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Uniqueness Theorem for the Inverse Aftereffect Problem and Representation the Nodal Points Form

A. Neamaty*
University of Mazandaran

Sh. Akbarpoor
University of Mazandaran

A. Dabbaghian
Islamic Azad University of Neka

Abstract. In this paper, we consider a boundary value problem with aftereffect on a finite interval. Then, the asymptotic behavior of the solutions, eigenvalues, the nodal points and the associated nodal length are studied. We also calculate the numerical values of the nodal points and the nodal length. Finally, we prove the uniqueness theorem for the inverse aftereffect problem by applying any dense subset of the nodal points.

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1. Introduction

In this work, we consider the equation

$$-y''(x) + q(x)y(x) + \int_0^x M(x-t)y(t)dt = \lambda y(x), \quad 0 \leqslant x \leqslant \pi, \tag{1}$$

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*Corresponding author

under the separated boundary conditions

$$U(y) := y'(0) - hy(0) = 0, \tag{2}$$

$$V(y) := y'(\pi) + Hy(\pi) = 0, \tag{3}$$

where $\lambda = \rho^2$ and $\rho = \sigma + i\tau$ is the spectral parameter and also $q, M \in W^{2,1}(0, \pi)$ are real functions. We denote the boundary value problem (1)-(3) by $L(q, M, h, H)$.

In fact, in this work, we consider the Sturm-Liouville operator disorganized by a Volterra integral operator. Uniqueness in the inverse aftereffect problem is the problem of chek the uniqueness of the function M. In this paper, we suppose that the function q is the known function and prove the uniqueness theorem for the solution of the inverse problem i.e. M.

Many authors studied the uniqueness of the inverse boundary value problem for the Sturm-Liouville equations lately (see [1, 3, 7, 9, 16]) but a few of them considered it for the differential equations with aftereffect (for example see [6]). In this paper, we obtain the nodal points and the associated nodal length and investigate the uniqueness of inverse aftereffect problem $L(q, M, h, H)$ with the separated boundary conditions by using any dense subset of the nodal points. Proof of the uniqueness and computation of the nodal points was studied for Sturm-Liouville equations in [2, 4, 8, 10, 11, 12, 13, 15, 17, 18] and other works but it was not considered for the differential equations with aftereffect.

The form of the differential equation with aftereffect without the computation of the nodal points were perused. Fereiling and Yurko In [6], considered the equation (1) under the Dirichlet boundary conditions, obtained the eigenvalues and proved the uniqueness theorem by using the transformation operator method. In [5], we studied the equation (1) under the separated boundary conditions on a finite interval with discontinuity conditions in an interior point and proved the uniqueness theorem by using the nodal points but we did not obtain the numerical values of the nodal points. Whereas in this paper, we consider the differential equation (1) under the boundary conditions (1)-(3) but without discontinuity conditions in an interior point and obtain the numerical values of the nodal points and the nodal length and prove the uniqueness theorem by applying any dense subset of the nodal points. In section 2, we obtain the asymptotic form of the solution, the characteristic function, the eigenvalues, the numerical values of the nodal points and the nodal length and present a uniqueness theorem for the solution of the inverse aftereffect problem.

2. Main Results

First, we present some definitions that will be applied in this paper.

Definition 2.1. ([6]) *The values of the parameter λ for which L has nonzero solutions are called eigenvalues and the corresponding nontrivial solutions are called eigenfunctions. The set of eigenvalues is called the spectrum of L.*

Definition 2.2. ([14]) *If $f(z)$ and $g(z)$, two functions of a complex number z, which may be a parameter of the problem or an independent variable defined on some domain D, possess limits as $z \to z_0$ in D, then we say that $f(z) = O(g(z))$ as $z \to z_0$ if there exist positive constants K and δ such that $|f| \leqslant K|g|$ whenever $0 < |z - z_0| < \delta$ and if $|f| \leqslant K|g|$ for all z in D, we say $f(z) = O(g(z))$.*

Definition 2.3. ([14]) *If $f(z)$ and $g(z)$ are such that, for any $\varepsilon > 0$, $|f| \leqslant \varepsilon|g|$ whenever z is in a small $\delta-$neighborhood of z_0, we say $f(z) = o(g(z))$ as $z \to z_0$.*

Definition 2.4. ([14]) *A finite or infinite sequence of functions $\phi_n(z), n = 1, 2, ...$ is an asymptotic sequence as $z \to z_0$ if, for all n, $\phi_{n+1}(z) = o(\phi_n(z))$ as $z \to z_0$, that is, $\lim_{z\to z_0} \frac{\phi_{n+1}}{\phi_n} = 0$.*

Definition 2.5. ([14]) *If $\phi_n(z), n = 1, 2, ...$ is an asymptotic sequence of functions as $z \to z_0$, we say that $\sum_{n=1} a_n\phi_n(z)$, where the a_n are constants (with the upper limit omitted), is an asymptotic expansion or asymptotic approximation of the function $f(z)$ if for each N*

$$f(z) = \sum_{n=1}^{N} a_n\phi_n(z) + o(\phi_n(z)), \quad as \quad z \to z_0.$$

2.1 The asymptotic form of the solution and the eigenvalues

Let $\varphi(x, \rho)$ be solutions of (1) under the initial conditions $\varphi(0, \rho) = 1$ and $\acute{\varphi}(0, \rho) = h$. In this case, the functions $\varphi(x, \rho)$ satisfies the following integral equations (see

$$\varphi(x, \rho) = \cos\rho x + h\frac{\sin\rho x}{\rho} + \int_0^x \frac{\sin\rho(x-t)}{\rho}$$

$$\times \left(q(t)\varphi(t, \rho) + \int_0^t M(t-s)\varphi(s, \rho)ds\right)dt, \tag{4}$$

and hence

$$\varphi'(x,\rho) = -\rho\sin\rho x + h\cos\rho x + \int_0^x \cos\rho(x-t)$$

$$\times\left(q(t)\varphi(t,\rho) + \int_0^t M(t-s)\varphi(s,\rho)ds\right)dt. \tag{5}$$

Lemma 2.1.1. *For $|\rho|\to\infty$, the formula*

$$\varphi(x,\rho) = \cos\rho x + O\left(\frac{1}{|\rho|}e^{|\tau|x}\right) = O\left(e^{|\tau|x}\right), \tag{6}$$

holds, uniformly with to $x\in[0,\pi]$ as $\tau = Im\rho$.

Proof. See [6]. $\square$

Substituting (6) into (4) and (5), we get

$$\varphi(x,\rho) = \cos\rho x + q_1(x)\frac{\sin\rho x}{\rho} + \frac{1}{2\rho}\int_0^x q(t)\sin\rho(x-2t)dt$$

$$+\frac{1}{\rho}\int_0^x \sin\rho(x-t)\int_0^t M(t-s)\cos\rho s ds dt + O\left(\frac{1}{|\rho|^2}e^{|\tau|x}\right), \tag{7}$$

and also

$$\varphi'(x,\rho) = -\rho\sin\rho x + q_1(x)\cos\rho x + \frac{1}{2}\int_0^x q(t)\cos\rho(x-2t)dt$$

$$+\int_0^x \cos\rho(x-t)\int_0^t M(t-s)\cos\rho s ds dt + O\left(\frac{1}{|\rho|}e^{|\tau|x}\right), \tag{8}$$

where

$$q_1(x) = h + \frac{1}{2}\int_0^x q(t)dt.$$

Integration by parts results

$$\int_0^x q(t)\sin\rho(x-2t)dt = -\frac{1}{2\rho}q(x)\cos\rho x + \frac{1}{2\rho}q(0)\cos\rho x$$

$$+\frac{1}{2\rho}\int_0^x q'(t)\cos\rho(x-2t)dt, \tag{9}$$

and

$$\int_0^x \sin\rho(x-t)\int_0^t M(t-s)\cos\rho s ds dt = \frac{1}{\rho}\int_0^x M(x-s)\cos\rho s ds$$

$$-\frac{1}{\rho}\int_0^x \cos\rho(x-t)\left(M(0)\cos\rho t + \int_0^t \frac{\partial M}{\partial t}(t-s)\cos\rho s ds\right)dt. \tag{10}$$

We obtain from (7)-(10) that

$$\varphi(x,\rho) = \cos\rho x + q_1(x)\frac{\sin\rho x}{\rho} + O\left(\frac{1}{|\rho|^2}e^{|\tau|x}\right), \tag{11}$$

and

$$\varphi'(x,\rho) = -\rho\sin\rho x + q_1(x)\cos\rho x + O\left(\frac{1}{|\rho|}e^{|\tau|x}\right). \tag{12}$$

Substituting (11) into (4) and applying integration by parts, we get

$$\varphi(x,\rho) = \cos\rho x + q_1(x)\frac{\sin\rho x}{\rho} - \frac{1}{4\rho^2}q(x)\cos\rho x + \frac{1}{4\rho^2}q(0)\cos\rho x$$

$$-\frac{\cos\rho x}{2\rho^2}\int_0^x q(t)q_1(t)dt - \frac{M(0)}{2\rho^2}x\cos\rho x + O\left(\frac{1}{|\rho|^3}e^{|\tau|x}\right), \tag{13}$$

similarly, using (12) and (5), we get

$$\varphi'(x,\rho) = -\rho\sin\rho x + q_1(x)\cos\rho x + \frac{3}{4\rho}q(x)\sin\rho x - \frac{1}{4\rho}q(0)\sin\rho x$$

$$+\frac{\sin\rho x}{2\rho}\int_0^x q(t)q_1(t)dt + \frac{M(0)}{2\rho}x\sin\rho x + O\left(\frac{1}{|\rho|^2}e^{|\tau|x}\right). \tag{14}$$

Let $\varphi(x,\rho)$ and $\psi(x,\rho)$ be solutions of (1) under the initial conditions $\varphi(0,\rho) = 1$, $\varphi'(0,\rho) = h$, $\psi(\pi,\rho) = 1$, $\psi'(\pi,\rho) = -H$. Denote

$$\Delta(\rho) := \langle\psi(x,\rho),\varphi(x,\rho)\rangle, \tag{15}$$

where

$$\langle \psi(x,\rho), \varphi(x,\rho) \rangle = \psi(x,\rho)\varphi'(x,\rho) - \psi'(x,\rho)\varphi(x,\rho),$$

is the Wronskian of $\psi(x,\rho)$ and $\varphi(x,\rho)$. Since the function $\Delta(\rho)$ called the characteristic function for the boundary value problem $L(q, M, h, H)$ does not depend on x, hence, substituting $x = \pi$ into (15), we get

$$\Delta(\rho) = V(\varphi) = \varphi'(\pi,\rho) + H\varphi(\pi,\rho). \tag{16}$$

Lemma 2.1.2. *For $|\rho| \to \infty$, the representation*

$$\Delta(\rho) = -\rho \sin\rho\pi + \omega\cos\rho\pi + \frac{3}{4\rho} q(\pi)\sin\rho\pi - \frac{1}{4\rho} q(0)\sin\rho\pi$$

$$+\frac{\sin\rho\pi}{2\rho}\int_0^\pi q(t)q_1(t)dt + \frac{M(0)}{2\rho}\pi\sin\rho\pi$$

$$+Hq_1(\pi)\frac{\sin\rho\pi}{\rho} + O\left(\frac{1}{|\rho|^2}e^{|\tau|\pi}\right), \tag{17}$$

holds where $\omega = H + q_1(\pi)$.

Proof. we arrive at (17) from (13), (14) and (16) by using straightforward calculations. □

Since the eigenvalues $\{\rho_n\}_{n\geqslant 1}$ of the boundary value problem coincide with the zeros of the function $\Delta(\rho)$, using some straightforward calculations ([6]), we obtain

$$\rho_n = n + \frac{\omega}{n\pi} + O\left(\frac{1}{n^2}\right), \quad n \to \infty. \tag{18}$$

2.2 The asymptotic form of the nodal points and the associated nodal length

Denote

$$\varphi_n(x) = \varphi(x, \rho_n),$$

where φ_n is the eigenfunction corresponding to the eigenvalue λ_n. Let $\lambda_0 < \lambda_1 < \ldots \to \infty$ be the eigenvalues of the aftereffect problem (1)-(3) and also let the nodal points of the nth eigenfunction φ_n be shown with $0 < x_n^1 < x_n^2 < \cdots < x_n^j < \pi$, $j = 1, 2, ..., n-1$. The set of all nodal points $\{x_n^j\}_{n>1, j=\overline{1,n-1}}$ is dense in $[0, \pi]$ (see [10]).

Theorem 2.2.1. *The nodal points of the aftereffect problem (1)-(3) are*

$$x_n^j = (j - \frac{1}{2})\frac{\pi}{n} + \frac{1}{n^2} q_1(x_n^j) + \frac{1}{2n^2}\int_0^{x_n^j} q(t)\cos 2nt dt$$

$$+\frac{1}{n^2}\int_0^{x_n^j}\int_0^t M(t-s)\cos nt \cos ns ds dt + O\left(\frac{1}{n^3}\right), \tag{19}$$

and the nodal length is

$$l_n^j = \frac{\pi}{n} + \frac{1}{2n^2}\int_{x_n^j}^{x_n^{j+1}} q(t)dt + \frac{1}{2n^2}\int_{x_n^j}^{x_n^{j+1}} q(t)\cos 2nt dt$$

$$+\frac{1}{n^2}\int_{x_n^j}^{x_n^{j+1}}\int_0^t M(t-s)\cos nt \cos ns ds dt + O\left(\frac{1}{n^3}\right). \tag{20}$$

Proof. Since the nodal points are the zeroes of the eigenfunctions, then we get from (4) that

$$\varphi_n(x) = \cos \rho_n x + h\frac{\sin \rho_n x}{\rho_n} + \int_0^x \frac{\sin \rho_n(x-t)}{\rho_n}$$

$$\times \left(q(t)\varphi_n(t) + \int_0^t M(t-s)\varphi_n(s)ds\right) dt$$

$$= \cos \rho_n x + h\frac{\sin \rho_n x}{\rho_n} + \frac{\sin \rho_n(x)}{\rho_n}\int_0^x \cos \rho_n(t)$$

$$\times \left(q(t)\varphi_n(t) + \int_0^t M(t-s)\varphi_n(s)ds\right) dt$$

$$-\frac{\cos \rho_n(x)}{\rho_n}\int_0^x \sin \rho_n(t)\left(q(t)\varphi_n(t) + \int_0^t M(t-s)\varphi_n(s)ds\right) dt.$$

Now, we set $\varphi_n(x) = 0$. Thus, we obtain

$$\cot \rho_n x + \frac{h}{\rho_n} + \frac{1}{\rho_n}\int_0^x \cos \rho_n(t)\left(q(t)\varphi_n(t) + \int_0^t M(t-s)\varphi_n(s)ds\right)dt$$

$$-\frac{\cot \rho_n(x)}{\rho_n}\int_0^x \sin \rho_n(t)\left(q(t)\varphi_n(t) + \int_0^t M(t-s)\varphi_n(s)ds\right)dt = 0,$$

then, for $n \to \infty$, we get

$$x_n^j = (j - \frac{1}{2})\frac{\pi}{\rho_n} + \frac{h}{\rho_n^2} + \frac{1}{\rho_n^2}\int_0^{x_n^j} \cos \rho_n(t) q(t)\varphi_n(t)dt$$

$$+\frac{1}{\rho_n^2}\int_0^{x_n^j} \cos \rho_n(t)\int_0^t M(t-s)\varphi_n(s)dsdt.$$

From (7) and (18), we obtain

$$x_n^j = (j - \frac{1}{2})\frac{\pi}{n} + \frac{1}{n^2}q_1(x_n^j) + \frac{1}{2n^2}\int_0^{x_n^j} q(t)\cos 2nt dt$$

$$+\frac{1}{n^2}\int_0^{x_n^j}\int_0^t M(t-s)\cos nt \cos ns ds dt + O\left(\frac{1}{n^3}\right).$$

Also, the nodal length is

$$l_n^j = x_n^{j+1} - x_n^j.$$

Therefore

$$l_n^j = \frac{\pi}{n} + \frac{1}{2n^2}\int_{x_n^j}^{x_n^{j+1}} q(t)dt + \frac{1}{2n^2}\int_{x_n^j}^{x_n^{j+1}} q(t)\cos 2nt dt$$

$$+\frac{1}{n^2}\int_{x_n^j}^{x_n^{j+1}}\int_0^t M(t-s)\cos nt \cos ns ds dt + O\left(\frac{1}{n^3}\right),$$

and consequently, the theorem is proved. □

Example 2.2.2. Let h=1 and q(x)=M(x)=x. Using (19) and (20), we obtain the numerical values of the nodal points and the nodal length (see Table 1 and Table 2).

Proof. Consider the problems

$$-y''(x) + q(x)y(x) + \int_0^x M(x-t)y(t)dt = \rho^2 y(x), \quad 0 \leqslant x \leqslant \pi, \tag{21}$$

$$y'(0) - hy(0) = 0, \qquad y'(\pi) + Hy(\pi) = 0, \tag{22}$$

and

$$-y''(x) + q(x)y(x) + \int_0^x \tilde{M}(x-t)y(t)dt = \tilde{\rho}^2 y(x), \quad 0 \leqslant x \leqslant \pi, \tag{23}$$

$$y'(0) - \tilde{h}y(0) = 0, \qquad y'(\pi) + \tilde{H}y(\pi) = 0. \tag{24}$$

Let $\varphi_n(x)$ and $\tilde{\varphi}_n(x)$ be the solutions of (21) and (23) under the initial conditions $\varphi_n(0,\rho) = 1$, $\varphi_n'(0,\rho) = h$ and $\tilde{\varphi}_n(0,\rho) = 1$, $\tilde{\varphi}_n'(0,\rho) = \tilde{h}$, respectively. Let $x_n^j = \tilde{x}_n^j$, for $n > 1$ and $j = 1, 2, ..., n-1$, be a dense set in $[0,\pi]$ (see [10]). Then from (21) and (23), we obtain

$$[\tilde{\varphi}_n'(x)\varphi_n(x) - \tilde{\varphi}_n(x)\varphi_n'(x)]' = \int_0^x [\tilde{M}(x-t)\varphi_n(x)\tilde{\varphi}_n(t)$$

$$-M(x-t)\varphi_n(t)\tilde{\varphi}_n(x)]dt + (\rho_n^2 - \tilde{\rho}_n^2)\varphi_n\tilde{\varphi}_n. \tag{25}$$

Integrating (25) from 0 to x_n^j and using (22) and (24), we get

$$(h - \tilde{h})\varphi_n(0)\tilde{\varphi}_n(0) = \int_0^{x_n^j} \int_0^x [\tilde{M}(x-t)\varphi_n(x)\tilde{\varphi}_n(t)$$

$$-M(x-t)\varphi_n(t)\tilde{\varphi}_n(x)]dtdx + (\rho_n^2 - \tilde{\rho}_n^2) \int_0^{x_n^j} \varphi_n(x)\tilde{\varphi}_n(x)dx. \tag{26}$$

Now, we select a subsequence of the nodal points from the dense set that tends to 0, then from (26), we get $h = \tilde{h}$. Integrating both sides of (25) from x_n^j to π, we obtain

$$(H - \tilde{H})\varphi_n(\pi)\tilde{\varphi}_n(\pi) = \int_{x_n^j}^{\pi} \int_0^x [\tilde{M}(x-t)\varphi_n(x)\tilde{\varphi}_n(t)$$

$$-M(x-t)\varphi_n(t)\tilde{\varphi}_n(x)]dtdx + (\rho_n^2 - \tilde{\rho}_n^2) \int_{x_n^j}^{\pi} \varphi_n(x)\tilde{\varphi}_n(x)dx.$$

We select a subsequence of the nodal points that tends to π. Thus, we get $H = \tilde{H}$. Consequently $\rho_n = \tilde{\rho_n}$. Now, Integrating (28) from 0 to x_n^j, we obtain

$$\int_0^{x_n^j} \int_0^x \left[\tilde{M}(x-t)\varphi_n(x)\tilde{\varphi}_n(t) - M(x-t)\varphi_n(t)\tilde{\varphi}_n(x) \right] dtdx = 0.$$

We take a sequence $\{x_n^j\}_{n>1, j=\overline{1,n-1}}$ accumulating at an arbitrary $b \in [0, \pi]$. Then, using (6) for $n \to \infty$, we get

$$\int_0^b \int_0^x \left[\tilde{M}(x-t) - M(x-t) \right] \cos nx \cos nt dtdx = 0.$$

Consequently, from the completeness of the function cosine, we can conclude that M is uniquely determined on $[0, \pi]$. $\square$

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Abdol Ali Neamaty
Department of Mathematics
Associate Professor of Mathematics
University of Mazandaran
Babolsar, Iran
E-mail: Namaty@umz.ac.ir

Shahrbanoo Akbarpoor
Department of Applied Mathematics
Ph.D Student of Mathematics
University of Mazandaran
Babolsar, Iran
E-mail: Sh.akbarpoor@stu.umz.ac.ir

Abdolhadi Dabbaghian
Department of Mathematics
Assistant Professor of Mathematics
Neka Branch, Islamic Azad University
Neka, Iran
E-mail: A.dabbaghian@iauneka.ac.ir