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Original Research Paper

Krasner H_v-homology Functors

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Abstract. We develop a structural theory for Krasner H_v-modules over a Krasner H_v-ring R and introduce the derived functors Txt_n^R and txt_n^R, and formulate a number of properties. The conditions we obtained allow us to apply results to sheaves and complexes.

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1 Introduction

We will be using the natural generalized notions of groups, namely (canonical) hypergroup "or multigroups", (canonical) H_v-group; among others. We start by reviewing those terms.

Let S be a non-empty set equipped with a multivalued binary operation, denoted by $\odot$, such that $x \odot y$ is a subset of S, for any two

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elements x and y of S. That is, $\odot$ is a map from $S \times S \to P^*(S)$, where $P^*(S)$ is the set of all non-empty subsets of S. For any x in S and any subset A, B of S, we consider x as a singleton set, and extend $\odot$ as $A \odot B = \bigcup\{a \odot b : a \in A, b \in B\}$. In other words, $\odot$ is a map from $\mathcal{P}^*(S) \times \mathcal{P}^*(S) \to \mathcal{P}^*(S)$. The operation $\odot$ is called a hyper-operation (or a hyper-composition).

Definition 1.1. (Marty [11, 13]) A *hypergroup* is a pair $(S, \odot)$, where S is a set, $\odot$ is a hyperoperation on S and the following axioms hold:

(a) (The associative law) $(x \odot y) \odot z = x \odot (y \odot z)$ for all $x, y, z \in S$, i.e., the sets on both sides are equal, equivalently,

$$\bigcup_{a \in x \odot y} a \odot z \;=\; x \odot \bigcup_{b \in y \odot z} b.$$

(b) (The reproduction law) $x \odot S = S = S \odot x$ for all $x \in S$. This is equivalent to the condition: for all $x, y \in S$, there exist $a, b \in S$ such that: $y \in x \odot a$ and $y \in b \odot x$.

Here we note that a hyperoperation is not necessarily associative.

Definition 1.2. A hypergroup $(S, \odot)$ is said to be *commutative* if $x \odot y = y \odot x$, for all $x, y \in S$.

Definition 1.3. (See Mittas [14]) A commutative hypergroup $(S, \odot)$ is said to be *canonical hypergroup* if the following axioms hold:

(a) (Existence of a scaler identity) There exist an element $e \in G$ with the property that $e \odot x = x = x \odot e$ for all $x \in H$.

(b) (The existence of an inverse) For each $x \in H$ there exists a unique element $x^{-1} \in H$ such that $e \in x \odot x^{-1}$ and $e \in x^{-1} \odot x$.

(c) (The reversible law) For every $x, y, z \in H$, if $x \in y \odot z$ then $y \in x \odot z^{-1}$ and $z \in y^{-1} \odot x$.

Definition 1.4. (See Vougiouklis [22]) An H_v-group is a pair $(S, \odot)$, where S is a set, $\odot$ is a hyperoperation on S and the following axioms hold:

(a) (The weak associative law) $((x \odot y) \odot z) \cap (x \odot (y \odot z)) \neq \phi$, for all $x, y, z \in S$.

(b) (The reproduction law) $x \odot S = S = S \odot x$ for all $x \in S$.

Moreover, if an H_v-group $(S, \odot)$ satisfies the conditions (a)-(c) as for the commutative hypergroup, and $x \odot y \cap y \odot x \neq \phi$, then we say that S is a *canonical H_v-group*.

The algebraic hyperstructures theory has its origin in Marty's paper [11] in 1934, in which the theory was initiated by generalizing the group axioms leading to the notion of hypergroups. He proved among other results that, the quotient of a group any subgroup is a hypergroup. Marty wrote two sequels to this paper ([12],[13]) where he developed a structure theory of hyperstructures by studying subhypergroups and applied their properties in algebraic functions and rational fractions. He also showed that, a necessary and sufficient for a hyperquotient group with respect to a subgroup to be a group is being invariant of the subgroup?. One should also mention the contribution of Wall, Kuntzmann, Ore, Griffiths, Krasner, Dresher [23, 9, 10, 17, 8, 4], around 1937, to the theory of hyperstructures. The construction of hypergroup has been used to generalize other algebraic structures, and the structural theory then seems to be of interest in its own rights with many useful properties and numerous applications to a variety of problems in science.

A natural question that arises is what can we say about the relation between the theories of classical structures and hyperstructures?

Based on Marty works, Koskas [5] answered this question positively by showing that hypergroups is in fact a generalization of groups and partitioning a hypergroup G into equivalence classes with respect to the equivalence relation β^* such that G/β^* is a group, where the relation β is reflexive and symmetric but not transitive in general, and β^* is fundamental relation. Particulary, for a hypergroup G and a set $\mathcal{U}$ of all finite products of elements in G, the relation β^* is defined as the transitive closure of β where $x\beta y$ if and only if there exist some u in $\mathcal{U}$ such that $\{x, y\} \subseteq u$. Freni [3] has shown that the relation β is transitive on hypergroups, i.e., $\beta = \beta^*$. We however still do not know if this property holds for H_v-groups.

In this article we continue developing a structure theory for Krasner H_v-modules. Before we discuss the contents of the paper we give a short overview.

H_v-structures, where initiated by Vougiouklis [22] as a new class of hyperstructures.

In particular, Vougiouklis considered a weaker associative hyperoperation by replacing the axiom's equality with the non-empty intersection. It follows that, we adopt the weak axioms such as weak distributivity and weak commutativity.

In recent years there have been much works in this area.

Since the first relation β of Koskas, various relations of a similar nature have been extensively studied (For instance see [1, 3, 22]). The smallest of these relations are called Fundamental and denoted by $\beta^*, \gamma^*, \varepsilon^*$. Thus, if H is an H_v-ring (H_v-module over H_v-ring), then the quotient $H/\gamma^*(\varepsilon^*)$ is a ring (module over the ring R/γ^*). An H_v-structure H is called proper if $H \neq H/\sim$, where $\sim$ is the associated fundamental relation. According to [21] the fundamental relation ε^* on an H_v-module can be defined as follows:

Consider a left H_v-module M over an H_v-ring R. If $\mathcal{U}$ denotes the set of all expressions consisting of finite hyperoperations of either on R and M or of the external hyperoperations applying on finite sets of elements of R and M. A relation ε can be defined on M whose transitive closure is the fundamental relation ε^*. Indeed, for every $x, y \in M$ we have

$$x\varepsilon y \Longleftrightarrow \{x, y\} \subseteq u, \text{ for some } u \in \mathcal{U}.$$

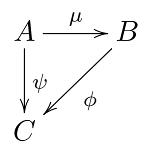
Suppose that $\gamma^*(r)$ is the equivalence class containing $r \in R$ and $\varepsilon^*(x)$ is the equivalence class containing $x \in M$. On M/ε^* the $\oplus$ and the external product $\odot$ using the γ^* classes in R are defined as follows:

$$\varepsilon^*(x) \oplus \varepsilon^*(y) = \varepsilon^*(c), \text{ for every } c \in \varepsilon^*(x) + \varepsilon^*(y),$$

$$\gamma^*(r) \odot \varepsilon^*(x) = \varepsilon^*(d), \text{ for every } d \in \gamma^*(r) \cdot \varepsilon^*(x),$$

where $x, y \in M$ and $r \in R$. For any H_v-modules M, N over a H_v-ring R, with the fundamental relations $\varepsilon^*_M, \varepsilon^*_N$ and ε^* on M, N and $M \times N$ respectively, clearly we have that $(x_1, x_2)\ \varepsilon^*(y_1, y_2)$ if and only if $x_1\varepsilon^*_M y_1$ and $x_2\varepsilon^*_N y_2$, for all $(x_1, x_2), (y_1, y_2) \in M \times N$ [6, 21]. In this paper we will adopt the *weakly equal* notion. Thus, the non- empty subsets X and Y of an H_v-module M are weakly equal ($X \overset{w}{=} Y$) if for every $x \in X$, there exists $y \in Y$ such that $\varepsilon^*_M(x) = \varepsilon^*_M(y)$ and for every $y \in Y$ there exists $x \in X$ such that $\varepsilon^*_M(x) = \varepsilon^*_M(y)$. Clearly this notion can be naturally extended to weak equal maps and weak commutative diagrams of H_v-modules. Many of the H_v-structure terms that we use in this paper are analogous to the corresponding terms for related classical structures and hyperstructures. Let R be an H_v-ring, a map $\phi\colon\ M \to N$ of H_v-modules M and N is said to be *star R-homomorphism* if $f(x \oplus y) \overset{w}{=} f(x) \oplus f(y)$ and $f(r \odot x) \overset{w}{=} r \odot f(x)$, that is, $\varepsilon^*_N(f(x+y)) = \varepsilon^*_N(f(x) + f(y))$ and $\varepsilon^*_N(f(rx)) = \varepsilon^*_N(rf(x)))$ for every $x, y \in M$ and every $r \in R$.

H_v-modules, star homomorphisms and their compositions form a category denoted as $\mathcal{HM}$, in which triangle diagrams are weak commutative, where for $A, B, C \in Obj(\mathcal{HM}))$, and $\mu, \psi, \phi \in Hom_{\mathcal{HM}}$, we can take $\psi \overset{w}{=} \phi \cdot \mu$



The subject of this article is a functor in a special class of H_v-modules, which requires the notion of exact sequences. In [2], the authors have defined exact sequences based on the fundamental relations above, strong homomorphism and the notion of weak equality. A sequence

$$M_o \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} M_{n-1} \xrightarrow{f_n} M_n$$

of H_v-modules $M_0, \ldots, M_n$ and star homomorphisms $f_i\colon M_i \to M_{i+1}$, $1 \leq i \leq n$, is said to be weakly exact at M_i if $Im\ (f_{i-1}) \overset{w}{=} Ker\ (f_i)$.

It is called exact if it is weakly exact at $M_1, \ldots, M_{n-1}$.

The outline of the paper is as follows. In Section 2, we characterize star projective Krasner H_v-modules as well as describe some general results that in particular lead to specific type of unitary Krasner H_v-modules over a Krasner H_v-field with identity that we call star free Krasner H_v-module. All Krasner H_v-vector spaces have basis and are in fact a star free Krasner H_v-modules. We will also describe how to construct star free modules. Moreover, we prove that if a start free Krasner H_v-module has an infinite basis, all its bases are infinite and have the same cardinality. Finally, we consider a Krasner H_v-module and apply $M[-]$ and $[-]M$ to its deleted resolution, for some certain Krasner H_v-module M. In Section 3, we shall describe the resulting Krasner H_v-homology on Krasner H_v-modules and define the left derived functors Txt_n^R and txt_n^R, where n is a positive integer and R is a Krasner H_v-ring.

We will see the derived functors Txt_n^R and txt_n^R are leading to a number of useful properties.

Many other Krasner H_v-structures terms that we use in this paper are analogous to the corresponding terms for related hyperstructures.

A triple $(R, \oplus, \cdot)$ is called a *Krasner H_v-ring* if $(R, \oplus)$ is a canonical H_v-group, $(R, \cdot)$ is a semigroup with the property that 0 is an absorbing element, that is, $x \cdot 0 = 0 = 0 \cdot x$ for all x in R, and $\cdot$ is weak distributive with respect to $\oplus$. In the following we introduce the notion of a (left) H_v-module over a Krasner H_v-ring.

Definition 1.5. Let $(R, +, \cdot)$ be a Krasner H_v-ring. A (left) *Krasner H_v-module over R* is the triple $(M, \oplus, \odot)$, where M is a set equipped with two hyperoperations $\oplus$ and $\odot$ called a hyperaddition and a hypermultiplication, respectively, such that the following properties hold:

(a) $(M, \oplus)$ is a canonical H_v-group.

(b) For all $a, b \in R$, $x \in M$:

1. $(a \cdot b) \odot x \ \cap \ a \odot (b \odot x) \neq \phi$.
2. $0_R \odot x = 0_M$.

(c) The hyperaddition $\oplus$ is weak distributive with respect to the hypermultiplication $\odot$: For all $a, b \in R$ and $x, y \in M$ we have

1. $a \odot (y \oplus z) \ \cap \ (a \odot y \oplus a \odot z) \neq \phi$.
2. $(a + b) \odot x \ \cap \ (a \odot x \oplus b \odot x) \neq \phi$.

Fundamental relations on canonical Krasner H_v-groups, Krasner H_v-rings, and Krasner H_v-modules are defined similarly to corresponding H_v-hyperstructures. Hence, a map $f \colon M_1 \longrightarrow M_2$ of Krasner H_v-modules over a Krasner H_v-ring R is called a *Krasner star homomorphism* if it is star homomorphism of Krasner H_v-modules. We also have that $(V, \oplus, \cdot)$ is a Krasner H_v-field if it is H_v-ring and F/γ^* is a field.

Throughout this article, the term Krasner H_v^R-module stands for a Krasner H_v-module over a Krasner H_v-rings R unless otherwise stated. The category of left Krasner H_v^R-modules with Krasner homomorphisms is denoted by $\mathcal{KM}$.

2 Homological Tools

In this section we develop some structural theory of star free Krasner H_v-modules, and give some preliminaries on complexs and Krasner H_v-homology functors. First we move onto describing star projectivity as a preparation of Krasner H_v-homology.

Following the standard definitions of the star projective [20] of H_v-modules, we will now concentrate on star projective Krasner H_v-modules, where the strong homomorphism and the unit ω_N of the group $(N/\epsilon^*, \oplus)$ are substituted by star homomorphism and $\{0\}$, respectively. Hence, a Krasner H_v-module P is *star projective* if given a Krasner star homomorphism $f\colon P \to N$, and surjective Krasner star homomorphism $g\colon M \to N$, there exists a Krasner star homomorphism $h\colon P \to M$ (not necessarily unique) such that $f \overset{w}{=} gh$. For the dual notion, as usual we reverse all arrows in the mapping diagram that defines a start projective Krasner H_v-module.

Before characterizing star projectivity, we will derive some preliminary facts.

Proposition 2.1. *A direct sum of Krasner H_v-modules is star projective if and only if each summand is star projective.*

Proof. Follows easily from the natural projection map π of $P = \bigoplus_{\lambda\in\Lambda} P_\lambda$ onto P_λ, and the natural injection map ι of P_λ into P. □

Theorem 2.2. *Let P be a Krasner H_v-modules. The following are equivalent:*

(1) *P is a star projective.*

(2) *Every short weakly exact sequence of Krasner H_v-modules split.*

Proof. (1) $\Rightarrow$ (2) Consider the diagram

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow{\scriptstyle id} & & \\ B & \xrightarrow{g} & P & \longrightarrow & 0 \end{array}$$

of Krasner star homomorphisms with bottom row exact. Since P is star projective, it follows that there exists a Krasner star homomorphism $h : P \longrightarrow B$ such that $gh \overset{w}{=} 1_P$. Therefore, the short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} P \longrightarrow 0$$

is split exact and $B \overset{w}{=} A \bigoplus P$.

(2) $\Rightarrow$ (1) We want to show that P is a star projective H_v-module. So, we show that for every diagram

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow{\scriptstyle f} & & \\ X & \xrightarrow{g} & Y & \longrightarrow & 0 \end{array}$$

of Krasner H_v-modules and Krasner star homomorphism such that bottom row is exact, there is a Krasner star homomorphism $\varphi : P \longrightarrow X$ such that $g\varphi \overset{w}{=} f$. Now, we take the mapping $h : X \longrightarrow P$ by $h(x) \in f^{-1}\big(g(x)\big)$ for every $x \in X$. Then, we have the exact sequence

$$0 \longrightarrow \ker h \longrightarrow X \xrightarrow{h} P \longrightarrow 0\ .$$

Since this exact sequence is split, it follows that there exists a Krasner star homomorphism $\psi : P \longrightarrow X$ such that $h\psi \overset{w}{=} 1_p$. Therefore $f(h\psi) \overset{w}{=} f1_p \overset{w}{=} f$. Then $(fh)\psi \overset{w}{=} f$. Therefore, $g\psi \overset{w}{=} f$. This yields that P is star projective. $\square$

Example 2.3. Let R be a Krasner H_v-ring such that $x = x^2$, for some $x \in R$. This implies that $\varepsilon^*(x) = \varepsilon^*(x) \odot \varepsilon^*(x)$. Moreover, Rx/ε^* is a Krasner H_v-module over the fundamental ring R/γ^*. Now, by Lemma 1.1 in [16], we conclude that Rx/ε^* is a projective module over R/γ^*. This yields that Rx is a Krasner star projective module over R.

Example 2.4. (1) Notice that every projective module can be considered as a star projective H_v-module. Let $R = \mathbb{Z}_6$, which is clearly a free module over R. Take $A = \{0,\ 2,\ 4\}$ and $B = \{0,\ 3\}$. Then, A and B are both R-submodule of R, and $R = A \oplus B$. Since $A \cong \mathbb{Z}_3$ and $B \cong \mathbb{Z}_2$, it follows that $\mathbb{Z}_2$ as well as $\mathbb{Z}_3$ are projective $\mathbb{Z}_6$-modules. Obviously, they are not $\mathbb{Z}_6$-modules.

(2) Consider $R = \mathbb{Z}_6$ as an H_v-ring and let $M = \{0,\ 1,\ 2\}$ together with the following hyperoperations:

$$\begin{array}{c|ccc} * & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 2 \\ 1 & 1 & \{0,\ 2\} & 1 \\ 2 & 2 & 1 & 0 \end{array} \quad \text{and} \quad \cdot_M : \underset{(r,m)\mapsto\ 0}{R \times M \to M}$$

Since $\{0,\ 2\} \subseteq 1 * 1$, $\{1\} \subseteq \{1\}$, $0 * 0 = 0$ and $r \cdot m = 0$ for every $r \in R$ and every $m \in M$, we obtain $\varepsilon^*(0) = \varepsilon^*(2) = \{0, 2\}$ and $\varepsilon^*(1) = \{1\}\}$. Thus,

$$M/\varepsilon^* = \Big\{\{0, 2\},\ \{1\}\Big\}.$$

Consequently, we have $M/\varepsilon^* \cong \mathbb{Z}_2$. Now, by part (1), M/ε^* is a non-free star projective $\mathbb{Z}_6$-module, and so M is a non-free star projective H_v-module.

Now, we turn to the subcategory of unitary left Krasner H_v-modules. Before identifying a special class, call star free whose structure is shown to be very restricted, we derive some standard terminology.

A subset X of a Krasner H_v-R-module M is said to be *linearly independent* over R if and only if

$$\gamma^*(r_1)\varepsilon^*(x_1) \oplus \gamma^*(r_2)\varepsilon^*(x_2) \oplus \cdots \oplus \gamma^*(r_n)\varepsilon^*(x_n) = 0 \Longrightarrow \gamma^*(r_i) = 0,$$

where $r_i \in R, x_i \in X, i = 1, 2, \ldots, n$. Moreover, if R has an identity and M is unitary, we say that X *spans* M if and only if every element of m can be written as a finite linear combination of elements of $\varepsilon^*(X)$ with coefficients in $\gamma^*(R)$, i.e., $m \overset{w}{=} \gamma^*(r_1)\varepsilon^*(x_1) \oplus \gamma^*(r_2)\varepsilon^*(x_2) \oplus \cdots \oplus \gamma^*(r_n)\varepsilon^*(x_n)$. A linearly independent subset of M that spans M is called a *basis* of M.

Proposition 2.5. *Let R be a Krasner H_v-field with identity and let F be a unitary Krasner H_v-module. The following conditions are equivalent:*

(1) *F has a non-empty basis.*

(2) *F is an internal weak direct sum of a family of cyclic Krasner H_v-modules, each of which is Krasner star isomorphic to R.*

(3) *F is Krasner star isomorphic to a star direct sum of copies of R.*

(4) *Let X be non-empty set, and let M be any unitary Krasner H_v-module and $f\colon X \longrightarrow M$ be any map. Then f extends uniquely to a Krasner star homomorphism $\bar{f}\colon F \longrightarrow M$ such that the diagram is weak commutative, that is, $\bar{f}\iota \overset{w}{=} f$.*

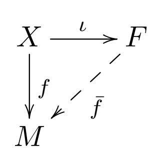
Proof. (1) $\Rightarrow$ (2) Let X be a basis of F and $x \in X$. Clearly, the map $R \longrightarrow Rx, r \longmapsto rx$, is a Krasner star monomorphism, since $\gamma^*(r)\varepsilon^*(x) \neq 0$, whenever $\gamma^*(r) \neq 0$. Hence, the result follows from $R \overset{w}{=} Rx$ as left Krasner H_v-modules.

(2) $\Rightarrow$ (3) It is straightforward.

(3) $\Rightarrow$ (1) Suppose that $F \overset{w}{=} \oplus R$, where the copies of R are indexed by a set X. For each $x \in X$, let θ_x be the element $\{r_i\}$ of $\oplus R$, such that $r_i \overset{w}{=} 0$ for $i \neq x$ and $r_x \overset{w}{=} 1_R$. Then $\{\theta_x \mid x \in X\}$ is a basis and the result follows.

(1) $\Rightarrow$ (4) Let X be a basis of F, $\iota : X \longrightarrow F$ be the inclusion map and $f\colon X \longrightarrow M$ be a map. For any $u \in F$, $u \in r_1x_1 + r_1x_2 + \cdots + r_nx_n$, for some $r_i \in R$ and $x_i \in X$ (Notice that if $u \in \sum\limits_{i=1}^{n} s_ix_i$, for some $s_i \in R$, then $u \in \sum\limits_{i}(r_i - s_i)x_i \overset{w}{=} 0$, and it follows that for every i, $r_i \overset{w}{=} s_i$). Therefore, the well defined map $\bar{f}\colon F \longrightarrow M$ given by

$$\bar{f}(u) \overset{w}{=} \bar{f}(\sum_{i=1}^{n} r_ix_i) \overset{w}{=} \sum_{i=1}^{n} f(r_ix_i),$$



where $\bar{f}\iota \overset{w}{=} f$ is the required star homomorphism map. It is unique since it can be determined by its action on the generators X of F. Indeed, if $g\colon F \longrightarrow M$ is a Krasner H_v-module star homomorphism such that $g\iota \overset{w}{=} f$, then for every $x \in X$, $\varepsilon^*(g(x)) = \varepsilon^*(g(\iota(x))) = \varepsilon^*(f(x)) = \varepsilon^*(\overline{f}(x)))$ and so $g \overset{w}{=} \overline{f}$.

(4) $\Rightarrow$ (3) Consider the map $\iota\colon X \to F$ and for each $x \in X$, construct the direct sum $\oplus R$ with copies of R. Let $Y = \{\theta_x \colon x \in X\}$ be the basis of the (unitary) Krasner H_v-module $\oplus R$. From the above argument, one can see that $\oplus R$ is a star free object on the set Y in the category $\mathcal{KM}$ of Krasner H_v-modules, with $Y \longrightarrow \oplus R$ being the inclusion map. As $|X| = |Y|$, $\varepsilon^*(f(\iota(X))) = \varepsilon^*(Y)$ and hence $F \overset{w}{=} \oplus R$. □

We call such F, satisfying one of the equivalent conditions of Proposition 2.5, a *star free Krasner H_v-module on the set X*. The last condition in Proposition 2.5 states that F is a star free object in the subcategory of unitary left Krasner H_v modules.

Remark 2.6. As usual any star free is star projective. But the converse is not true.

Remark 2.7. As an immediate consequence of Proposition 2.5 we see that every (unitary) Krasner H_v-module M over R (with identity) is the image of the star homomorphism of a star free Krasner H_v-module F. Furthermore if M is finitely generated, then F may be chosen to be finitely generated. Indeed, if X is a generating set of M and F the free Krasner H_v-module on the set X. Then it follows that the inclusion map $X \longrightarrow M$ induces a Krasner star homomorphism $\bar{f}\colon F \longrightarrow M$ such that $X \subseteq Im\bar{f}$. Therefore $Im\bar{f} \overset{w}{=} M$.

From the proof of Proposition 2.5 we know how to construct a star free Krasner H_v-module F over R with identity on any non-empty set X, where a typical element of $F \overset{w}{=} \bigoplus_{x\in X} Rx$ has the form

$$\gamma^*(r_1)\varepsilon^*(x_1) \oplus \cdots \oplus \gamma^*(r_n)\varepsilon^*(x_n),$$

for some $r_i \in R, x_i \in X$. We now focus on the Krasner H_v-vector spaces over a Krasner H_v-field R. It turns out that every such space is in fact a star free Krasner H_v-module.

Lemma 2.8. *Let V be a Krasner H_v-vector space over a Krasner H_v-field F. Let X be maximal linearly independent subset of V over F. Then X is a basis for V.*

Proof. Let W be the subspace of V spanned by X. Suppose that $a \in V$ is a non-zero element such that $a \notin W$ and consider the set $X \cup \{a\}$. If

$$\gamma^*(r)\varepsilon^*(a) + \gamma^*(r_1)\varepsilon^*(x_1) + \cdots + \gamma^*(r_n)\varepsilon^*(x_n) = 0,$$

for some $\gamma^*(r) \neq 0$, and $r, r_i \in R, x_i \in X$, then

$$\varepsilon^*(a) = -(\gamma^*(r)^{-1}\gamma^*(r_1)\varepsilon^*(x_1) \oplus \cdots \oplus \gamma^*(r)^{-1}\gamma^*(r_n)\varepsilon^*(x_n)),$$

and we get the contradiction $a \in W$. Hence we must have $\gamma^*(r) = 0$, and so $\gamma^*(r_i) = 0$ for all i. This however contradicts the maximality of X. Therefore, $W = V$ and X is a basis. □

Corollary 2.9. *Let V be a Krasner H_v-vector space over a Krasner H_v-field F. Every such space V has a basis and is therefore a star free Krasner H_v-module. In particular any linearly independent set is contained in a basis.*

Next we prove that, analogously to the case of free R-modules, if F is star free with an infinite basis X, then any other basis Y for F has the same cardinality.

Theorem 2.10. *Let R be a Krasner H_v-ring with identity and let F be a star free Krasner H_v-module over R with an infinite basis X. Then any basis for F has the same cardinality as X.*

Proof. Let Y be another basis of F. Assume that Y has finite cardinality. Then every element of Y is a linear combination of finitely many elements of X, and thus there is a finite subset $\{x_1, \ldots, x_m\}$ of X that generates F. Thus any $x \in X \setminus \{x_1, \ldots, x_m\}$ is a linear combination of $x_1, \ldots, x_m$, and so

$$\varepsilon^*(x) = \gamma^*(r_1)\varepsilon^*(x_1) + \cdots + \gamma^*(r_m)\varepsilon^*(x_m),$$

for some $r_i \in R$, which contradicts the linear independence of X. Therefore, Y has infinite cardinality. Now let Γ_Y be the set of all finite subsets of Y. Define a map $\psi\colon X \longrightarrow \Gamma_Y$ by $x \mapsto \{y_1, \ldots, y_n\}$, where $\varepsilon^*(x) = \gamma^*(r_1)\varepsilon^*(y_1) \oplus \cdots \oplus \gamma^*(r_n)\varepsilon^*(y_n)$ and $\gamma^*(r_i) \neq 0$ for all i. An easy argument shows that the set $\psi^{-1}(Z)$ is a finite subset, for every $Z \in Im\ \psi$. This follows that $\bigcup_{Z\in Im\ \psi} \psi^{-1}(Z)$ partitions X and therefore

$$|X| = |\sum_{Z\in Im\ \psi} \psi^{-1}(Z)| \leq |Im\ \psi| \times |N| \leq |\Gamma_Y| \times |N| = |\Gamma_Y| = |Y|.$$

□

Remark 2.11. The presentation of a Krasner H_v-module M is defined analogously by the pair (X, Y), where F is a star free Krasner H_v-module mapping onto M, X is a basis of F, and Y generates $K = \ker(F \to M)$. A presentation allows us to treat equations in M as if they were equations in the star free Krasner H_v-module F.

For the remainder of this section we introduce some preliminaries on H_v-homology functors. We first introduce the nth Krasner H_v-homology for chain complexes as well as some other interesting results on Krasner H_v-module following [18]. As usual, a *complex* of Krasner H_v-modules $(\mathcal{M}_\bullet, d_\bullet)$ (or simply $\mathcal{M}_\bullet$) is a sequence of star morphisms

$$\cdots \longrightarrow M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \xrightarrow{d_{n-1}} \cdots$$

such that for all $n \in Z$, $d_n \cdot d_{n+1} \overset{w}{=} 0$.

Definition 2.12. The Krasner H_v-homology of a complex $(\mathcal{M}_\bullet, d_\bullet)$ in $\mathcal{C}_\bullet(\mathcal{KM})$ is the sequence $H(\mathcal{M}_\bullet, d_\bullet)$, where

$$H_n(\mathcal{M}_\bullet, d_\bullet) = \frac{ker\,(d_n\colon M_n \to M_{n-1})}{im\,(d_{n+1}\colon M_{n+1} \to M_n)}.$$

As usual, elements of $ker\,(d_n\colon M_n \to M_{n-1})$ are called n-cycles of $\mathcal{M}_\bullet$, elements of $im\,(d_{n+1}\colon M_{n+1} \to M_n)$ are called n-boundaries of $\mathcal{M}_\bullet$. The Krasner H_v-module of n-cycles is denoted by $Z_n(\mathcal{M}_\bullet, d_\bullet)$, and the subspace of n-boundaries is denoted by $B_n(\mathcal{M}_\bullet, d_\bullet)$. If all differentials are 0, then $H_n(\mathcal{M}_\bullet, d_\bullet) = Z_n(\mathcal{M}_\bullet)/B_n(\mathcal{M}_\bullet) \overset{w}{=} \ker d_n \overset{w}{=} M_n$. That is, for any $m \in H_n(\mathcal{M}_\bullet, d_\bullet)$, there exists $m^{'} \in M_n$ such that $\epsilon^*(m) = \epsilon^*(m^{'})$, and for every $m^{'} \in M_n$, there exists $m \in H_n(\mathcal{M}_\bullet, d_\bullet)$ such that $\epsilon^*(m) = \epsilon^*(m^{'})$.

Furthermore, we say that $\psi_\bullet : (\mathcal{M}_\bullet, d_\bullet) \longrightarrow (\mathcal{M}'_\bullet, d'_\bullet)$, where $\psi_n : M_n \longrightarrow M'_n$ is a sequence of star homomorphisms and such that for all $n \in Z$, the diagrams are weak *star morphism of complexes.*

$$\begin{array}{ccc} M_{n+1} & \xrightarrow{d_{n+1}} & M_n \\ \psi_{n+1}\downarrow & & \downarrow\psi_n \\ M'_{n+1} & \xrightarrow[d'_{n+1}]{} & M'_n \end{array}$$

The category of complexes over $\mathcal{KM}$ is denoted by $\mathcal{C}_\bullet(\mathcal{KM})$, where direct sum of complexes is defined coordinate-wise, and inverse limits and their dual, direct limits exist in $\mathcal{C}_\bullet(\mathcal{KM})$. As for any weakly exact complex $(\mathcal{M}_\bullet, d_\bullet)$ we have that the quotient of $ker\, d_n$ by $im\, d_{n+1}$ is 0, we next turn onto measuring the deviation of a complex in $\mathcal{C}_\bullet(\mathcal{KM})$ from being a weakly exact sequence.

Let $f : (\mathcal{M}_\bullet, d_\bullet) \to (\mathcal{M}_\bullet{}', d'_\bullet)$ be a chain map, let $H_n(f) : H_n(\mathcal{M}_\bullet) \to H_n(\mathcal{M}_\bullet{}')$, $cls(z_n) := z_n + B_n(\mathcal{M}_\bullet) \mapsto cls(f_n z_n) := f_n(z_n) + B_n(\mathcal{M}_\bullet{}')$ be the induced map, and consider the following weak commutative diagram

$$\begin{array}{ccccc} M_{n+1} & \xrightarrow{d_{n+1}} & M_n & \xrightarrow{d_n} & M_{n-1} \\ f_{n+1}\downarrow & & \downarrow f_n & & \downarrow f_{n-1} \\ M'_{n+1} & \xrightarrow[d'_{n+1}]{} & M'_n & \xrightarrow[d'_n]{} & M'_{n-1} \end{array}$$

Let $z \in Z_n(\mathcal{M}_\bullet)$ be an n-cycle with $d_n z \overset{w}{=} 0$. Then clearly $f_n z$ is an n-cycle as $d'_n f_n z \overset{w}{=} f_{n-1} d_n z \overset{w}{=} 0$. We also have that if $z + B_n(\mathcal{M}_\bullet) \overset{w}{=} y + B_n(\mathcal{M}_\bullet)$, then there exists $c \in M_{n+1}$ such that $z - y \overset{w}{=} d_{n+1}c$. Therefore, $f_n z - f_n y \overset{w}{=} f_n d_{n+1} c \overset{w}{=} d^{n+1} f_{n+1} c \subseteq B_n(\mathcal{M}_\bullet{}')$ and so $H_n(f)$ is well-defined.

Now for any chain maps f, g where gf is defined and any $z \in Z_n(\mathcal{M}_\bullet)$

$$H_n(gf)(z) \overset{w}{=} (gf)_n(z) + B_n(z) \overset{w}{=} H_n(g)H_n(f)(z),$$
$$H_n(f+g)(z) \overset{w}{=} (f_n + g_n)(z) \overset{w}{=} (H_n(f) + H_n(g))(z).$$

As $H_n(1_{\mathcal{M}_\bullet})$ is the identity, an additive functor is obtained, as follows.

Proposition 2.13. *$H_n\colon \mathcal{C}_\bullet(\mathcal{KM}) \longrightarrow \mathcal{KM}, \mathcal{M}_\bullet \mapsto H_n(\mathcal{M}_\bullet)$ is an additive functor.*

Clearly, if $\mathcal{KM}$ is the category of all sheaves of Krasner H_v-modules over a space X, then $H_n(\mathcal{M}_\bullet)$ is a sheaf. We next see the relation between different Krasner H_v-homologies. Now, consider the weak commutative diagram below with a weakly exact rows.

Fix $n \in \mathbb{Z}$. Then the diagram is a well-defined star homomorphism since $\delta_n(cls(z^{''})) \overset{w}{=} cls(c^{'})$ where $z^{''} \in M_n^{''}$ and $m^{'} \in M_{n-1}^{'}$ with $i_{n-1}m^{'} \overset{w}{=} dc$ such that i_{n-1} is a weak monic. Moreover, it is not difficult to see that the map $Z_n^{''} \to M_{n-1}^{'}/B_{n-1}^{'}$ is a well-defined star homomorphism.

$$\begin{array}{ccccccccc}
0 & \to & M_{n+1}^{'} & \xrightarrow{i_{n+1}} & M_{n+1} & \xrightarrow{p_{n+1}} & M_{n+1}^{''} & \to & 0 \\
 & & \downarrow{\scriptstyle d_{n+1}^{'}} & & \downarrow{\scriptstyle d_{n+1}} & & \downarrow{\scriptstyle d_{n+1}^{''}} & & \\
0 & \to & M_n^{'} & \xrightarrow{i_n} & M_n & \underset{p_n}{\xrightarrow{\curvearrowleft}} & M_n^{''} & \to & 0 \\
 & & \downarrow{\scriptstyle d_n^{'}} & \curvearrowleft & \downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle d_n^{''}} & & \\
0 & \to & M_{n-1}^{'} & \underset{i_{n-1}}{\xrightarrow{\curvearrowleft}} & M_{n-1} & \xrightarrow{p_{n-1}} & M_{n-1}^{''} & \to & 0
\end{array}$$

We next see that there exists a weakly exact sequence in $\mathcal{KM}$ and determine a connecting star homomorphisms

$$H_n(\mathcal{M}_\bullet{}^{'}) \longrightarrow H_n(\mathcal{M}_\bullet) \longrightarrow H_n(\mathcal{M}_\bullet{}^{''}) \xrightarrow{\delta_n} H_{n-1}(\mathcal{M}_\bullet{}^{'})$$

To see this, notice first that $Z^{''} \to Z^{'}/B^{'} = H_{n-1}$ is a star homomorphism for if $i_{n-1}m^{'} \overset{w}{=} dc$, for some $m^{'} \in M_{n-1}^{'}$, then $ddc \overset{w}{=} dic^{'} \overset{w}{=} idc^{'} \overset{w}{=} 0$, and as i is a weak monic, $d^{'}c^{'} \overset{w}{=} 0$. Finally, the canonical H_v-subgroup $B_n^{''}$ goes into $B_{n-1}^{'}$. Suppose that $z^{''} \overset{w}{=} d^{''}c^{''}$, where $c^{''} \in C_{n+1}^{''}$, and let $pu \overset{w}{=} c^{''}$, where $u \in C_{n+1}$. Weak commutativity gives $pdu \overset{w}{=} d^{''}pu \overset{w}{=} d^{''}c^{''} \overset{w}{=} z^{''}$. Since $\delta(z^{''})$ is well-defined, we choose du with $pdu \overset{w}{=} z^{''}$, and so

$$\delta(cls(z^{''})) \overset{w}{=} cls(i^{-1}d(du)) = 0.$$

This follows that we have a star homomorphism $\delta_n\colon H_n(C^{''}) \to H_{n-1}(C^{'})$. Thus we have arrived at

Proposition 2.14. *Let* $0 \to \mathcal{M_\bullet}' \xrightarrow{i} \mathcal{M}_\bullet \xrightarrow{P} \mathcal{M_\bullet}'' \to 0$ *be a weakly exact sequence in* $\mathcal{C}_\bullet(\mathcal{KM})$*. Then there is a long weakly exact sequence*

$$\longrightarrow H_{n+1}(\mathcal{M_\bullet}'') \xrightarrow{\delta_{n+1}} H_n(\mathcal{M_\bullet}') \xrightarrow{H_n(i)} H_n(\mathcal{M}_\bullet) \xrightarrow{H_n(p)} H_n(\mathcal{M_\bullet}'') \xrightarrow{\delta_n} H_{n-1}(\mathcal{M_\bullet}') \longrightarrow \ ,$$

where for each $n \in \mathbb{Z}$*,*

$$\delta_n\colon H_n(\mathcal{M_\bullet}'') \to H_{n-1}(\mathcal{M_\bullet}'),\ cls(z_n^{''}) \mapsto cls(i_{n-1}^{-1} d_n p_n^{-1} z_n^{''}),$$

is a star morphism in $\mathcal{KM}$*.*

Proof. We have already seen that δ_n is a star morphism. Straightforward calculations show that any such sequence is a long exact sequence. □ Next we see the naturality of the connecting morphisms. In particular, consider a weak commutative diagram in the category of complexes $\mathcal{M}_\bullet(\mathcal{KM})$ over the category of left Krasner H_v-modules with weakly exact rows.

Notice that if $cls(z^{''}) \subseteq H_n(\mathcal{M_\bullet}'')$, then $f_*\delta\, cls(z^{''}) \overset{w}{=} \delta' h_*\, cls(z^{''})$. Indeed, if $c \in M_n$ as a lifting of $z^{''}$, then $\delta\, cls(z^{''}) \overset{w}{=} cls(z^{'})$, where $iz^{'} \overset{w}{=} dc$ and so $f_*\delta\, cls(z^{''}) \overset{w}{=} cls\, cls(fz^{'})$. Conversely as

$$qgc \overset{w}{=} hpc \overset{w}{=} hz^{''},\ \delta^{'} cls(hz^{''}) \overset{w}{=} cls(u^{'})$$

where gc is the lifting of $hz^{''}$ and $ju^{'} \overset{w}{=} \delta gc$.

Now as $jfz^{'} \overset{w}{=} giz^{'} \overset{w}{=} gdc \overset{w}{=} \delta gc \overset{w}{=} ju^{'}$, it follows that $fz^{'} \overset{w}{=} u^{'}$.

$$\begin{array}{ccccccccc}
0 \to \mathcal{M_\bullet}' & \xrightarrow{i} & \mathcal{M}_\bullet & \xrightarrow{p} & \mathcal{M_\bullet}'' \to 0 & \quad & H_n(\mathcal{M_\bullet}'') & \xrightarrow{\delta} & H_{n-1}(\mathcal{M_\bullet}') \\
\downarrow f & & \downarrow g & & \downarrow h & & \downarrow H_n(h) & & \downarrow H_n(f) \\
0 \to \mathcal{N_\bullet}' & \xrightarrow{j} & \mathcal{N}_\bullet & \xrightarrow{q} & \mathcal{N_\bullet}'' \to 0 & \quad & H_n(\mathcal{N_\bullet}'') & \xrightarrow{\delta'} & H_{n-1}(\mathcal{N_\bullet}')
\end{array}$$

Therefore, the diagram involving the star connecting homomorphism is a weak commutative and as a consequence we arrive at

$$\begin{array}{ccccccccc} \longrightarrow & H_n(\mathcal{M}_\bullet{}') & \xrightarrow{H_n(i)} & H_n(\mathcal{M}_\bullet) & \xrightarrow{H_n(p)} & H_n(\mathcal{M}_\bullet{}'') & \xrightarrow{\delta} & H_{n-1}(\mathcal{M}_\bullet{}') & \longrightarrow \\ & \downarrow{\scriptstyle H_n(f)} & & \downarrow{\scriptstyle H_n(g)} & & \downarrow{\scriptstyle H_n(h)} & & \downarrow{\scriptstyle H_n(f)} & \\ \longrightarrow & H_n(\mathcal{N}_\bullet{}') & \xrightarrow[H_n(j)]{} & H_n(\mathcal{N}_\bullet) & \xrightarrow[H_n(q)]{} & H_n(\mathcal{N}_\bullet{}'') & \xrightarrow[\delta']{} & H_{n-1}(\mathcal{N}_\bullet{}') & \longrightarrow \end{array}$$

Remark 2.15. Homotopic chain maps induce the same star morphism in Krasner H_v-homology, for if $f, g\colon (\mathcal{M}_\bullet, d_\bullet) \longrightarrow (\mathcal{M}_\bullet{}', d'_\bullet)$ are chain maps and $f \cong g$, then for all n,

$$f_{*n} \overset{w}{=} g_{*n}\colon H_n(\mathcal{M}_\bullet) \longrightarrow H_n(\mathcal{M}_\bullet{}').$$

This is because if z is an n-cycle, then $d_n z \overset{w}{=} 0$ and $f_n z - gnz \overset{w}{=} d'_{n+1} s_n z + s_{n-1} d_n z \overset{w}{=} d'_{n+1} s_n z$, and hence $f_{*n} \overset{w}{=} g_{*n}$.

Next we show among others that if $\mathcal{KM}$ has star projective, then there must exist a star projective resolution for any object from $\mathcal{KM}$.

For a Krasner H_v-module M, a star free resolution of M is defined analogously by a weakly exact sequence

$$P = \cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

where each F_i is a star free Krasner H_v-submodule. Moreover, P is said to be a *star projective resolution* of $M \in obj(\mathcal{KM})$ if each P_i is star projective. We also define the nth syzygy of P by $K_i = ker(P_i \to P_{i-1})$.

Remark 2.16. (1) Notice that as Krasner H_v-submodule of a star free Krasner H_v-module need not be a star free, we naturally think of generators and relations: We map a free Krasner H_v-module F_1 onto K, and let (X_1, Y_1) be a presentation of K; that is, X_1 is a basis of F_1 and Y_1 generates $K_1 = \ker(F_1 \to K)$. If K_1 is star free, we stop; otherwise, we continue with a presentation of it. Hence, a star free resolution of a Krasner H_v-modules M is a generalized presentation.

(2) Every left Krasner H_v-module M has a star free resolution. In order to see this, suppose that F_0 is a star free Krasner H_v-module and let $0 \to K_1 \xrightarrow{i_1} F_0 \xrightarrow{\epsilon} M \to 0$ be a weakly exact sequence. Then we arrive at

$$\begin{array}{ccccccc} & & F_1 & \xrightarrow{d_1} & F_0 & \xrightarrow{\epsilon} & M \to 0 \\ & \nearrow & & \searrow^{\epsilon_1} & \uparrow{\scriptstyle i_1} & & \\ 0 \to K_2 & & & & K_1 & & \end{array}$$

where F_1 is a star free Krasner H_v-module with a weak surjective $\epsilon_1 \colon F_1 \to K_1$, and a weakly exact sequence. From this it follow that $d_1 \colon F_1 \to F_0$ with $d_1 = i_1\epsilon_1$ such that $im\, d_1 \overset{w}{=} K_1 \overset{w}{=} \ker \epsilon$ and $\ker d_1 \overset{w}{=} K_2$, yielding the weakly exact row.

3 Left Derived Functors

In this section we use the machinery that we have developed to define derived functors and prove some useful properties.

From our previous work we have that Krasner H_v-module has many presentations. In fact like in H_v-modules, it is not difficult to see the following:

Let $f \colon M \longrightarrow M^{'}$ be a star homomorphism in $\mathcal{KM}$, and P_n be a star projective in the diagram with complex rows and exact row bottom. Then there exists a chain map $f^{'}$ such that the completed diagram is a weak commutative. Moreover, any two such chain maps are homotopic.

$$\begin{array}{ccccccccc} \to P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\epsilon} & M & \to 0 \\ {\scriptstyle f_2'}\downarrow & & \downarrow{\scriptstyle f_1'} & & \downarrow{\scriptstyle f_0'} & & \downarrow{\scriptstyle f} & \\ \to P_2' & \xrightarrow{d_2'} & P_1' & \xrightarrow{d_1'} & P_0' & \xrightarrow{\epsilon'} & M & \to 0 \end{array}$$

Now, consider the singular complex $S_\bullet(X)$ of a Krasner H_v-topological space X and we get $\to M[S_{n+1}(X)] \xrightarrow{f'_{n+1}} M[S_n(X)] \xrightarrow{f'_n} M[S_{n-1}(X)] \to$ by applying the functor $M[-]$ for an H_v-module M. The Krasner H_v-homology H_v-module $H_n(M, X) = H_n(M[S_\bullet(X)])$ is called the *Krasner H_v-homology* of X with coefficients in M. Consider the additive functors $M[-]$ and $[-]M$ between Krasner H_v-module categories $\mathcal{KM}$ over any Krasner H_v-rings, where $\mathcal{KM}$ has enough star projective. We now see how to obtain the value of left derived functors at some Krasner H_v-modules by considering star projective resolution of this module.

Let M be a right Krasner H_v^R-module and N be a left Krasner H_v^R-module. Let $P = \twoheadrightarrow P_2 \overset{d_2}{\twoheadrightarrow} P_1 \overset{d_1}{\twoheadrightarrow} P_0 \overset{\epsilon}{\twoheadrightarrow} N \twoheadrightarrow 0$ be a star projective resolution of a left Krasner H_v^R-module N, then the Krasner H_v^R-module $txt_n^R(M, N)$ is defined to be

$$txt_n^R(M, N) := H_n(M, P_N) = \frac{1_M[ker\, d_n^{'}]}{1_M[Img\, d_{n+1}^{'}]}.$$

Notice that the domain of $txt_n^R(M, -)$ is $\mathcal{KM}$; the category of left Krasner H_v^R-modules over a Krasner H_v-ring R. In particular, if R is commutative, then $M[N]$ is a Krasner H_v^R-module and $txt_n^R(M, N) = txt_n^R(N, M)$ for any Krasner H_v^R-modules M, N and any $n \geq 0$.

Moreover, if $Q = \twoheadrightarrow Q_2 \overset{d_2}{\twoheadrightarrow} Q_1 \overset{d_1}{\twoheadrightarrow} Q_0 \overset{\eta}{\twoheadrightarrow} M \twoheadrightarrow 0$ is a star projective resolution of M over R, then

$$Txt_n^R(M, N) := H_n(Q_M, N) = \frac{[ker\, d_n^{'}]1_N}{[Img\, d_{n+1}^{'}]1_N}.$$

Definition 3.1. Let R be a Krasner H_v-ring, M be a right Krasner H_v-module over R, and $M[-]$ be the covariant additive functor. The functor

$$txt_n^R(M[-]) := txt_n^R(M, -)$$

is called the n-th left derived functor of M. Moreover, for a left Krasner H_v-module N over R, we consider the contravariant additive functor $[-]N$ and define nth left derived functor of N to be the the functor

$$Txt_n^R([-]N) := Txt_n^R(-, N)$$

Remark 3.2. One can check that the left derived functors $txt_n^R(M, -)$ and $Txt_n^R(-, N)$ are well-defined. Moreover, for any right Krasner H_v-module M and any left Krasner H_v-module N with star projective P, we have that $txt_n^R(M, P) = \{0\}$, for all $n \geqslant 1$, and thus only non-trivial value may occur at $n = 0$.

Remark 3.3. Let M be a right Krasner H_v^R-module, and consider the star projective resolution $P = \cdots \to P_2 \overset{d_2}{\to} P_1 \overset{d_1}{\to} P_0 \overset{\epsilon}{\to} N \to 0$ of a left Krasner H_v-module N. Define $K_0 = Ker\, \epsilon'$ and $K_n = Ker\, d_n$ for all $n \geqslant 1$. Then

$$txt_{n+1}^R(M,N) \overset{w}{=} txt_n^R(M,K_0) \overset{w}{=} \cdots \overset{w}{=} txt_1^R(M,K_{n-1}).$$

Indeed, as P is exact, $K_0 = Ker\, \epsilon \overset{w}{=} Img\, d_1$, and so we have a star projective resolution, $Q = \cdots \to Q_2 \overset{d_2}{\to} Q_1 \overset{d_1}{\to} K_0 \to 0$, of K_0. The result follows by applying $Q_n = P_{n+1}$ and $\delta_n = d_{n+1}$ iteratively to get

$$\begin{aligned} txt_n^R(M,K_0) \quad & \overset{w}{=} H_n(M,Q_{K_0}) = \frac{Ker\ M[-]\delta_n}{Img\, M[-]\delta_{n+1}} \overset{w}{=} \frac{Ker\, M[-]d_{n+1}}{Img\, M[-]d_{n+2}} \\ & \overset{w}{=} txt_{n+1}^R(M,N). \end{aligned}$$

From our previous work,

$$\begin{array}{ccccccccc} 0 & \to & M' & \overset{i}{\to} & M & \overset{p}{\to} & M'' & \to & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \to & N' & \overset{j}{\to} & N & \overset{q}{\to} & N'' & \to & 0 \end{array}$$

one can easily see that there are functors $txt_n^R(M,N)$, where M is a right Krasner H_v-module and N is a left Krasner H_v-module and a natural connecting homomorphisms $txt_n^R(M'',N) \to txt_{n-1}^R(M',N)$ and $txt_n^R(M,N'') \to txt_{n-1}^R(M,N')$ for the first two rows of short exact sequences of the weak commutative diagram such that the sequences

$$txt_n^R(M,N') \to txt_n^R(M,N) \to txt_n^R(M,N'') \to txt_{n-1}^R(M,N'),$$

$$txt_n^R(M',N) \to txt_n^R(M,N) \to txt_n^R(M'',N) \to txt_{n-1}^R(M',N)$$

are weakly exact. Next result shows that one can construct a new weak commutative diagram.

Lemma 3.4. *Consider the weak commutative diagram above of right Krasner H_v^R-modules with weakly exact rows. Then there exists a weak* *commutative diagram with weakly exact rows for every right Krasner H_v^R-module T,*

$$\begin{array}{ccccccc} txt_n^R(T,M') & \xrightarrow{i_*} & txt_n^R(T,M) & \xrightarrow{p_*} & txt_n^R(T,M'') & \xrightarrow{\delta_*} & txt_{n-1}^R(T,M') \\ \downarrow f_* & & \downarrow g_* & & \downarrow h_* & & \downarrow f_* \\ txt_n^R(T,N') & \xrightarrow{j_*} & txt_n^R(T,N) & \xrightarrow{q_*} & txt_n^R(T,N'') & \xrightarrow{\delta'_*} & txt_{n-1}^R(T,N'). \end{array}$$

Proof. This follows from the weakly exactness of the deleted complexes $0 \longrightarrow P_{M'} \longrightarrow P_M \longrightarrow P_{M''} \longrightarrow 0$ and $0 \longrightarrow TP_{M'} \longrightarrow TP_M \longrightarrow TP_{M''} \longrightarrow 0$, and then by applying (1). □

Lemma 3.5. *Let R be Krasner H_v-ring, M be a right Krasner H_v-module over R, and N be a left Krasner H_v-module over R. The functors $M[-]$ and $[-]N$ are naturally star isomorphic to $txt_0^R(M,-)$ and $Txt_0^R(-,N)$, respectively. Moreover, we have star isomorphisms*

$$Txt_0^R(M,N) \cong M[N] \cong txt_0^R(M,N).$$

Proof. Follows from the fact that $M[-]$ and $[-]N$ are additive covariant right exact functors. □

Now consider the short exact sequence $0 \longrightarrow L \longrightarrow K \longrightarrow N \longrightarrow 0$ of left Krasner H_v^R-modules over R. For every right Krasner H_v^R-module M, one can get a long exact sequences by applying $M[-]$ and $txt_n^R(M,-)(Txt_n^R(M,-))$ where $n \geq 1$. Next we show that, as expected we have $Txt_n^R(M,N) \overset{w}{=} txt_n^R(M,N)$.

Theorem 3.6. *Let M be a right Krasner H_v-module on a Krasner H_v-ring R, let N be a left Krasner H_v-module on Krasner H_v-ring R, and let*

$$\mathbf{P} = \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \to 0 \quad and \quad \mathbf{Q} = \cdots \to Q_1 \xrightarrow{d_1'} Q_0 \xrightarrow{\epsilon'} N \to 0$$

be star projective resolutions. Then $H_n([\mathbf{P}_M]N) \overset{w}{=} H_n(M[\mathbf{Q}_N])$ for all $n \geqslant 0$, or equivalently,

$$Txt_n^R(M,N) \overset{w}{=} txt_n^R(M,N).$$

Proof. The proof will be accomplished by induction on $n \geq 0$. When $n = 0$ the claim follows from a result from Lemma 3.5 and that if we factorize the syzygies of $\mathbb{P}$ and $\mathbb{Q}$ into short exact sequences. Then we get the exact sequences $0 \longrightarrow K_i \longrightarrow P_i \longrightarrow K_{i-1} \longrightarrow 0$ and $0 \longrightarrow V_j \longrightarrow Q_j \longrightarrow V_{j-1} \longrightarrow 0$ respectively, for all $i, j \geq 0$ and where $M = K_{-1}$ and $V_{-1} = N$. As $M[-]$ and $[-]N$ are functors of two variables, it follows that, for each $i, j \geqslant 0$, we have the following weak commutative diagram with a weakly exact rows and columns

$$\begin{array}{ccccccccc}
 & & Txt_1(K_{i-1},V_j) & \longrightarrow & 0 & & Txt_1(K_{i-1},V_{j-1}) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
txt_1(K_i,V_{j-1}) & \longrightarrow & K_i[V_j] & \longrightarrow & K_i[Q_j] & \longrightarrow & K_i[V_{j-1}] & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_i[V_j] & \longrightarrow & P_i[Q_j] & \longrightarrow & P_i[V_{j-1}] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
txt_1(K_{i-1},V_{j-1}) & \longrightarrow & K_{i-1}[V_j] & \longrightarrow & K_{i-1}[Q_j] & \longrightarrow & K_{i-1}[V_{j-1}] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

It follows that $Txt_1(K_{i-1}, V_{j-1}) \overset{w}{=} txt_1(K_{i-1}, V_{j-1})$, for all $i, j \geq -1$ and so $Txt_1(M, N) \overset{w}{=} txt_1(M, N)$. Now assume that $n \geq 1$. From Theorem 3.6 one can see that

$$txt_{n+1}(M, N) \overset{w}{=} txt_1(M, V_{n-1}) \overset{w}{=} txt_1(K_{-1}, V_{n-1}),$$

$$Txt_{n+1}(M, N) \overset{w}{=} Txt_1(K_{n-1}, N) \overset{w}{=} Txt_1(K_{n-1}, V_{-1}),$$

and thus $Txt_1(K_{i-1}, V_j) \overset{w}{=} txt_1(K_i, V_{j-1})$.

We can now apply the claim when $n = 1$ iteratively to get the required equation $Txt_1(K_{n-1}, V_{-1}) \overset{w}{=} txt_{n+1}(M, N)$. □

Let M be a left Krasner H_v-module over a Krasner H_v-ring R. Now consider a sequence of additive covariant functors. We next determine a conditions under which this sequence is star isomorphic to $Txt_n^R(-, M)$.

Theorem 3.7. *Let $T_n \colon \mathcal{KM}_1 \longrightarrow \mathcal{KM}_2$ be a sequence of additive covariant functors where $n \neq 0$. Then T_n is star isomorphic to $Txt_n^R(-, M)$ for all $n \neq 0$, if the following conditions hold:*

(i) *For every short exact sequence $0 \to L \to N \to K \to 0$ of right Krasner H_v-module over a Krasner H_v-ring R, there is a long exact sequence with connecting star homomorphisms*

$$\longrightarrow T_{n+1}(K) \xrightarrow{\delta_{n+1}} T_n(L) \longrightarrow T_n(N) \longrightarrow T_n(K) \xrightarrow{\delta_n} T_{n-1}(L) \longrightarrow$$

(ii) *T_0 is naturally star isomorphic to $[-]M$ for some left Krasner H_v-module.*
(iii) *$T_n(P) = \{0\}$ for any star projective right Krasner H_v-module P over R and all $n \neq 1$.*

Proof. Suppose that (i), (ii) and (iii) hold. We show that T_n is star isomorphic to $Txt_n^R(-, M)$ for all $n \neq 0$. The proof will be accomplished by induction on $n \neq 0$. When $n = 0$ the claim follows from a result from (ii). When $n = 1$, given a right Krasner H_v-module L on Krasner H_v-module R, there is an exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow L \longrightarrow 0$ where P is star projective. Now axiom (i) implies that we have the following diagram with exact rows

$$\begin{array}{ccccccc}
\longrightarrow T_1(P) & \longrightarrow & T_1(L) & \xrightarrow{\delta_1'} & T_0(K) & \longrightarrow & T_0(P) \\
 & & \downarrow{\scriptstyle \tau_{1A}} & & {\scriptstyle \tau_{0K}}\downarrow & & {\scriptstyle \tau_{0P}}\downarrow \\
\to Txt_1^R(P, M) & \to & Txt_1^R(L, M) & \xrightarrow[\delta_1]{} & Txt_0^R(K, M) & \to & Txt_0^R(P, M)
\end{array}$$

where the maps τ_{0K} and τ_{0P} are the natural star isomorphisms, and the naturally gives weak commutativity of the square on the right. Moreover $T_1(P) \overset{w}{=} \{0\} \overset{w}{=} Txt_1^R(P, M)$ by (iii) and thus the maps δ_1' and δ_1 are weak injective. Thus $\tau_1 L$ is a star isomorphism and the result follows. Now assume $n \neq 1$. By inductive hypothesis we have that $\tau_{nK} \colon T_n(K) \longrightarrow Txt_n^R(K, M)$ is a star isomorphism. This follows that both $\delta_n' \colon T_{n+1}(L) \to T_n(K)$ and $\delta_n \colon Txt_{n+1}^R(L, M) \to Txt_n^R(K, M)$ are star isomorphisms and so the composite $\tau_{n+1,L} = \delta_{n+1}^{-1} \tau_{nK} \delta_{n+1}' \colon T_{n+1}(L)$

$\longrightarrow F_{n+1}(L)$ is a star isomorphism. Hence by the naturality of the connecting star homomorphisms $\delta^{'}$ and δ, the claim follows. $\square$

Remark 3.8. Consider the sequences $(T_n)_{n\neq 0}, (T^{'}_n)_{n\neq 0}$ of additive covariant functors $\mathcal{KM}_1 \longrightarrow \mathcal{KM}_2$, where $\mathcal{A}$ has enough star projective. Then T_n is naturally star isomorphic to $T^{'}_n$ for all $n \neq 0$ if for every short exact sequence $0 \longrightarrow M \longrightarrow N \longrightarrow K \longrightarrow 0$ in $\mathcal{KM}_1$, there are long exact sequences with natural connecting star homomorphisms, T_0 is naturally star isomorphic to $T^{'}_0$, and $T_n(P) \overset{w}{=} 0 \overset{w}{=} T^{'}_n(P)$ for all star projective P and $n \neq 1$.

Finally we now examine Txt more closely and write down some properties.

Theorem 3.9. *Let R be a Krasner H_v-ring, M be a right Krasner H_v-module on Krasner H_v-ring R, and N be a left Krasner H_v-module on Krasner H_v-ring R. Then*

$$\mathrm{Txt}^R_n(M,N) \overset{w}{=} \mathrm{Txt}^{R^{op}}_n(M,N)$$

for all $n \geq 0$, where R^{op} is the opposite ring of R.

Proof. Pick a deleted star projective resolution $\mathbf{P}_M$ of M. Then, $t\colon [\mathbf{P}_M]_R N \rightarrow N[\mathbf{P}_M]_{R^{op}}$ is a chain map such that $t_n\colon [\mathbf{P}_n]_R N \rightarrow N[\mathbf{P}_n]_{R^{op}}, [x_n]b \mapsto b[x_n]$. As star isomorphic complexes have the same Krasner H_v-homology, we have

$$\mathrm{Txt}^R_n(M,N) = H_n([\mathbf{P}_M]N) \overset{w}{=} H_n(N[\mathbf{P}_M)]_{R^{op}})$$

for all $n \geq 0$. We however have that $\mathbf{P}_M$ is a deleted projective resolution of left Krasner H_v-module M on R^{op}. Therefore, $H_n(N[\mathbf{P}_M]_{R^{op}})$ $\overset{w}{=} Txt^{R^{op}}_n(N,M)$. $\square$

Definition 3.10. If R is a Krasner H_v-ring, then a right Krasner H_v-module A on R is star flat,if $A[-]$ is an exact functor; that is, whenever

$$0 \longrightarrow B' \xrightarrow{i} B \xrightarrow{P} B'' \longrightarrow 0$$

is an exact sequence of left Krasner H_v-module on R, then

$$0 \longrightarrow A[B^{'}] \xrightarrow{1_A[i]} A[B] \xrightarrow{1_A[P]} A[B''] \longrightarrow 0$$

is an exact sequence of canonical H_v-groups.

We next show that $\mathrm{Txt}_n(M,-)$ and $\mathrm{Txt}_n(-,N)$ vanish on star flat Krasner H_v-modules, in other words, if M is a right Krasner H_v^R-module, then $M[-]$ is an exact functor.

Lemma 3.11. *Let R be a Krasner H_v-ring, F be a right Krasner H_v-module over R and M be left Krasner H_v-module over R. If F is star flat, then $\mathrm{Txt}_n^R(F,M)=\{0\}$ for all $n\geq 1$. Conversely, if $\mathrm{Txt}_1^R(F,M)=\{0\}$ for every such M, then F is star flat.*

Proof. Let $\mathbf{P}$ be a star projective resolution of M. As F is star flat, the functor $F[-]$ is weakly exact, and so the complex

$$F[\mathbf{P}_M]=\twoheadrightarrow F[P_2]\twoheadrightarrow F[P_1]\twoheadrightarrow F[P_0]\twoheadrightarrow 0$$

is weakly exact for all $n\geq 1$. Therefore, $\mathrm{Txt}_n(F,M)=\{0\}$ for all $n\geq 1$. Conversely, if $0\to M\xrightarrow{i}N$ is weak exact then

$$0=\mathrm{Txt}_1^R(F,N/M)\twoheadrightarrow F[A]\overset{1[i]}{\twoheadrightarrow}F[M]$$

is weakly exact. Hence, $1[i]$ is an weakly injection, and so F is star flat. □

Remark 3.12. (1) Let M be a right Krasner H_v-module. Then N'' is a left Krasner star flat H_v-module if and only if every exact sequence $0\to N'\to N\to N''\to 0$ of left Krasner H_v-modules is pure exact, that is, the sequence $0\to M[N']\to M[N]\to M[N'']$ is weakly exact. To see this, pick an exact sequence $0\to N'\xrightarrow{i}N\to N''\to 0$ with a star free N. Then $\mathrm{Txt}_1^R(M,N)\twoheadrightarrow\mathrm{Txt}_1^R(M,N'')\twoheadrightarrow M[N']\overset{1[i]}{\twoheadrightarrow}M[N]$ is a weakly exact sequence. As $\mathrm{Txt}_1^R(M,N'')=Ker\,1[i]$, where N is a star free, $\mathrm{Txt}_1^R(M,N)=\{0\}$. This follows that $\mathrm{Txt}_1^R(M,N'')=\{0\}$ for all M, and thus N'' is star flat. Conversely, a weakly exact sequence $\mathrm{Txt}_n^R(M,N'')\twoheadrightarrow M[N']\twoheadrightarrow M[N]\twoheadrightarrow M[N'']\twoheadrightarrow 0$ is pure exact.

(2) Let $0\to M\to N\to T\to 0$ be a weakly exact sequence of right Krasner H_v^R-module, and suppose that T is star flat. Then M is star flat if and only if N is star flat. This follows from the existence of an exact sequence $\mathrm{Txt}_2^R(T,X)\twoheadrightarrow\mathrm{Txt}_1^R(M,X)\twoheadrightarrow\mathrm{Txt}_1^R(N,X)\twoheadrightarrow\mathrm{Txt}_1^R(T,X)$, for any left Krasner H_v-module X.

Theorem 3.13. *The functors* $\mathrm{Txt}_n^R(M,-)$ *and* $\mathrm{Txt}_n^R(-,N)$ *can be computed using star flat resolutions of either variable. More precisely,*

$$H_n([\mathbf{F}_M]N) \overset{w}{=} \mathrm{Txt}_n^R(M,N) \overset{w}{=} H_n(M[\mathbf{G}_N]),$$

for all $n \geq 0$ *and for all star flat resolutions* $\mathbf{F}$ *and* $\mathbf{G}$ *of* M *and* N*, respectively.*

Proof. It suffices to show that $H_n([\mathbf{F}_M]N) \overset{w}{=} \mathrm{Txt}_n^R(M,N)$. Consider the star flat resolution $\twoheadrightarrow F_2 \overset{d_2}{\twoheadrightarrow} F_1 \overset{d_1}{\twoheadrightarrow} F_0 \twoheadrightarrow A \twoheadrightarrow 0$. It is easy to see that $H_0([\mathbf{F}_M]N) = Coker\,([d_1]1) \overset{w}{=} [M]N \overset{w}{=} \mathrm{Txt}_0^R(M,N)$. An easy induction gives the following diagram

$$\begin{array}{c} [F_2]N \xrightarrow{[d_2]1} [F_1]N \xrightarrow{[d_1]1} [F_0]N \\ \downarrow \alpha \\ \dfrac{([F_1]N)}{Img\,([d_2]1)} \\ \downarrow \beta \\ \dfrac{([F_1]N)}{Ker\,([d_1]1)} \end{array}$$

where α is the natural map, β is surjective since $Im\,g([d_2]1) \subseteq Ker([d_1]1)$, and $\gamma\colon ([F_1]N)/Ker([d_1]1) \to [F_0]N$ is weakly injective. This follows that $H_1([\mathbf{F}_A]N) \overset{w}{=} Im\,g([d_1]1) \overset{w}{=} Im\,g([i]1)$.

From the weakly exact sequence for Txt we have that

$$\mathrm{Txt}_1^R(F_0,N) \longrightarrow \mathrm{Txt}_1^R(M,N) \longrightarrow [Ker\,d_1]N \xrightarrow{[i]1} [F_0]N.$$

By Lemma 3.4, it follows $\mathrm{Txt}_1^R(F_0,N) = \{0\}$ and thus $\mathrm{Txt}_1^R(M,N) \overset{w}{=} Ker([i]1) \overset{w}{=} H_1([\mathbf{F}_M]N)$. Finally as F_0 is star flat and for any $n \geq 1$ the sequence

$$\mathrm{Txt}_{n+1}(F_0,N) \twoheadrightarrow \mathrm{Txt}_{n+1}(M,N) \twoheadrightarrow \mathrm{Txt}_n(Kerd_1,N) \twoheadrightarrow \mathrm{Txt}_n(F_0,N)$$

is a weakly exact, we get $F' = \twoheadrightarrow F_2 \twoheadrightarrow F_1 \twoheadrightarrow Ker\,d_1 \twoheadrightarrow 0$ is a star flat resolution of $Ker\,d_1$, and so by the induction hypothesis we have that

$$\mathrm{Txt}_n(Ker\,d_1,N) \overset{w}{=} H_n([\mathbf{F}'_{Ker\,d_1}]N) \overset{w}{=} H_{n+1}([\mathbf{F}_M]N) = \frac{[Ker\,d_{n+1}]1}{[Img\,d_{n+2}]1}.$$

□

Proposition 3.14. *Let R be Krasner H_v-ring and M be a Krasner H_v-modules over R. If $(N_i)_{i\in I}$ is a family of left Krasner H_v-modules, then there are natural star isomorphisms*

$$\mathrm{Txt}_n^R(M, \bigoplus_{i\in I} N_i) \overset{w}{=} \bigoplus_{i\in I} \mathrm{Txt}_n^R(M, N_i),$$

for all $n \geq 0$. Moreover if (N_i, φ_j^i) is a direct system of Krasner H_v-modules over an H_v-ring R indexed by a directed set I, then for all $n \geq 0$, there is a star isomorphism

$$\mathrm{Txt}_n^R(M, \varinjlim N_i) \overset{w}{=} \varinjlim \mathrm{Txt}_n^R(M, N_i).$$

Proof. The proof can be accomplished by an easy induction on n. The second part follows from this, and the facts that $\varinjlim N_i$ is a Krasner H_v^R-module and that $\varinjlim$ is an exact functor. □

Remark 3.15. The star isomorphism in the previous lemma also exists if the direct sum is in the first variable. A moment reflection also shows that, like in modules and H_v-modules, the condition where every direct product of star flat right Krasner H_v-modules on R is star flat, can be expressed equivalently by saying that the right Krasner H_v-modules on R, R^X, is star flat for every set X. This is equivalent to the condition that every finitely generated Krasner H_v-submodule of a star free left Krasner H_v-modules on R is finitely presented.

The next following propositions describes interaction between Txt and the localization.

Proposition 3.16. *Let R and S be Krasner H_v-rings, and let $T\colon {}_R\mathcal{KM} \longrightarrow {}_S\mathcal{KM}$ be an exact additive functor. Then there is a star isomorphism*

$$H_n(T\mathcal{M}_\bullet, Td_\bullet) \overset{w}{=} TH_n(\mathcal{M}_\bullet, d_\bullet)$$

for every complex $(\mathcal{M}_\bullet, d_\bullet) \in \mathcal{C}_\bullet(\mathcal{KM})$ and for every $n \in \mathbb{Z}$.

Proof. The proof is routine. □

We finally show that localization commutes with the functor Txt.

Proposition 3.17. *Let S be a multiplicative subset of a commutative Krasner H_v-ring R, and M, N be Krasner H_v-modules on R. Then, there is a natural star isomorphism*

$$S^{-1}\mathrm{Txt}_n^R(M,N) \overset{w}{\cong} \mathrm{Txt}_n^{S^{-1}R}(S^{-1}M, S^{-1}N),$$

for all $n \geq 0$.

Proof. This follows from the fact that $S^{-1}R$ is a star flat Krasner H_v^R-module. $\square$

Example 3.18. (1) Let p is a prime number. It is well known that $\mathbb{Q}$, rational numbers, is flat over $\mathbb{Z}_p$, but not projective. This yields that $\mathbb{Q} \times \mathbb{Q}$ is flat over $\mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$ but not projective.

(2) Now, consider the H_v-module M defined in Example 2.4 (2). Since M is star projective, it follows that it is star flat. So, by using part (1) we conclude that $\mathbb{Q} \times \mathbb{Q} \times M$ is a star flat H_v-module over $\mathbb{Z}_6$, but it is not star projective.

4 Conclusion

Star projective Krasner H_v-modules are characterized. General results that lead to specific kinds of unitary Krasner H_v-modules, called star free Krasner Hv-modules, are described. All Krasner H_v-vector spaces have bases and are star free Krasner Hv-modules. Moreover, Krasner H_v-homology on Krasner H_v-modules is described. The left derived functors Txt_n^R and txt_n^R are defined, where n is a positive integer and R is a Krasner H_v-ring. These derived functors are shown to be natural and possess several notable properties.

For future research, we suggest to study Krasner H_v-cohomology on Krasner H_v-modules. Krasner H_v-cohomology arises by dualizing the construction of Krasner H_v-homology.

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