 h'(\beta) = -\frac{1}{2} r_{n(k)} < 0$. By this, it is easy to conclude that $g(\beta)$ is strictly increasing function and, hence, $l_p(\beta)$ constantly increases with respect to β. This indicates that $\widehat{\beta}$ has no unique value, and, hence, the MLE method is not applicable in this case.

6.2 Proof of the posterior propriety of $\pi(\alpha, \beta | r_{(k)})$:

The joint posterior density of (α, β) based on the prior $\pi(\alpha, \beta)$ is given by [**?**]

$$\begin{aligned}\pi(\alpha,\beta|r_{1(k)}, r_{2(k)}, \dots, r_{n(k)}) &\propto \pi(\alpha,\beta)\cdot L(r_{1(k)}, r_{2(k)}, \dots, r_{n(k)}|\alpha,\beta)\\ &= k^n \beta^{-(n+1)} \alpha^{n+\tau-1} e^{-\alpha}\left(1+\frac{r_{n(k)}}{\beta}\right)^{-k\alpha}\\ &\times \prod_{i=1}^{n}\left(1+\frac{r_{i(k)}}{\beta}\right)^{-1}.\end{aligned}$$

From here we have

$$\begin{aligned}&\int_0^\infty \int_0^\infty \pi(\alpha,\beta|r_{1(k)}, r_{2(k)}, \dots, r_{n(k)})\, d\beta\, d\alpha\\ &\le \int_0^\infty k^n \alpha^{n+\tau-1} e^{-\alpha} \int_0^\infty \beta^{-(n+1)}\left(1+\frac{r_{n(k)}}{\beta}\right)^{-(k\alpha+n)} d\beta\, d\alpha\\ &\propto \int_0^\infty \alpha^{n+\tau-1} e^{-\alpha} B(k\alpha, n)\, d\alpha\\ &\propto \int_0^\infty \alpha^{n+\tau-1} e^{-\alpha} \frac{1}{k\alpha(k\alpha+1)\cdots(k\alpha+n-1)}\, d\alpha,\end{aligned} \tag{11}$$

where $B(a,b)$ is the Beta function. Let $h_1(\alpha) = \alpha^{n+\tau-1}e^{-\alpha}\frac{1}{k\alpha(k\alpha+1)\cdots(k\alpha+n-1)}$. It can be seen that $h_1(\alpha) = O(\alpha^{n+\tau-2})$ as $\alpha \to 0$ and $h_1(\alpha) = O(\alpha^{\tau-1}e^{-\alpha})$ as $\alpha \to \infty$. We can therefore conclude that integral (11) is finite for $n \ge 2$.

6.3 Proof of Theorem 2.1

Let S be the inverse of the FI (8). Then,

$$S \propto \begin{pmatrix} \beta^2\alpha^{2-3n}(\alpha+1)^{2n}(\alpha+2)^n & \beta\alpha^{3-2n}(\alpha+2)^n(\alpha+1)^n \\ \beta\alpha^{3-2n}(\alpha+2)^n(\alpha+1)^n & \alpha^{4-2n}(\alpha+1)^{2n} \end{pmatrix}. \tag{12}$$

From (12) and by following the procedures proposed in [? ?], we can state that the second-order probability matching prior $\pi_{M_1}(\alpha,\beta)$ satisfies the following partial differential equations:

$$\begin{aligned} &\frac{\partial}{\partial\beta}\Big(\beta\alpha^{1-3n/2}(\alpha+1)^n(\alpha+2)^{n/2}\pi_{M_1}(\alpha,\beta)\Big) \\ &+\frac{\partial}{\partial\alpha}\Big(\alpha^{2-n/2}(\alpha+2)^{n/2}\pi_{M_1}(\alpha,\beta)\Big)=0, \end{aligned} \tag{13}$$

for which we have the solution (5). Alternatively, the second-order probability prior $\pi_{M_2}(\alpha,\beta)$ is of the form

$$\frac{\partial}{\partial\beta}\left(\beta\alpha^{1-n}(\alpha+2)^n\pi_{M_2}(\alpha,\beta)\right)+\frac{\partial}{\partial\alpha}\left(\alpha^{2-n}(\alpha+1)^n\pi_{M_2}(\alpha,\beta)\right)=0, \tag{14}$$

for which we have the solution given by the formula (6). This proofs (a) and (b). Part (c) follows the same steps as in the proof of Theorem 2.5.

6.4 Proof of Theorem 2.2

(a) The MDI prior for (α,β) has the following form:

$$\pi_M(\alpha,\beta)\propto\exp\{H(\alpha,\beta)\},$$

where $H(\alpha,\beta)=E(\ln f_n(x))$ and $f_n(x)$ is the density function of the nth upper kth record value from Lomax distribution. Then it follows quite directly

$$\pi_M(\alpha,\beta)\propto\frac{\alpha^n}{\beta e^{1/\alpha}}.$$

(b) The joint posterior distribution of (α, β) based on π_M is given as

$$\begin{aligned}\pi_M(\alpha, \beta | r_{1(k)}, r_{2(k)}, \ldots, r_{n(k)}) &\propto \pi_M(\alpha, \beta) \cdot L(r_{1(k)}, r_{2(k)}, \ldots, r_{n(k)} | \alpha, \beta) \\ &= \alpha^{2n} \beta^{-(n+1)} e^{-\frac{1}{\alpha}} \left(1 + \frac{r_{n(k)}}{\beta}\right)^{-k\alpha} \\ &\times \prod_{i=1}^{n} \left(1 + \frac{r_{i(k)}}{\beta}\right)^{-1}.\end{aligned}$$

Then, we have

$$\begin{aligned}&\int_0^\infty \int_0^\infty \alpha^{2n} \beta^{-(n+1)} e^{-\frac{1}{\alpha}} \left(1 + \frac{r_{n(k)}}{\beta}\right)^{-k\alpha} \prod_{i=1}^{n} \left(1 + \frac{r_{i(k)}}{\beta}\right)^{-1} d\beta\, d\alpha \geq \\ &\int_0^\infty \int_0^\infty \alpha^{2n} \beta^{-(n+1)} e^{-\frac{1}{\alpha}} \left(1 + \frac{r_{n(k)}}{\beta}\right)^{-(k\alpha+n)} d\beta\, d\alpha \propto \\ &\int_0^\infty \int_0^\infty \alpha^{2n} e^{-\frac{1}{\alpha}} \frac{1}{k\alpha(k\alpha+1)\cdots(k\alpha+n-1)} d\beta\, d\alpha.\end{aligned}$$

Let us denote the function $h_2(\alpha) = \alpha^{2n} e^{-\frac{1}{\alpha}} \frac{1}{k\alpha(k\alpha+1)\cdots(k\alpha+n-1)}$. It can be observed that $h_2(\alpha) \propto \alpha^n e^{-\frac{1}{\alpha}}$ as $\alpha \to \infty$, and since $\int_0^\infty \alpha^n e^{-\frac{1}{\alpha}}\, d\alpha = \infty$, $n \geq 1$, we have the result.

6.5 Proof of Theorem 2.3

(a) When α is the parameter of interest, the conditional prior distribution of β given α can be defined on the FI matrix (8) as

$$\pi(\beta|\alpha) = \sqrt{I_{11}} \approx \frac{\alpha^{n/2}}{(\alpha+2)^{n/2}\beta},$$

where I_{11} is the upper left part of FI (8). Then, by choosing a sequence of compact sets $\Omega_i = (d_{1i}, d_{2i}) \times (d_{3i}, d_{4i})$ for (α, β) such that $d_{1i}, d_{3i} \to$

$0, d_{2i}, d_{4i} \to \infty$ as $i \to \infty$, it follows that

$$k_{1i}(\alpha)^{-1} = \int_{d_{3i}}^{d_{4i}} \pi(\beta|\alpha)\, d\beta = \int_{d_{3i}}^{d_{4i}} \frac{\alpha^{n/2}}{(\alpha+2)^{n/2}\beta}\, d\beta$$
$$= \frac{\alpha^{n/2}}{(\alpha+2)^{n/2}} \left(\ln(d_{4i}) - \ln(d_{3i})\right),$$

and

$$p_i(\beta|\alpha) = k_{1i}(\alpha)\pi(\beta|\alpha) = \frac{1}{\beta}\left(\ln(d_{4i}) - \ln(d_{3i})\right)^{-1}. \tag{15}$$

In addition, the marginal reference prior α can be defined based on FI matrix (8) and (15) as

$$\pi_i(\alpha) = \exp\left[\frac{1}{2}\int_{d_{3i}}^{d_{4i}} p_i(\beta|\alpha) \log\left(\frac{|I|}{I_{11}}\right) d\beta\right] = \frac{1}{\alpha(\alpha+1)^n},$$

which produces the following reference prior:

$$\pi_{R1}(\alpha,\beta) = \lim_{i\to\infty}\left[\frac{k_{1i}(\alpha)\pi_i(\alpha)}{k_{1i}(\alpha_0)\pi_i(\alpha_0)}\right]\pi(\beta|\alpha) \propto \frac{1}{\alpha\beta(\alpha+1)^n},$$

for any fixes point α_0. When the β is the parameter of interest, the same procedure holds. Let

$$\pi(\alpha|\beta) = \sqrt{I_{22}} = \frac{1}{\alpha},$$

where I_{22} is the bottom right part of FI (8). Then,

$$k_{2i}^{-1}(\beta) = \int_{d_{1i}}^{d_{2i}} \pi(\alpha|\beta)\, d\alpha = \ln(d_{2i}) - \ln(d_{1i})$$

and

$$p_i(\alpha|\beta) = k_{2i}\pi(\alpha|\beta) = \frac{1}{\alpha(\ln(d_{2i}) - \ln(d_{1i}))}.$$

Therefore, the marginal reference prior for β can be produced as

$$\pi(\beta) = \exp\left[\frac{1}{2}\int_{d_{1i}}^{d_{2i}} p_i(\alpha|\beta) \log\left(\frac{|I|}{I_{22}}\right) d\alpha\right] \propto \frac{1}{\beta},$$

by which we obtain the reference prior as

$$\pi_{R_2}(\beta,\alpha) = \lim_{i\to\infty}\left[\frac{k_{2i}(\beta)\pi_i(\beta)}{k_{2i}(\beta_0)\pi_i(\beta_0)}\right]\pi(\alpha|\beta) \propto \frac{1}{\alpha\beta},$$

for any fixed point β_0.

(b) Let $\pi_{R_1}(\alpha,\beta|r_{1(k)},r_{2(k)},\ldots,r_{n(k)})$ be the posterior density on the prior $\pi_{R_1}(\alpha,\beta)$. Then, it holds

$$\begin{aligned}\pi_{R_1}(\alpha,\beta|r_{1(k)},r_{2(k)},\ldots,r_{n(k)}) &\propto \pi_{R_1}(\alpha,\beta)\cdot L(r_{1(k)},r_{2(k)},\ldots,r_{n(k)}|\alpha,\beta)\\ &= k^n\beta^{-(n+1)}\frac{\alpha^{n-1}}{(\alpha+1)^n}\Big(1+\frac{r_{n(k)}}{\beta}\Big)^{-k\alpha}\\ &\times\prod_{i=1}^{n}\Big(1+\frac{r_{i(k)}}{\beta}\Big)^{-1}.\end{aligned}$$

From here we have

$$\begin{aligned}&\int_0^\infty\int_0^\infty \pi_{R_1}(\alpha,\beta|r_{1(k)},r_{2(k)},\ldots,r_{n(k)})\,d\beta\,d\alpha\\ &=\int_0^\infty\int_0^\infty k^n\beta^{-(n+1)}\frac{\alpha^{n-1}}{(\alpha+1)^n}\Big(1+\frac{r_{n(k)}}{\beta}\Big)^{-k\alpha}\prod_{i=1}^{n}\Big(1+\frac{r_{i(k)}}{\beta}\Big)^{-1}d\beta\,d\alpha\\ &\le\int_0^\infty\int_0^\infty k^n\beta^{-(n+1)}\frac{\alpha^{n-1}}{(\alpha+1)^n}\Big(1+\frac{r_{n(k)}}{\beta}\Big)^{-k\alpha-n}d\beta\,d\alpha\\ &\propto\int_0^\infty\frac{\alpha^{n-1}}{(\alpha+1)^n}\frac{1}{k\alpha(k\alpha+1)\cdots(k\alpha+n-1)}\,d\alpha. \qquad (16)\end{aligned}$$

Let $h_3(\alpha) = \frac{\alpha^{n-1}}{(\alpha+1)^n}\frac{1}{k\alpha(k\alpha+1)\cdots(k\alpha+n-1)}$. It can be seen that $h_3(\alpha) = O(\alpha^{n-2})$ as $\alpha\to 0$ and $h_3(\alpha) = O(\alpha^{-2})$ as $\alpha\to\infty$. We can then conclude that integral (16) is finite for $n \ge 2$.

For the prior π_{R_2}, we have the posterior density

$$\pi_{R_2}(\alpha,\beta|r_{1(k)},r_{2(k)},\ldots,r_{n(k)})$$

as

$$
\begin{aligned}
\pi_{R_2}(\alpha, \beta | r_{1(k)}, r_{2(k)}, \ldots, r_{n(k)}) &\propto \pi_{R_2}(\alpha, \beta) \cdot L(r_{1(k)}, r_{2(k)}, \ldots, r_{n(k)} | \alpha, \beta) \\
&= k^n \beta^{-(n+1)} \alpha^{n-1} \Big(1 + \frac{r_{n(k)}}{\beta}\Big)^{-k\alpha} \\
&\times \prod_{i=1}^{n} \Big(1 + \frac{r_{i(k)}}{\beta}\Big)^{-1}.
\end{aligned}
$$

From here we have

$$
\begin{aligned}
&\int_0^\infty \int_0^\infty \pi_{R_1}(\alpha, \beta | r_{1(k)}, r_{2(k)}, \ldots, r_{n(k)})\, d\beta\, d\alpha \\
&= \int_0^\infty \int_0^\infty k^n \beta^{-(n+1)} \alpha^{n-1} \Big(1 + \frac{r_{n(k)}}{\beta}\Big)^{-k\alpha} \prod_{i=1}^{n} \Big(1 + \frac{r_{i(k)}}{\beta}\Big)^{-1} d\beta\, d\alpha \\
&\geq \int_0^\infty \int_0^\infty k^n \beta^{-(n+1)} \frac{1}{\alpha} \Big(1 + \frac{r_{1(k)}}{\beta}\Big)^{-k\alpha - n} d\beta\, d\alpha \\
&\propto \int_0^\infty \frac{1}{\alpha}\, d\alpha = \infty, \text{for all } n \geq 1. \qquad (17)
\end{aligned}
$$

(c) This follow directly from (5) and (6).

6.6 Proof of Theorem 2.4

(a) According to the definition of the Jeffrey's prior (see [?]), it follows that

$$
\pi_J(\alpha, \beta) \propto \sqrt{|I|} \propto \alpha^{3n/2-2} (\alpha + 1)^{-n} (\alpha + 2)^{-n/2} \beta^{-1},
$$

where $|I|$ denotes the determinant of the FI matrix (8). This completes this part of the proof.
(b) Let $\pi_J(\alpha, \beta | r_{1(k)}, r_{2(k)}, \ldots, r_{n(k)})$ be the posterior density based on the prior π_J. Then, it holds

$$
\pi_J(\alpha, \beta | r_{1(k)}, r_{2(k)}, \ldots, r_{n(k)}) \propto \pi_J(\alpha, \beta) \cdot L(r_{1(k)}, r_{2(k)}, \ldots, r_{n(k)} | \alpha, \beta)
$$

and hence

$$\int_0^\infty \int_0^\infty \alpha^{5n/2-2}(\alpha+1)^{-n}(\alpha+2)^{-n/2}\beta^{-(n+1)}\left(1+\frac{r_{n(k)}}{\beta}\right)^{-k\alpha} \tag{18}$$
$$\times \prod_{i=1}^{n}\left(1+\frac{r_{i(k)}}{\beta}\right)^{-1} d\beta\, d\alpha$$
$$\le \int_0^\infty \alpha^{5n/2-2}(\alpha+1)^{-n}(\alpha+2)^{-n/2}\int_0^\infty \beta^{-(n+1)}\left(1+\frac{r_{n(k)}}{\beta}\right)^{-k\alpha} \tag{19}$$
$$\times \left(1+\frac{r_{1(k)}}{\beta}\right)^{-n} d\beta\, d\alpha$$
$$\le \int_0^\infty \alpha^{5n/2-2}(\alpha+1)^{-n}(\alpha+2)^{-n/2}\int_0^\infty \beta^{-(n+1)}\left(1+\frac{r_{n(k)}}{\beta}\right)^{-n-k\alpha} d\beta\, d\alpha$$
$$\propto \int_0^\infty \alpha^{5n/2-2}(\alpha+1)^{-n}(\alpha+2)^{-n/2}\frac{1}{k\alpha(k\alpha+1)\cdots(k\alpha+n-1)}\, d\alpha$$
$$\propto \int_0^\infty \alpha^{3n/2-2}(\alpha+1)^{-n}(\alpha+2)^{-n/2}\, d\alpha. \tag{20}$$

Let $h_4(\alpha) = \alpha^{3n/2-2}(\alpha+1)^{-n}(\alpha+2)^{-n/2}$. As it is valid that $h_4(\alpha) = O(\alpha^{3n/2-2})$ as $\alpha \to 0$ and $h_4(\alpha) = O(\alpha^{-2})$ as $\alpha \to \infty$. We can then conclude that integral (18) is finite for $n > 4/3$, i.e. $n \ge 2$.

(c) This follow directly from (5) and (6).

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