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Original Research Paper

On Submodules of the Set of Rational Numbers

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Abstract. In this note, we completely determine all submodules of the set of rational numbers.

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1 Introduction

The study of the set of rational numbers and its submodules is an interesting subject for mathematicians. In this work, we investigate submodules of the set of rational numbers in some new aspect. First, we recall some basic terminologies and results. We denote the set of rational numbers and integers respectively by $\mathbb{Q}$ and $\mathbb{Z}$. Note that every abelian group can be viewed as a $\mathbb{Z}$-module and so its subgroups are exactly its $\mathbb{Z}$-submodules. We refer the reader to [1] and [2] for undefiend terms and notions. A submodule K of a nonzero $\mathbb{Z}$-module M is said to be essential in M, in case for any nonzero submodule L of M one has

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$K \cap L \neq (0)$. Recall that a $\mathbb{Z}$-module M is called torsion, if for every $x \in M$, there exist a positive integer n such that $nx = 0$.

It is well known that any torsion $\mathbb{Z}$-module can be decomposed into a direct sum of its p-primary components.

Proposition 1.1. *Let M be a nonzero torsion $\mathbb{Z}$-module and P be the set of prime numbers, then $M = \bigoplus_{p \in P} M(p)$, where*

$$M(p) = \{x \in M \mid p^n x = 0 \textit{ for some positive integer } n\}.$$

In the proposition above, $M(p)$ is called the p-primary component of M. For an explicite example, consider the torsion $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$. Then we have $\mathbb{Q}/\mathbb{Z} = \bigoplus_{p \in P} \mathbb{Z}_{p^\infty}$, where

$$\mathbb{Z}_{p^\infty} = \frac{\mathbb{Q}}{\mathbb{Z}}(p) = \{\frac{m}{p^n} + \mathbb{Z} \mid m, n \in \mathbb{Z} \text{ and } n \geq 0\}.$$

We know that for any number p, the proper submodules of $\mathbb{Z}_{p^\infty}$ are cyclic and form a chain

$$H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots,$$

where $H_0 = (0)$ and $H_n = (\frac{1}{p^n} + \mathbb{Z})$, for each $n \geq 1$. Note that for each positive inreger n, we have $H_n \cong \mathbb{Z}/p^n\mathbb{Z} \cong \mathbb{Z}_{p^n}$.

Let $f \colon \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ be the natural epimorphism and p be a prime number, then

$$f^{-1}(\mathbb{Z}_{p^\infty}) = \{\frac{m}{p^n} \mid m, n \in \mathbb{Z} \text{ and } n \geq 0\}$$

and for each positive integer k,

$$f^{-1}(H_k) = \{\frac{m}{p^n} \mid m, n \in \mathbb{Z} \text{ and } 0 \leq n \leq k\}.$$

2 Main Results

Now we are ready to characterize the submodules of $\mathbb{Q}$ elementwise. Let K be a proper submodule of $\mathbb{Q}$ containing $\mathbb{Z}$. Then $K/\mathbb{Z}$ is a proper submodule of $\mathbb{Q}/\mathbb{Z}$ and so $K/\mathbb{Z}$ is a torsion $\mathbb{Z}$-module and we have

$$\frac{K}{\mathbb{Z}} = \bigoplus_{p\in P} \frac{K}{\mathbb{Z}}(p).$$

It is easily seen that $\frac{K}{\mathbb{Z}}(p) \subseteq \frac{\mathbb{Q}}{\mathbb{Z}}(p) = \mathbb{Z}_{p\infty}$. Thus $\frac{K}{\mathbb{Z}}(p) = \mathbb{Z}_{p\infty}$ or for some integer $n \geq 0$, we have $\frac{K}{\mathbb{Z}}(p) = (\frac{1}{p^n} + \mathbb{Z})$. For each prime number p let $g(K,p) = \infty$ if $\frac{K}{\mathbb{Z}}(p) = \mathbb{Z}_{p\infty}$ and $g(K,p) = n$ if $\frac{K}{\mathbb{Z}}(p) = (\frac{1}{p^n} + \mathbb{Z})$ for some integer $n \geq 0$. So,

$$\frac{K}{\mathbb{Z}}(p) = \{\frac{m}{p^n} + \mathbb{Z} \mid m, n \in \mathbb{Z}, 0 \leq n \leq g(K,p)\}.$$

Thus by considering the natural map $f : \mathbb{Q} \to \frac{\mathbb{Q}}{\mathbb{Z}}$ we have

$$K = f^{-1}(\frac{K}{\mathbb{Z}}) = f^{-1}(\bigoplus_{p\in P} \frac{K}{\mathbb{Z}}(p)) = \sum_{p\in P} f^{-1}(\frac{K}{\mathbb{Z}}(p))$$

and so

$$K = \sum_{p\in P} \{\frac{m}{p^n} \mid m, n \in \mathbb{Z}, 0 \leq n \leq g(K,p)\}.$$

Hence

$$K = \{\frac{m}{{p_1}^{n_1}{p_2}^{n_2}\dots{p_r}^{n_r}} \mid m, n_1, n_2, \dots, n_r \in \mathbb{Z}, 0 \leq n_i \leq g(K,p_i), r \geq 1\},$$

where $P = \{p_1, p_2, p_3, \dots\}$ is the set of prime numbers. Therefore, we have poved the following theorem.

Theorem 2.1. *Let K be a submodule of $\mathbb{Q}$ containing $\mathbb{Z}$. Then*

$$K = \{\frac{m}{{p_1}^{n_1}{p_2}^{n_2}\dots{p_r}^{n_r}} \mid m, n_1, n_2, \dots, n_r \in \mathbb{Z}, 0 \leq n_i \leq g(K,p_i), r \geq 1\},$$

where $P = \{p_1, p_2, p_3, \dots\}$ is the set of prime numbers and for any $p \in P$, we have $\frac{K}{\mathbb{Z}}(p) = \{\frac{m}{p^n} + \mathbb{Z} \mid m, n \in \mathbb{Z}, 0 \leq n \leq g(K,p)\}$.

Corollary 2.2. *Let K be a submodule of $\mathbb{Q}$ containing $\mathbb{Z}$. Then K is cyclic if and only if for each prime number p, one has $g(K,p) \neq \infty$ and $\{p \in P \mid g(K,p) \neq 0\}$ is a finite set.*

Note that in the corollary above, if $\{p \in P \mid g(K,p) \neq 0\} = \{q_1, q_2, \dots, q_s\}$, then

$$K = (\frac{1}{{q_1}^{g(K,q_1)}{q_2}^{g(K,q_2)} \dots {q_s}^{g(K,q_s)}}).$$

Now we consider the general case for submodules of $\mathbb{Q}$. Let K be a nonzero submodule of $\mathbb{Q}$. Since $\mathbb{Z}$ is an essential submodule of $\mathbb{Q}$, there exist a positive integer t such that $K \cap \mathbb{Z} = t\mathbb{Z}$. Thus $t\mathbb{Z} \leq K$ and so $\mathbb{Z} \leq t^{-1}K$ and $t^{-1}K$ is a submodule of $\mathbb{Q}$ containing $\mathbb{Z}$, then by Theorem 2.1, we have

$$t^{-1}K = \{\frac{m}{{p_1}^{n_1}{p_2}^{n_2} \dots {p_r}^{n_r}} \mid m, n_1, n_2, \dots, n_r \in \mathbb{Z}, 0 \leq n_i \leq g(t^{-1}K, p_i), r \geq 1\}.$$

So we have the following theorem.

Theorem 2.3. *Let K be a nonzero submodule of $\mathbb{Q}$ such that $K \cap \mathbb{Z} = t\mathbb{Z}$ for some positive integer t. Then*

$$K = \{\frac{tm}{{p_1}^{n_1}{p_2}^{n_2} \dots {p_r}^{n_r}} \mid m, n_1, n_2, \dots, n_r \in \mathbb{Z}, 0 \leq n_i \leq g(t^{-1}K, p_i), r \geq 1\},$$

where $P = \{p_1, p_2, p_3, \dots\}$ is the set of prime numbers and for any $p \in P$, we have $\frac{t^{-1}K}{\mathbb{Z}}(p) = \{\frac{m}{p^n} + \mathbb{Z} \mid m, n \in \mathbb{Z}, 0 \leq n \leq g(t^{-1}K, p)\}$.

Similar to Corollary 2.2, we have the following result.

Corollary 2.4. *Let K be a nonzero submodule of $\mathbb{Q}$ such that $K \cap \mathbb{Z} = t\mathbb{Z}$ for some positive integer t. Then K is cyclic if and only if for each prime number p, one has $g(t^{-1}K, p) \neq \infty$ and $\{p \in P \mid g(t^{-1}K, p) \neq 0\}$ is a finite set.*

Observing that in the above corollary, if $\{p \in P \mid g(t^{-1}K, p) \neq 0\} = \{q_1, q_2, \dots, q_s\}$, then

$$K = (\frac{t}{{q_1}^{g(t^{-1}K,q_1)}{q_2}^{g(t^{-1}K,q_2)} \dots {q_s}^{g(t^{-1}K,q_s)}}).$$

In this case the integer t is relatively prime to each of $q_1, q_2, \dots, q_s$.

References

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[2] L. Fuchs, *Infinite Abelian groups*, vol II, Academic Press, (1970).

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