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Semi-Centralizing Maps and k-Commuting Maps of Module Extension Algebras

S. Ebrahimi*
Payame Noor University

S. Talebi
Payame Noor University

Abstract. We investigate the structure of semi-centralizing and k-commuting maps of module extension algebras. In particular, we give conditions that every semi-centralizing and k-commuting map L of such an algebra is of the form $L(c) = cx + h(c)$, where x lies in the center of the algebra and h is a linear map from the algebra to its center.

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1. Introduction

Let A be an algebra and M be an A -bimodule. The module extension of A by M is the algebra $A \times M$ with underlying k-vector space $A \times M = \{(a, m) | a \in A, m \in M\}$ and the multiplication defined by

$$(a, m) \cdot (b, n) = (ab, an + mb), \quad (a, b \in A, m, n \in M).$$

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*Corresponding author

Recall that a triangular algebra $Tri(A, M, B)$ is an algebra of the form

$$\mathrm{Tri}(A, M, B) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \mid a \in A,\ m \in M,\ b \in B \right\},$$

whose algebra operations are just like $2 \times 2-$matrix operations; where A and B are unital algebras and M is a unital $(A, B)-$bimodule which is faithful as a left $A-$module and right $B-$module. One can easily check that $Tri(A, M, B)$ is isomorphic to the module extension algebra $(A \oplus B) \ltimes M$, where the algebra $A \oplus B$ has its usual pairwise operations and M as an $(A \oplus B)-$module is equipped with the module operations

$$(a, b)m = am \quad \text{and} \quad m(a, b) = mb, \qquad (a \in \mathcal{A}, b \in B, m \in \mathcal{M}).$$

Every triangular algebra $Tri(A, M, B)$ can be identified with the module extension algebra $(A \oplus B) \times M$.
Module extension algebras have been studied by many authors, see [1, 12, 14, 21, 23].
Let R be a commutative ring with identity, A be a unital algebra over R and $Z(A)$ be the center of A. An R-linear map $L : A \to A$ is called semi-centralizing if either
$L(a)a - aL(a) \in Z(A)$ or $L(a)a + aL(a) \in Z(A)$ for all $a \in A$. Furthermore, the map L is called centralizing (resp. skew-centralizing) if $L(a)a - aL(a) \in Z(A)$ (resp. $L(a)a + aL(a) \in Z(A)$) for all $a \in A$. In the special case where $L(a)a - aL(a) = 0$.
Let R be a commutative ring with identity element and A a unital associative (resp. $L(a)a + aL(a) = 0$) for all $a \in A$, the map L is said to be commuting (resp. skew-commuting) R-algebra. For arbitrary elements $a, b \in A$, we set $[a, b]_0 = a$, $[a, b]_1 = ab - ba$. When we treat a semi-centralizing map of an arbitrary algebra, the principal task is to describe its inductively $[a, b]_k = [[a, b]_{k-1}, b]$, where k is a fixed positive integer. Denote by $Z(A)$ the center of A. Define

$$Z(A)_k = \{a \in A | [a, x]_k = 0, \forall x \in A\}.$$

Clearly, $Z(A)_1 = Z(A)$. Each R-linear mapping $L : A \to A$ is said to be k-commuting on A if $[L(a), a]_k = 0$ for all $a \in A$. In particular, an

R-linear mapping $L : A \to A$ is called commuting on A if $[L(a), a] = 0$ for all $a \in A$. Let L be a k-commuting mapping of an R-algebra A. Then L will be called proper if it has the form

$$L(a) = ca + h(a)$$

for all $a \in A$, where $h \in Z(A)$ and $h : A \to Z(A)$ is an R-linear mapping.

Results related to commuting maps on prime or semiprime rings are considered in [4, 5, 15, 16]. Bresar [3] considered Von Neumann algebras, and showed that every commuting map is, according to our definition, proper.

It was Cheung who initiated the study of commuting maps of triangular algebras (e.g., of upper triangular matrix algebras and nest algebras) in [8, 9], where he determined the class of triangular algebras for which every commuting mapping is proper. Xiao and Wei [24] extended Cheung results to the generalized matrix algebra case. They established sufficient conditions for each commuting mapping of a generalized matrix algebra $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ to be proper. Benkovic and Eremita [2] considered commuting traces of bilinear mappings on a triangular algebra $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$. They gave conditions under which every commuting trace of a triangular algebra $\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}$ is proper.

Ebrahimi [11] studied commuting map on module extension algebras. In particular, she gave conditions that every commuting map L on such an algebra is of the form L(c) = cx+h(c), where x lies in the center of the algebra and h is a linear map from the algebra to its center. Du and Wang [13] proved that under certain conditions, each k-commuting map on a triangular algebra is proper. Also Li and Wei [19] extended the main results of Cheung, Du and Wang [9, 13] to the case of generalized matrix algebras.

Brear [6] showed that every skew-commuting map on a 2-torsion free (semi-) prime ring is zero. Posners Theorem [22] states that zero is the only centralizing derivation on a noncommutative prime algebra. Mayne

[20] proved that any centralizing automorphism on a noncommutative prime algebra is trivial, that is, any centralizing automorphism on a noncommutative prime algebra is an identical map. Chung and Luh [10] showed any semi-commuting automorphism on a noncommutative prime algebra is trivial, that is, any semi-commuting automorphism on a noncommutative prime algebra is an identical map. In [15], Hirano et al. jointly obtained that every semi-centralizing derivation on a noncommutative prime algebra is zero. Brear in [7] gave a much more generalization concerning centralizing derivations. He proved that if d and g are a pair of derivations of noncommutative prime algebra A such that $d(a)a - ag(a) \in Z(A)$ for all $a \in A$, then $d = 0$ or $g = 0$. In [17] Li and Wei studied semi-centralizing maps of generalized matrix algebras and describe its general form by routine and complicated computations. They proved that any skew-commuting map on a class of generalized matrix algebras is zero and that any semi-centralizing derivation on a generalized matrix algebra is zero. We introduce the general form of centralizing maps of a module extension algebra and k-commuting mapping of a 2-torsion free module extension algebra and provide a sufficient condition which enables every centralizing maps and k-commuting mapping to be proper.

2. Centralizing Maps on Module Extension Algebras

Let A be an algebra and M be a faithful A -bimodule. Recall the module extension algebra is of the form

$$A \times M = \{(a, m) | a \in A, m \in M\}.$$

Let us define two projections $\pi_A : (a, m) \mapsto a$ and $\pi_M : (a, m) \mapsto m$. Note that the center of $A \times M$ is equal to

$$\begin{aligned} Z(A \times M) &= \{(a, m) : a \in Z(A), [a, n] = [m, b] = 0 \ \forall b \in A, \forall n \in M\} \\ &= \pi_A(Z(A \times M) \times \pi_M(Z(A \times M). \end{aligned}$$

Theorem 2.1. *A centralizing map L on $A \times M$ is of the form*

$$L(a,m) = (L_A(a) + T(m), L_M(a) + S(m)) \tag{1}$$

where $L_A : A \to A$, $T : M \to A$, $L_M : A \to M$, $S : M \to M$ *are linear maps satisfying the following conditions :*

(i) L_A *is centralizing map on* A *;*

(ii) $[L_M(a), a] \in \pi_M(Z(A \times M))$*;*

(iii) $[T(m), m] \in Z(A)$*;*

(iv) $[S(m), a] + [m, L_A(a)] \in \pi_M(Z(A \times M))$.

Proof. Suppose that L is a centralizing map of form (1).
Linearizing $[L(X), X] \in Z(A \times M)$, leads to

$$[L(X), Y] - [X, L(Y)] \in Z(A \times M), \tag{2}$$

for all $X, Y \in A \times M$. For any $a \in A$, taking $X = (0, m)$ and $Y = (a, 0)$ in (2) yields

$$\begin{aligned}[L(X), Y] &= [(T(m), S(m)), (a, 0)] \\ &= ([T(m), a], [S(m), a]),\end{aligned}$$

and

$$\begin{aligned}[X, L(Y)] &= [(0, m), (L_A(m), L_M(m))] \\ &= (0, [m, L_A(a)]),\end{aligned}$$

Thus $[T(m), a] \in \pi_A(Z(A \times M)) \subseteq Z(A)$ and $[S(m), a] + [m, L_A(a)] \in \pi_M(Z(A \times M))$.
Since for all $X \in A \times M$

$$[L(X), X] \in Z(A \times M) \tag{3}$$

We have

$$\begin{aligned}[L(a,0), (a,0)] &= [(L_A(a) + T(m), L_M(a) + S(m)), (a, 0)] \\ &= ([L_A(a), a], [L_M(a), a]) \in Z(A \times M), \end{aligned} \tag{4}$$

Thus $[L_A(a), a] \in \pi_A(Z(A \times M)) \subseteq Z(A)$ and $[L_M(a), a] \in \pi_M(Z(A \times M))$, so L_A is centralizing map on A and (ii) holds.
But by (3), we have

$$[L(0, m), (0, m)] \quad = \quad (0, [T(m), m]) \in Z(A \times M).$$

So we conclude that $[T(m), m] \in \pi_A(Z(A \times M)) \subseteq Z(A)$ and (iii) holds. $\square$

Corollary 2.2. *A commuting map L on $A \times M$ is of the form*

$$L(a, m) = (L_A(a) + T(m), L_M(a) + S(m)),$$

where $L_A : A \to A$, $T : M \to A$, $L_M : A \to M$, $S : M \to M$ are linear maps satisfying the following conditions:

(i)L_A *and* L_M *are commuting maps on A;*

(ii) $[T(m), m] = 0$*;*

(iii) $[a, S(m)] = [L_A(a), m]$*;*

(iv) $T(m) \in Z(A)$*.*

Let p, $q = 1-p$ be two nontrivial idempotents in A. In view of Theorem 2.1, we have the following result.

Theorem 2.3. *Every centralizing map L of $A \times M$ is proper,if the following conditions holds:*

(i) Every centralizing map on A is proper;

(ii) $\pi_A(Z(A \times M)) = Z(A)$*;*

(iii) $pmq = m$ *(for all $m \in M$);*

(iv) $pap = 0$, $qaq = 0$ *(for all $a \in A$);*

(v) $MA = 0$*.*

Let us next describe skew-centralizing maps of module extension algebra $A \times M$ and its general forms.

Theorem 2.4. *A skew-centralizing map L on $A \times M$ is of the form*

$$L(a, m) = (L_A(a) + T(m), L_M(a) + S(m)) \qquad (\clubsuit)$$

where $L_A : A \to A$, $T : M \to A$, $L_M : A \to M$ and $S : M \to M$ are linear maps satisfying the following conditions :

(i) L_A, L_M are skew-centralizing maps on A that is: $L_A(a)a + aL_A(a) \in Z(A)$, $L_M(a)a + aL_M(a) \in Z(A)$;

(ii) $T(m)a + aT(m) \in Z(A)$;

(iii) $aS(m) + S(m)a + L_A(a)m + mL_A(a) \in \pi_M(Z(A \times M))$.

Proof. For computational convenience, we will adopt the notation $L < X, Y >= L(X)Y + XL(Y)$. Let L be a skew-centralizing map of form (♣). Linearizing $< L(X), X >\in Z(A \times M)$, leads to

$$< L(X), Y > + < X, L(Y) >\in Z(A \times M) \tag{5}$$

for all $X, Y \in A \times M$. for any $a \in A$, taking $X = (0, m)$ and $Y = (a, 0)$ in (5) yields

$$\begin{aligned} < L(X), Y > &= < (T(m), S(m)), (a, 0) > \\ &= (< T(m), a >, < S(m), a >) \end{aligned}$$

and

$$\begin{aligned} < X, L(Y) > &= < (0, m), (L_A(m), L_M(m)) > \\ &= (0, < m, L_A(a) >). \end{aligned}$$

Thus $< T(m), a >= T(m)a + aT(m) \in \pi_A(Z(A \times M)) \subseteq Z(A)$ and

$$< S(m), a > + < m, L_A(a) >\in \pi_M(Z(A \times M)).$$

We know that $< L(X), X >\in Z(A \times M)$
for all $X \in A \times M$, thus

$$\begin{aligned} < L(a, 0), (a, 0) > &= < (L_A(a) + T(m), L_M(a) + S(m)), (a, 0) > \\ &= (< L_A(a), a >, < L_M(a), a >) \in Z(A \times M) \end{aligned}$$

which leads to L_A, L_M are skew-centralizing maps on A. So the proof is complete. $\square$

Corollary 2.5. *A skew-centralizing map L on $\boldsymbol{\tau} = Tri(A, M, B)$ is of the form*

$$L\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} g_A(a) + h_B(b) + K_A(m) & S(m) \\ 0 & h_A(a) + L_B(b) + K_B(m) \end{pmatrix},$$

where $g_A : A \to A$ with $j_A = g_A(1_A)$, $g_B : B \to B$, $h_A : A \to Z(B)$ with $j_B = h_A(1_A)$, $h_B : B \to Z(A)$, $k_A : M \to A$, $k_B : M \to B$ are linear mappings satisfying the following conditions :

(i) g_A, g_B are skew-centralizing maps on A and B, respectively;

(ii) $S(m) = -g_A(1)m - mh_A(1) = -h_B(1)m - mg_B(1)$ for all $m \in M$;

(iii) $k_A(m)a + ak_A(m) \in Z(A)$, $h_B(b)a + ah_B(b) \in Z(A)$;

(iv) $k_B(m)b + bk_B(m) \in Z(B)$, $h_A(b)b + bh_A(b) \in Z(B)$.

Proof. Let $A \oplus B$ be the direct sum of A and B as R-algebras. Consider $\boldsymbol{\tau} = Tri(A \oplus B, M)$, so it can be regarded as a module extension algebra. Fix $(a, b) \in A \oplus B$ and $m \in M$. Let L be a skew-centralizing from $\boldsymbol{\tau}$ into itself, then by replacing $A \oplus B$ by A in Theorem 2.4, $L_{A\oplus B}$ is skew-centralizing map on $A \oplus B$, but

$$L_{A\oplus B}(a, b) = (g_A(a) + h_B(b), h_A(a) + g_B(b))$$

and $L_M(a, b) = 0$. Also $T(m) = (k_A(m), k_B(m))$. Replacing $a \oplus b$ by a in Theorem 2.4 (ii) we have

$$\begin{aligned}(k_A(m), k_B(m))(a, b) + (a, b)(k_A(m), k_B(m)) &= (k_A(m)a, k_B(m)b) + (ak_A(m), bk_B(m)) \\ &= (k_A(m)a + ak_A(m), k_B(m)b + bk_B(m)) \\ &\in Z(A \oplus B) = Z(A) \oplus Z(B),\end{aligned}$$

which leads to $k_A(m)a + ak_A(m) \in Z(A)$ and $k_B(m)b + bk_B(m) \in Z(B)$. Also by Theorem 2.4 (i) and replacing $a \oplus b$ by a in it we have

$$\begin{aligned}L_{A\oplus B}(a, b)(a, b) + (a, b)L_{A\oplus B}(a, b) &= (g_A(a) + h_B(b), h_A(a) + g_B(b))(a, b) \\ &\quad + (a, b)(g_A(a) + h_B(b), h_A(a) + g_B(b)) \\ &= (g_A(a)a + h_B(b)a, h_A(a)b + g_B(b)b) \\ &\quad + (ag_A(a) + ah_B(b), bh_A(a) + bg_B(b)) \\ &\in Z(A) \oplus Z(B).\end{aligned}$$

So $g_A(a)a + ag_A(a) \in Z(A)$ and $h_B(b)a + ah_B(b) \in Z(A)$.
Also $h_A(a)b + h_A(a) \in Z(B)$ and $g_B(b)b + bg_B(b) \in Z(B)$. Thus (i),(iii), (iv) holds.

Applying Theorem 2.4 (iii), we have

$$aS(m) + S(m)a + L_A(a)m + mL_A(a) \in \pi_M(Z(A \times M)).$$

Putting (a, b) in place of a and $A \oplus B$ in place of A, we obtain that

$$(a, b)S(m) + S(m)(a, b) + L_{A\oplus B}(a, b)m + mL_{A\oplus B}(a, b) =$$

$$aS(m) + S(m)a + g_A(a)m + h_B(b)m + mh_A(a) + mg_B(b)$$

$$\in \pi_M(Z((A \oplus b) \times M)) = 0.$$

Take $(a, b) = (1, 0)$ in above equality we have $S(m) = -g_A(1)m - mh_A(1)$ and by putting $(a, b) = (0, 1)$ we obtain $S(m) = -h_B(1)m - mg_B(1)$. So condition (ii) is proved. $\square$

3. k-Commuting Maps on Module Extension Algebras

In this section we introduce k-commuting maps of module extension algebras in Theorem 3.1 and then we obtain k-commuting maps of triangular algebra by this theorem that previously was gained by Dua and Wang in [13].

Theorem 3.1. *Let A be a unital algebra with two idempotent p and $q = 1 - p$ that $pmq = m$ (for all $m \in M$), $pap = 0$ and $qaq = 0$ (for all $a \in A$) which M is $A-$ bimodule, then every k-commuting map L on $A \times M$ is of the form*

$$L(a, m) = (L_A(a) + T(m), L_M(a) + S(m)), \qquad (\star)$$

where $L_A : A \to A$, $T : M \to A$, $L_M : A \to M$, $S : M \to M$ are linear maps satisfying the following conditions :

(2) L_M is k-commuting map on A;

(3) $[L_A(1), m] + 2[T(m), m] = 0$;

(4) $[S(m), a] + [L_A(a), m] \in \pi_M(Z(A \times M))_{k-1}$;

(5) $T(m) \in Z(A)_k$.

Proof. Suppose that L is a k-commuting map of form $(\star)$. For any $X \in A \times M$, we have

$$0 = [L(X), X]_k = ([f, a]_k, h_k), \tag{6}$$

where

$$f = L_A + T, h_k = [g, a]_k + [[f, a]_{k-1}, m] + \sum_{j=1}^{k} [[f, a]_j, m, a]_{k-j-1}, \tag{7}$$

and

$$g = L_M + S.$$

Taking $X = (a, 0)$ into (6), an inductive approach gives that

$$\begin{aligned} 0 &= [L(X), X]_k \\ &= ([L_A(a), a]_k, [L_M(a), a]_k). \end{aligned}$$

Thus

$$[L_A(a), a]_k = 0, \tag{8}$$

and

$$[L_M(a), a]_k = 0, \tag{9}$$

for any $a \in A$. So L_A and L_M are k-commuting maps on A. Substituting $a + 1$ for a in (8) we get $[L_A(1), a]_k = 0$ for all $a \in A$. Therefore $L_A(1) \in Z(A)_k$.

Take $X = (1, m)$, let $Y_i = [L(X), X]_i = (X_{iA}, X_{iM})$ for each $1 \leqslant i \leqslant k$. Then

$$\begin{aligned} Y_{i+1} &= (X_{(i+1)A}, X_{(i+1)M}) \\ &= [Y_i, X] \\ &= [(X_{iA}, X_{iM}), (1, m)] \\ &= (0, [X_{iA}, m]). \end{aligned}$$

This implies that $X_{(i+1)A} = 0$ and $X_{(i+1)M} = [X_{iA}, m]$. Also we conclude that $X_{kA} = X_{0A} = 0$. But

$$Y_0 = (L_A(1) + T(m), L_M(1) + S(m)),$$

and

$$\begin{aligned} Y_1 &= [Y_0, X] \\ &= (0, [L_A(1), m] + [T(m), m]). \end{aligned} \tag{10}$$

Using induction computation for each $i \geqslant 0$, we have $X_{iA} = 0$, $X_{iM} = [X_{(i-1)A}, m] = 0$ and $Y_k = (X_{kA}, [X_{kA}, m]) = 0$ which implies that

$$Y_1 = (X_{1A}, [X_{1A}, m]) = 0.$$

So,

$$[L_A(1), m] + [T(m), m] = 0. \tag{11}$$

Likewise, if $X = (1, m)$ we have

$$[T(m), m] = 0. \tag{12}$$

Combining (11) and (12) we have $[L_A(1), m] + 2[T(m), m] = 0$. Using (5) we have

$$0 = [L(a, m), (a, m)]_k = ([f, a]_k, h_k).$$

So $0 = [f, a]_k = [L_A(a), a]_k + [T(m), a]_k = 0$. Since L_A is k-commuting maps, we deduce that $[T(m), a]_k = 0$ and this implies that $T(m) \in Z(A)_k$. By (7),

$$h_i = [h_{i-1}, a] + [[f, a]_{i-1}, m]. \tag{13}$$

Setting $m = 0$ in (13) we get

$$h_i = [h_{i-1}, a] = 0. \tag{14}$$

By induction on (14) we have

$$h_i = [h_1, a]_{i-1}.$$

In view of $h_k = 0$ we infer that

$$h_k = [h_1, a]_{k-1} = 0. \tag{15}$$

Using (5) we have
$h_1 = [h_0, a] + [[f, a]_0, m] = [L_M, a] + [S(m), a] + [L_A(a), m] + [T(m), m] = 0.$
Combining (14) and (15) we have

$$\begin{aligned}0 &= [[L_M, a] + [S(m), a] + [L_A(a), m] + [T(m), m], a]_{k-1} \\ &= [[L_M, a], a]_{k-1} + [[S(m), a], a]_{k-1} + [[L_A(a), m], a]_{k-1} + [[T(m), m], a]_{k-1} \\ &= [L_M, a]_k + [[S(m), a], a]_{k-1} + [L_A(a), m], a]_{k-1} + [[T(m), m], a]_{k-1} \end{aligned} \tag{16}$$

Using (8), $[L_M, a]_k = 0$. By hypothesis $pmq = m$ which implies that $mp = 0$, $pm = p$, $qm = 0$ and $mq = m$. Applying this relations, $pap = 0$ and $qap = 0$ in $[[T(m), m], a]_{k-1}$ we deduce that $[[T(m), m], a]_{k-1} = 0$. Thus by (16), $[[S(m), a], a]_{k-1} + [L_A(a), m], a]_{k-1} = 0$ which implies that

$$[S(m), a] + [L_A(a), m] \in \pi_M(Z(A \times M))_{k-1}.$$

This proves (4). $\square$

Corollary 3.2. *A k-commuting map L on $Tri(A, M, B)$ is of the form*

$$L\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} g_A(a) + h_B(b) + K_A(m) & aj_A - j_B b \\ 0 & h_A(a) + g_B(b) + K_A(m) \end{pmatrix},$$

where $g_A : A \to A$ with $j_A = g_A(1_A)$, $g_B : B \to B$, $h_A : A \to Z(B)$ with $j_B = h_A(1_A)$, $h_B : B \to Z(A)$, $k_A : M \to Z(A)$ and $k_B : M \to Z(B)$ are linear mappings satisfying the following conditions :

(1) g_A is a k-commuting map on A, $g_A(1) \in Z(A)_k$;

(2) g_B is a k-commuting map on B, $g_B(1) \in Z(B)_k$;

(3) $(g_A(1) + h_B(1) + 2k_A(m))m = m(h_A(1) + g_B(1) + 2k_B(m))$;

(4) $2(aj_A - j_Bb) = (g_A(1) - h_B(1))m - m(h_A(1) - g_B(1))$.

Theorem 3.3. *Let p and $q = 1 - p$ be two nontrivial idempotents in A. Then every k-commuting map L of $A \times M$ is proper; that is, L can be written as $L(c) = cx + h(c)$, where $x \in Z(A \times M)$ and h maps $A \times M$ into $Z(A \times M)$, if the following conditions hold:*

(i) Every k-commuting map on A is proper;

(ii) $\pi_A(Z(A \times M)) = Z(A)_k$,

(iii) $pmq = m$ (for all $m \in M$);

(iv) $pap = 0$, $qaq = 0$ (for all $a \in A$);

(v) $MA = 0$.

Proof. Let L be k-commuting map on $A \times M$ and conditions (i) to (iv) hold, we show that L is proper. Using Theorem 3.1 then L_A and L_M are commuting maps on A, and so by (i), L_A is proper. Thus there exist $y \in Z(A)$ and $l : A \to Z(A)$ such that $L_A(c) = cy + h(c)$. Also L_M is proper, so there exist $z \in Z(A)$ and $k : A \to Z(A)$ such that $L_M(c) = cz + k(c)$. Define $X = (y, z) \in Z(A \times M)$ and $h(a, m) = (l(a) + T(m), k(a) + S(m))$. Now we want to show that $h(a, m) \in A \times M$. By Theorem 3.1, $T(m) \in Z(A)_k$ which implies that $T(m) \in \pi_A(Z(A \times M)_k)$. Also $l(a) \in Z(A)_k$, then by using (ii) we have $l(a) \in \pi_A(Z(A \times M)_k)$.
Similarly $k(a) \in Z(A)_k$ leads to $[k(a), b]_k = 0$ for all $b \in A$. Since $pmq = m$, then $S(m) = pS(m)q$.
Also condition (iv) implies that $b = pbq + qbp$. So we have

$$[S(m), b] \quad = \quad [S(pmq), pbq + qbp] = 0$$

and then $h(a,m) \in Z(A\times M)_k$. Using (v),$mx = 0$ that leads to $(a,m)(x,y) = (ax, ay)$. Thus $L(a,m) = (a,m)(x,y)+h(a,m)$ where $(x,y) \in Z(A\times M)_k$ and $h(a,m) \in Z(A \times M)_k$. Then L is proper. $\square$

Corollary 3.4. *Every k-commuting map L of $\boldsymbol{\tau} = Tri(A,M,B)$ of the form*

$$L\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} g_A(a) + h_B(b) + K_A(m) & aj_A - j_B b \\ 0 & h_A(a) + L_B(b) + K_B(m) \end{pmatrix},$$

is proper, if the following conditions hold:

(I) Every k-commuting map on A, B is proper;

(II) $\pi_A(Z(\boldsymbol{\tau})_k) = Z(A)_k$, $\pi_B(Z(\boldsymbol{\tau}))_k) = Z(B)_k$;

(III) $MB = 0$.

Proof. Let L be k-commuting map on $Tri(A,M,B)$, so g_A and g_B are k-commuting maps. By (I) g_A and g_B are proper. Thus we can write $g_A(a) = ay+h_1(a)$ and $g_B(b) = bz+h_2(b)$ where $y \in Z(A)_k$, $h_1 \in Z(A)_k$, $z \in Z(B)_k$, $h_2 \in Z(B)_k$. Define $X = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in Z(\boldsymbol{\tau})_k$ and

$$h(\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}) = \begin{pmatrix} h_1(a) + h_B(b) + k_A(m) & 0 \\ 0 & h_2(b) + h_A(a) + k_B(m) \end{pmatrix}.$$

Using [9, Proposition 4(iii)] $k_A(m) \in Z(A)_k$ and $k_B(m) \in Z(B)_k$. Condition (II) implies that $k_A(m) \in \pi_A(Z(\boldsymbol{\tau})_k)$, $k_B(m) \in \pi_B(Z(\boldsymbol{\tau})_k)$, $h_A(a) \in \pi_B(Z(\boldsymbol{\tau})_k)$, $h_B(b) \in \pi_A(Z(\boldsymbol{\tau})_k)$, $h_1(a) \in \pi_A(Z(\boldsymbol{\tau})_k)$ and $h_2(b) \in \pi_B(Z(\boldsymbol{\tau})_k)$. Thus we conclude that $h(\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}) \in Z(\boldsymbol{\tau})_k$.

Using (III), $mb = 0$ implies that $\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} ax & 0 \\ 0 & by \end{pmatrix}$.

So we have

$$L\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & m \\ 0 & b \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} + h\begin{pmatrix} a & m \\ 0 & b \end{pmatrix},$$

where $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in Z(\boldsymbol{\tau})_k$ and $h(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in Z(\boldsymbol{\tau})_k$. Thus L is proper. $\square$

4. Conclusion

In the present paper, we have shown that under some conditions every semi-centralizing maps of a module extension algebra and k-commuting mappings of a 2-torsion free module extension algebra is proper. According to routine analysis, semi-centralizing automorphisms which quite often associate with semi-centralizing derivations should be at our hand. We can not make any progress in this regards, although the structures and properties of automorphisms of some other matrix algebras are clear, introduction of the structure of semi-centralizing automorphism of module extension algebras is an open problem in this field and can be considered in future research works.

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Sepideh Ebrahimi
Department of Mathematics
Ph.D of Pure Mathematics
Payame Noor University
P. O. Box: 19395-3697
Tehran, Iran
E-mail: seebrahimi2272@gmail.com

Sorayya Talebi
Department of Mathematics
Associate Professor of Mathematics
Payame Noor University
P. O. BOX: 19395-3697
Tehran, Iran
E-mail: talebi_s@pnu.ac.ir