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Ker-Coker Lemma on Hypermodules

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Abstract. In this work, we examine the idea of exact sequences on the hypermodule category and derive some fundamental properties of them. Specifically, we demonstrate the Snake Lemma, also called the ker-coker Lemma, for hypermodules, which has numerous applications in homological algebra.

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1 Introduction

In 1934, Marty presented the theory of hyperstructures at the 8^{th} congress of the Scandinavian Mathematicians [15]. Since Marty first proposed the idea of a hypergroup, other scholars have developed and worked on this novel area of contemporary algebra. Following Krasner [11], several authors examined the concepts of hyerfields and hyperrings.

One type of hypergroup is the canonical hypergroup. They were initially acquired from the hyperring and hyerfield's additive section. The word "canonical" was coined and J. Mittas was the first mathematician to study them in depth (see [18]). Additionally, some authors studied

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hypermodules whose additive structure is just a canonical hypergroup in the context of canonical hypergroups (see, for instance, [16], [13], [17], and [10]).

Hyperstructures have many applications in several branches of both pure and applied sciences. The hyperstructures book [2] describes how they are used in binary relations, graphs, lattices, probability, codes, automata, and hypergraphs.

The concepts of H_v-structures in which the axioms are replaced by the weak ones, that is, instead of the equality on sets one has non-empty intersections, were introduced first by T. Vougioklis in the Forth AHA Congress (1990) [20]. These concepts have been further investigated by many authors in [3, 4, 5, 12, 14], and [22]. A comprehensive review of the theory of H_v-structures appears in [21]. For example, in [6] Davvaz and Ghadiri defined the notion of exact sequences in H_v-modules and investigated some properties of these sequences concerning to fundamental relations on H_v-modules (see also, [8, 9]). A recent book [7] is devoted exclusively to the study of hyperring theory; it introduces and analyzes various types of hyperrings and concludes with an overview of applications in physics and chemistry, analyzing various special types of hyperstructures, such as transposition hypergroups and e-hyperstructures. In essence, hyperrings are rings with slightly altered axioms that treat addition as a hyperoperation, that is, $a + b$ is a set. Numerous authors have examined this idea.

Consider the category **Hmod** whose objects are hypermodules over a fixed hyperring R. Define $\mathrm{Mor}_{\mathbf{Hmod}}(M, N)$ to be the set of all homomorphisms $h : M \to N$. In this paper, we examine the idea of exact sequences on the hypermodule category and derive some fundamental properties of them. Specifically, we demonstrate the Snake Lemma, also called the ker-coker Lemma, for hypermodules, which has numerous applications in homological algebra.

2 Preliminaries and Basic Results

In this section, we fix our notations and compile a few fundamental definitions from the theory of hyperstructures. The reader may refer to [1] in this regard.

We start with the definition of canonical hypergroup.

Definition 2.1. *(See also [16, 17].) A set M equipped with a hyperoperation $+$ is said to be a canonical hypergroup if the following axioms hold:*
(i) $(x+y)+z = x+(y+z)$, for all $x, y, z \in M$;
(ii) $x+y = y+x$, for all $x, y \in M$;
(iii) there exists an element $0 \in M$ such that for every $x \in M, x+0 = \{x\}$;
(iv) for every $x \in M$ there exists a unique element $x' \in M$ such that $0 \in x+x'$. From now on, we denote x' by $-x$ and call it the opposite of x; instead of $x+(-y)$ we shall write $x-y$.
(iv) $z \in x+y$ implies that $x \in z-y$, for all $x, y, z \in M$.

Remark 2.2. i) In general, for a hyperoperation $+$ on the set M and $N, K \subseteq M$, we shall write $N+K$ to denote the set $N+K = \bigcup_{x\in N, y\in K} x+y$.
ii) It can be easily seen that the element 0 is unique with the property given by (iii).

We direct the reader to [18] for research on canonical hypergroups.

A well-known type of a hyperring is called the Krasner hyperring.

Definition 2.3. ([11]) *A non-empty set R equipped with a hyperoperation $(+)$ and an operation $(\cdot)$ is called a Krasner hyperring (hyperring for short) if:*
(i) the pair $(R, +)$ is a canonical hypergroup;
(ii) the pair $(R, \cdot)$ is a multiplicative semigroup with an element 0 such that $x \cdot 0 = 0 \cdot x = 0$ for every $x \in R$.
(iii) The distributivity laws hold in R. That is,

$$z \cdot (x+y) = z \cdot x + z \cdot y, \text{ and } (x+y) \cdot z = x \cdot z + y \cdot z,$$

for all $x, y, z \in R$.

According to [1], by a (left) *hypermodule* over a fixed hyperring R we shall mean a canonical hypergroup $(M, +)$ equipped with a function $\cdot : R \times M \to M$ which associates to each pair $(r, x) \in R \times M$, an element $rx \in M$ such that for all $x, y \in M$, and $r, s \in R$:

(i) $r(x+y) = rx + ry$;
(ii) $(r+s)x = rx + sx$;
(iii) $(rs)x = r(sx)$;
(iv) $0x = x$.

For any $a, b \in R$, let S be a subgroup of the multiplicative semigroup of R, satisfying the formula $aSbS = abS$. Let M be an R-module, where R is a unitary ring. Keep in mind that this requirement equals the normality of S only in the division ring scenario, when $R \setminus \{0\}$ is a group. We now establish an equivalence relation $\sim$ on M, which has the following definition:

$$x \sim y \Leftrightarrow \exists t \in S, \quad x = ty.$$

The set of all the equivalence classes of M modulo $\sim$ is denoted by $\overline{M}$. Afterwards, $\bar{x} \oplus \bar{y} = \{\bar{w} \in \overline{M} \mid \bar{w} \subseteq \bar{x} + \bar{y}\}$ defines a hyperoperation $\oplus$ in $\overline{M}$. That is, $\bar{x} \oplus \bar{y}$ consists of all classes $\bar{w} \in \overline{M}$ which are contained in the set-wise sum of $\bar{x}$ and $\bar{y}$. After that, $(\overline{M}, \oplus)$ is established as a canonical hypergroup. Let R be the quotient hyperring of R by S and consider the function $\cdot : \overline{R} \times \overline{M} \longrightarrow \overline{M}$ described as follows;

$$\bar{a}\bar{x} = \overline{ax} \quad \text{for every } \bar{r} \in \overline{R}, \bar{x} \in \overline{M}.$$

Since this composition fits the requirements of the hypermodule definition, $\overline{M}$ over $\overline{R}$ becomes a hypermodule. Massouros has shown in [16] that how closely this hypermodule relates to both the Euclidean spherical geometries and the analytic projective geometries.

Below, all the hypermodules that are taken into consideration are left hypermodules.

Definition 2.4. *Given a fixed hyperring R, let M and N be two hypermodules over it. A mapping $f : M \longrightarrow N$ is called a* homomorphism *if $f(x+y) = f(x) + f(y)$, and $f(rx) = rf(x)$ for all $r \in R$ and for all $x, y \in M$.*

As usual, we define the kernel of a homomorphism $f : M \longrightarrow N$ between two hypermodules M and N over a hyperring R in the usual manner, i.e.,

$$\ker f = \{a \in M \mid f(a) = 0\}.$$

Then we have

Proposition 2.5. *Let $f : M \longrightarrow N$ be a homomorphism of hypermodules over a hyperring R. Then f is one to one if and only if* $\ker f = \{0\}$.

Proof. It is straightforward (see also [1]). $\square$

3 Main Results

Throughout this section, we will be concerned almost exclusively with hypermodules. Consider a fixed hyperring R. The class of all hypermodules over the hyperring R along with all homomorphisms creates a category denoted by **Hmod**, as we have already established in the introduction. Let M be a hypermodule over the hyperring R, $\mathcal{C}(M)$ the class of M's subobjects in **Hmod**, and $\mathcal{C}_0(M)$ the set of M's subhypermodules. Let us examine the mapping $\varphi : \mathcal{C}_0(M) \to \mathcal{C}(M)$, which is defined as follows: $\varphi(X) = [X, i_X]$, where $i_X : X \to M$ is the inclusion homomorphism. After that, it is evident that φ is a bijection (see [13] for further information). It is easily checked that $[f(M), i_{f(M)}] \in \mathcal{C}(N)$ for any homomorphism $f : M \longrightarrow N$ of hypermodules, where $i_{f(M)} : f(M) \to N$ is the inclusion. It is worth to mention that if $f : M \to N$ is a monomorphism in the category of hupermodules, then $\mathrm{Im} f = [M, f]$ (see also remark 3.2 below).

We begin with the following:

Definition 3.1. *A sequence of homomorphisms of hypermodules*

$$\cdots \longrightarrow M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \longrightarrow \cdots \tag{1}$$

is said to be exact at M_i if $\mathrm{Im} f_i = \ker f_{i+1}$ *(as subobjects of M_i in* **Hmod***). The sequence is said to be exact if it is exact at each M_i.*

Remark 3.2. It is easy to see that a sequence:

1. $0 \longrightarrow M_1 \xrightarrow{f} M_2$ is an exact sequence if and only if f is a homomorphism that is one to one.

2. $M_1 \xrightarrow{f} M_2 \longrightarrow 0$ is an exact sequence if and only if f is a homomorphism that is onto.

3. $0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$ is an exact sequence if and only if f is a one to one homomorphism, g is an onto homomorphism, and $\ker g = [M_1, f]$.

Theorem 3.3. (Five Lemma) *Let the following diagram of hypermodules and homomorphisms over the hyperring R be given*

$$\begin{array}{ccccccccc}
M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{f_2} & M_3 & \xrightarrow{f_3} & M_4 & \xrightarrow{f_4} & M_5 \\
\downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3} & & \downarrow{\scriptstyle h_4} & & \downarrow{\scriptstyle h_5} \\
N_1 & \xrightarrow[g_1]{} & N_2 & \xrightarrow[g_2]{} & N_3 & \xrightarrow[g_3]{} & N_4 & \xrightarrow[g_4]{} & N_5.
\end{array}$$

Assume that the diagram is commutative and that the rows in it are exact sequences. Then

1. *if h_1 is an epimorphism, h_2 and h_4 are monomorphisms, then h_3 is a monomorphism;*
2. *if h_5 is a monomorphism, h_2 and h_4 are epimorphisms, then h_3 is an epimorphism;*
3. *if h_1 is an epimorphism, h_2 and h_4 are isomorphisms, and h_5 is a monomorphism, then h_3 is an isomorphism.*

Proof. 1) This is straightforward.
2) Suppose $b_3 \in N_3$; since h_4 is surjective, there exists $a_4 \in M_4$ such that $h_4(a_4) = g_3(b_3)$; it follows

$$(h_5 \circ f_4)(a_4) = (g_4 \circ h_4)(a_4) = (g_4 \circ g_3)(b_3) = 0,$$

and as h_5 is injective, $f_4(a_4) = 0$; the top row in the diagram being an exact sequence, there exists $a_3 \in M_3$ such that $f_3(a_3) = a_4$; we have

$$g_3(b_3) = h_4(a_4) = (h_4 \circ f_3)(a_3) = g_3(h_3(a_3)),$$

and so $0 \in g_3(b_3 - h_3(a_3))$. This means that there exists $a_3' \in b_3 - h_3(a_3)$ such that $g_3(a_3') = 0$, hence there exists $b_2 \in N_2$ such that $a_3' = g_2(b_2)$. Since, h_2 is surjective, it guarantees that there exists $a_2 \in M_2$ such that

$b_2 = h_2(a_2)$, hence $g_2(b_2) \in b_3 - h_3(a_3)$. But the diagram is commutative, so

$$g_2(b_2) = (g_2 \circ h_2)(a_2) = (h_3 \circ f_2)(a_2).$$

It follows that $h_3(f_2(a_2)) \in b_3 - h_3(a_3)$ or equivalently, $b_3 \in h_3(f_2(a_2)) + h_3(a_3)$. Since h_3 is a homomorphism, there exists $x \in f_2(a_2) + a_3$ such that $b_3 = h_3(x)$, i.e, h_3 is surjective.
3) follows from 1) and 2). □

Corollary 3.4. (Short Five Lemma) *Let the following commutative diagram of hypermodules and homomorphisms over the hyperring R be given:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{f_2} & M_3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3} & & \\
0 & \longrightarrow & N_1 & \xrightarrow[g_1]{} & N_2 & \xrightarrow[g_2]{} & N_3 & \longrightarrow & 0
\end{array}$$

The following assertions are then true:

1. *If h_1, h_3 are monomorphisms then so is h_2;*
2. *If h_1, h_3 are epimorphisms then so is h_2;*
3. *If h_1, h_3 are isomorphisms then so is h_2.*

Proof. It is straightforward. □

Theorem 3.5. *Let the following commutative diagram of hypermodules and homomorphisms over the hyperring R be given:*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E' & \xrightarrow{u'} & E & \xrightarrow{u} & E'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\
0 & \longrightarrow & F' & \xrightarrow{v'} & F & \xrightarrow{v} & F'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g''} & & \\
0 & \longrightarrow & G' & \xrightarrow{w'} & G & \xrightarrow{w} & G'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

If the columns and the two bottom rows in this diagram are exact sequences, then the top row is also an exact sequence.

Proof. We must show that:
1) u' is injective: This is trivial (see [19]).
2) $\mathrm{Im}u' = \ker u$:
This is analogous to the classical case. For the sake of completeness, we give here more details. We have $f'' \circ u \circ u' = v \circ f \circ u' = v \circ v' \circ f' = 0$ and since f'' is a monomorphism, it follows that $u \circ u' = 0$, hence $\mathrm{Im}u' \subseteq \ker u$. Conversely, if $x \in \ker u$, then

$$v(f(x)) = f''(u(x)) = 0,$$

and there exists $y' \in F'$ such that $f(x) = v'(y')$. We have

$$w'(g'(y')) = g(v'(y')) = g(f(x)) = 0,$$

and as w' is injective, $g'(y') = 0$. Consequently, there exists $x' \in E'$ such that $y' = f'(x')$; but then

$$f(x) = v'(y') = v'(f'(x')) = f(u'(x')).$$

As f is injective, it follows that $u'(x') = x$. Hence, $x \in \mathrm{Im}u'$.
3) u is surjective:
Let $x'' \in E''$. Being v surjective implies that there exists $y \in F$ such that $v(y) = f''(x'')$. We have

$$w(g(y)) = g''(v(y)) = g''(f''(x'')) = 0,$$

hence there exists $z' \in G'$ such that $g(y) = w'(z')$. As g' is surjective, there exists $y' \in F'$ such that $z' = g'(y')$. We have $g(y) = w'(g'(y')) = g(v'(y'))$. Hence, since $0 \in g(y - v'(y'))$, there exists $t \in y - v'(y')$ such that $g(t) = 0$. Also, there exists $x \in F$ such that $f(x) = t$. But, then $f(x) \in y - v'(y')$ and so $v(f(x)) \in v(y) - v(v'(y'))$. Since $v \circ v' = 0$, this is equivalent to $v(f(x)) = v(y)$ and so $f''(u(x)) = f''(x'')$. Since f'' is injective, we get $u(x) = x''$, as desired. $\square$

Let us now assume that M is a hypermodule over the hyperring R and N a subhypermodule of M. There is only one hypermodule hyperstructure on $M/N = \{a + N \mid a \in M\}$, which makes it easy to see that the mapping $\pi : M \longrightarrow M/N$, given by $\pi(a) = a + N$, is a homomorphism. This is given by the hyperoperation

$$(x+N)+(y+N) = \{z+N \mid z \in x+y\}$$

and the operation

$$r(x+N) = rx+N.$$

The homomorphism π is onto. We call it the *canonical projection* (for more details, see [1]). In particular, if $f : M \longrightarrow K$ is a homomorphism of hypermodules then

$$\mathrm{Im} f = \{b \in K \mid f(a) = b \text{ for some } a \in M\}$$

is a subhypermodule of K and we can construct the quotient hypermodule $K/\mathrm{Im} f$ that we denote it by $\mathrm{coker} f$. Now, we can prove the following.

Theorem 3.6. (Ker-Coker Lemma) *Let the following commutative diagram of hypermodules and homomorphisms over the hyperring R be given in which the rows are exact:*

$$\begin{array}{ccccccc} & & M' & \xrightarrow{f'} & M & \xrightarrow{f} & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha''} & & \\ 0 & \longrightarrow & N' & \xrightarrow[g']{} & N & \xrightarrow[g]{} & N'' & & \end{array}$$

Then one may canonically construct a homomorphism of hypermodules over the hyperring R such that the following is an exact sequence:

$$\ker\alpha' \xrightarrow{f'^{*}} \ker\alpha \xrightarrow{f^{*}} \ker\alpha'' \xrightarrow{\delta} \mathrm{coker}\alpha' \xrightarrow{\bar{g'}} \mathrm{coker}\alpha \xrightarrow{\bar{g}} \mathrm{coker}\alpha''. \quad (2)$$

Proof. The construction of δ is as follows:
Let $x'' \in \ker\alpha''$. Being f surjective implies that there exists $x \in M$ such that $x'' = f(x)$. We have $g(\alpha(x)) = \alpha''(f(x)) = \alpha''(x'') = 0$, hence, from the exactness of the bottom row in the above diagram we get an element $y' \in N'$ such that $g'(y') = \alpha(x)$. We claim that the image of y' under the canonical projection $\pi : N' \longrightarrow N'/\mathrm{Im}\alpha' = \mathrm{coker}\alpha'$ does not depend

on the chosen element x: indeed, let $x_1 \in M$ be such that $f(x_1) = x''$; from $f(x) = f(x_1)$ it follows that $0 \in f(x) + (-f(x_1)) = f(x - x_1)$, hence there exists $t \in x - x_1$ such that $0 = f(t)$. But then, since the top row of the diagram is exact, there exists an element $a' \in M'$ such that $f'(a') = t$. If $y'_1 \in N'$ has the property $g'(y'_1) = \alpha(x_1)$, then

$$g'\alpha'(a') = \alpha f'(a') \in \alpha(x - x_1) = g'(y' - y'_1),$$

so there exists $b' \in y' - y'_1$ such that $g'(b') = g'(\alpha'(a'))$. As g' is a monomorphism, we have $\alpha'(a') = b' \in y' - y'_1$; and therefore $y' + \mathrm{Im}\alpha' = y'_1 + \mathrm{Im}\alpha'$; hence the mapping δ defined by

$$\delta(x'') = y' + \mathrm{Im}\alpha',$$

is well defined. It is easily seen that δ is a homomorphism.
Next, we must show that the sequence (2) is exact. First of all, note that it is routine to check that the induced sequences

$$\ker \alpha' \xrightarrow{f'^*} \ker \alpha \xrightarrow{f^*} \ker \alpha''' \tag{3}$$

and

$$\mathrm{coker}\alpha' \xrightarrow{\bar{g'}} \mathrm{coker}\alpha \xrightarrow{\bar{g}} \mathrm{coker}\alpha'' \tag{4}$$

are exact.
To complete the proof, it remains to prove the following two parts.
1) $\ker \delta = \mathrm{Im} f^*$:
First, by the definition of δ we have $\delta f^* = 0$ and so $\mathrm{Im} f^* \subseteq \ker \delta$. Conversely, let $\delta(x'') = y' + \mathrm{Im}\alpha' = 0$, where $f(x) = x'', g'(y') = \alpha(x)$ for some $x \in M$. It follows that $y' = \alpha'(x')$ for some $x' \in M'$, but then $\alpha(x) = g'(y') = g'(\alpha'(x')) = \alpha(f'(x'))$ and so we have $0 \in \alpha(x - f'(x'))$. Therefore, there exists $t \in x - f'(x')$ such that $0 = \alpha(t)$. Next, we have

$$f^*(t) = f(t) \in f(x - f'(x')) = f(x) - f(f'(x')) = f(x).$$

Because the upper row is exact $f(f'(x')) = 0$. It implies that $f^*(t) = f(x) = x''$. Thus $x'' \in \mathrm{Im} f^*$ and we have the other-side inclusion.
2) $\mathrm{Im}\delta = \ker \bar{g'}$:
The proof is similar to the classical case. $\square$

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