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Quintic Functional Equations in Non-Archimedean Normed Spaces

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Abstract. The Hyers-Ulam stability of a quintic functional equation in the normed spaces and non-Archimedean normed spaces by direct method are proved.

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1. Introduction

In 1940, S. M. Ulam [20] asked the first question on the stability problem. In 1941, D. H. Hyers [15] solved the problem of Ulam. This result was generalized by Aoki [1] for additive mappings and by Rassias [19] for linear mappings by considering an unbounded Cauchy difference. The paper of Rassias [19] has provided a lot of influence in the development of what we now call Hyers-Ulam-Rassias stability of functional equations. Since then, several stability problems for various functional equations have been investigated by numerous mathematicians; cf e.g. [2], [3], [4], [5], [8], [12], [15], [16], [17] and [23].
In [22], Xu et al. obtained the general solution and investigated the Ulam stability problem for the following quintic functional equation

$$f(x+3y)-5f(x+2y)+10f(x+y)-10f(x)+5f(x-y) \\ -f(x-2y)=120f(y)$$

in quasi-β-normed spaces via fixed point method. This method which is different from the "*direct method*", initiated by Hyers in [15], had been applied by

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Cădariu and Radu for the first time in [10]. In other words, they employed this fixed point method to the investigation of the Cauchy functional equation [11] and for the quadratic functional equation [10] (see also [6] and [7]).
Recently, in [18], Park et al. introduced the following new form of quintic functional equation

$$f(3x+y)-5f(2x+y)+f(2x-y)+10f(x+y)-5f(x-y)$$
$$=10f(y)+f(3x)-3f(2x)-27f(x). \quad (1)$$

It is easily verified that that the function $f(x)=\alpha x^5$ satisfies the functional equation (1). In other words, every solution of the quintic functional equation is called a quintic mapping. In [18], the authors applied the fixed point method to establish the Hyers-Ulam stability of the orthogonally quintic functional equation (1) in Banach spaces and in non-Archimedean Banach spaces.
In this paper, we prove the Hyers-Ulam stability of the quintic functional equation (1) in the normed spaces and non-Archimedean normed spaces via direct way.

2. Stability of (1) in Real Normed Spaces

Throughout this paper, we use the abbreviation for the given mapping $f:\mathcal{X}\longrightarrow\mathcal{Y}$ as follows:

$$\mathcal{D}_qf(x,y):=f(3x+y)-5f(2x+y)+f(2x-y)+10f(x+y)-5f(x-y)$$
$$-10f(y)-f(3x)+3f(2x)+27f(x)$$

for all $x,y\in\mathcal{X}$.
Throughout this section, we assume that $\mathcal{X}$ is a real normed space with norm $\|.\|_{\mathcal{X}}$ and $\mathcal{Y}$ is a real Banach space with norm $\|.\|_{\mathcal{Y}}$. We are going to prove the stability of the quintic functional equation (1) in real normed spaces.

Theorem 2.1. *Let α be a real number and let $f:\mathcal{X}\longrightarrow\mathcal{Y}$ be a mapping for which there exists a function $\phi:\mathcal{X}\times\mathcal{X}\to[-\alpha,\infty)$ such that*

$$\widetilde{\phi}(x,y):=\sum_{n=0}^{\infty}\frac{1}{32^n}\phi(2^nx,2^ny)<\infty, \quad (2)$$

$$\|\mathcal{D}_qf(x,y)\|_{\mathcal{Y}}\leqslant\alpha+\phi(x,y), \quad (3)$$

for all $x,y\in\mathcal{X}$. Then there exists a unique quintic mapping $\mathcal{Q}:\mathcal{X}\longrightarrow\mathcal{Y}$ such that

$$\left\|f(x)-\mathcal{Q}(x)-\frac{10}{31}f(0)\right\|_{\mathcal{Y}}\leqslant\frac{\alpha}{31}+\frac{\widetilde{\phi}(x,0)}{32}, \quad (4)$$

for all $x \in \mathcal{X}$.

Proof. Setting $y = 0$ in (3), we have

$$\|32f(x) - f(2x) - 10f(0)\|_{\mathcal{Y}} \leqslant \alpha + \phi(x, 0),$$

for all $x \in \mathcal{X}$. Hence

$$\left\| f(x) - \frac{1}{32} f(2x) - \frac{10}{32} f(0) \right\|_{\mathcal{Y}} \leqslant \frac{\alpha}{32} + \frac{1}{32}\phi(x, 0), \tag{5}$$

for all $x \in \mathcal{X}$. Replacing x by $2x$ in (5) and continuing this method, we obtain

$$\left\| f(x) - \frac{f(2^n x)}{32^n} - \frac{10}{32}\sum_{k=0}^{n-1} \frac{1}{32^k} f(0) \right\|_{\mathcal{Y}} \leqslant \frac{1}{32}\sum_{k=0}^{n-1} \frac{\alpha}{32^k} + \frac{1}{32}\sum_{k=0}^{n-1} \frac{\phi(2^k x, 0)}{32^k}. \tag{6}$$

Also, we can use induction and triangular inequality to get

$$\begin{aligned} \left\| \frac{f(2^m x)}{32^m} - \frac{f(2^n x)}{32^n} \right\|_{\mathcal{Y}} &\leqslant \frac{10}{32}\sum_{k=m}^{n-1} \frac{1}{32^k} \|f(0)\|_{\mathcal{Y}} \\ &\quad + \frac{1}{32}\sum_{k=m}^{n-1} \frac{\alpha}{32^k} + \frac{1}{32}\sum_{k=m}^{n-1} \frac{\phi(2^k x, 0)}{32^k}, \end{aligned} \tag{7}$$

for all $x \in \mathcal{X}$, and $n > m \geqslant 0$. Thus the sequence $\left\{ \frac{f(2^n x)}{32^n} \right\}$ is Cauchy by (2) and (7). Since $\mathcal{Y}$ is a Banach space, there exists a map $\mathcal{Q}$ so that

$$\lim_{n\to\infty} \frac{f(2^n x)}{32^n} = \mathcal{Q}(x). \tag{8}$$

Letting the limit as n tends to infinity in (6) and applying (8), we can observe that the inequality (4) is true. Now, by replacing x, y by $2^n x, 2^n y$, respectively in (2), we deduce that

$$\mathcal{D}_q\mathcal{Q}(x, y) = \lim_{n\to\infty} \left\| \frac{1}{32^n}\mathcal{D}_q f(x, y) \right\|_{\mathcal{Y}} \leqslant \lim_{n\to\infty} \frac{1}{32^n}\alpha + \lim_{n\to\infty} \frac{\phi(2^n x, 2^n y)}{32^n} = 0,$$

for all $x, y \in \mathcal{X}$. It implies that $\mathcal{Q} : \mathcal{X} \longrightarrow \mathcal{Y}$ is a quintic mapping. For the uniqueness of $\mathcal{Q}$, let $\mathcal{Q}' : \mathcal{X} \longrightarrow \mathcal{Y}$ be another quintic mapping satisfying

(4). Then we have

$$\begin{aligned}\|\mathcal{Q}(x)-\mathcal{Q}'(x)\|_{\mathcal{Y}} &= \frac{1}{32^n}\,\|\mathcal{Q}(2^nx)-\mathcal{Q}'(2^nx)\|_{\mathcal{Y}}\\ &\leqslant \frac{1}{32^n}\left\|\mathcal{Q}(2^nx)-f(2^nx)+\frac{10}{31}f(0)\right\|_{\mathcal{Y}}\\ &+\frac{1}{32^n}\left\|f(2^nx)-\mathcal{Q}'(2^nx)-\frac{10}{31}f(0)\right\|_{\mathcal{Y}}\\ &\leqslant \frac{1}{32^n}\left[\frac{\alpha}{31}+\frac{\widetilde{\phi}(x,0)}{32}\right]\\ &= \frac{\alpha}{31}\frac{1}{32^n}+\frac{1}{32}\sum_{k=0}^{\infty}\frac{1}{32^{n+k}}\phi(2^{n+k}x,0)\\ &= \frac{\alpha}{31}\frac{1}{32^n}+\frac{1}{32}\sum_{k=n}^{\infty}\frac{1}{32^{k}}\phi(2^{k}x,0),\end{aligned}$$

for all $x\in\mathcal{X}$. Taking $n\to\infty$ in the preceding inequality, we have $\mathcal{Q}=\mathcal{Q}'$. This completes the proof. □

Corollary 2.2. *Let α,β,γ,r and s be non-negative real numbers such that $s>0$ and $r,s<5$. Suppose that $f:\mathcal{X}\longrightarrow\mathcal{Y}$ is a mapping fulfilling*

$$\|\mathcal{D}_qf(x,y)\|_{\mathcal{Y}}\leqslant \alpha+\beta\|x\|^r_{\mathcal{X}}+\gamma\|y\|^s_{\mathcal{X}}, \tag{9}$$

for all $x,y\in\mathcal{X}$. Then there exists a unique quintic mapping $\mathcal{Q}:\mathcal{X}\longrightarrow\mathcal{Y}$ such that

$$\|f(x)-\mathcal{Q}(x)\|_{\mathcal{Y}}\leqslant \frac{10\alpha}{651}+\frac{\beta\|x\|^r_{\mathcal{X}}}{32-2^r}, \tag{10}$$

for all $x\in\mathcal{X}$, and all $x\in\mathcal{X}\backslash\{0\}$ if $-5<r<0$. Also, if for each fixed $x\in\mathcal{X}$ the mappings $t\mapsto f(tx)$ from $\mathbb{R}$ to $\mathcal{Y}$ is continuous, then $\mathcal{Q}(tx)=t^5\mathcal{Q}(x)$ for all $x\in\mathcal{X}$ and all $t\in\mathbb{R}$.

Proof. Letting $\phi(x,y)=\beta\|x\|^r_{\mathcal{X}}+\gamma\|y\|^s_{\mathcal{X}}$ in Theorem 3.1, we have

$$\|f(x)-\mathcal{Q}(x)-\frac{10}{31}f(0)\|_{\mathcal{Y}}\leqslant\frac{\alpha}{31}+\frac{\beta\|x\|^r_{\mathcal{X}}}{32-2^r}.$$

It follows from (9) that $\|f(0)\|_{\mathcal{Y}}\leqslant\frac{\alpha}{21}$. By these statements we can get the result. Obviously, if $-5<r<0$, the inequality (10) holds for all $x\in\mathcal{X}\backslash\{0\}$. Now, suppose that $\mathcal{F}$ is any continuous linear functional on $\mathcal{X}$ and x is a fixed element in $\mathcal{X}$. Define the mapping $h : \mathbb{R} \to \mathbb{R}$ via $h(t) = \mathcal{F}[\mathcal{Q}(tx)]$ for each $t \in \mathbb{R}$. It is easily verified that h is a quintic function. Under the hypothesis that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{Y}$, the function h is the pointwise limit of the sequence of continuous functions $\{h_n\}$ in which $h_n(t) = \frac{\mathcal{F}(2^n tx)}{32^n}$, where $n \in \mathbb{N}$ and $t \in \mathbb{R}$. Thus h is a contiuous function and has the form $h(t) = t^5 h(1)$ for all $t \in \mathbb{R}$. Therefore

$$\mathcal{F}[\mathcal{Q}(tx)] = h(t) = t^5 h(1) = t^5 \mathcal{F}[\mathcal{Q}(x)] = \mathcal{F}[t^5 \mathcal{Q}(x)].$$

Since $\mathcal{F}$ is an arbitrary continuous linear functional on $\mathcal{X}$, we have $\mathcal{Q}(tx) = t^5 \mathcal{Q}(x)$ for all $t \in \mathbb{R}$ and $x \in \mathcal{X}$. $\square$

We have the following result which is analogous to Theorem 3.1 for the quintic functional equation (1). The proof is similar but we include its proof.

Theorem 2.3. *Suppose that $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is a mapping for which there exists a function $\phi : \mathcal{X} \times \mathcal{X} \longrightarrow [0, \infty)$ such that*

$$\widetilde{\phi}(x, y) := \sum_{k=1}^{\infty} 32^k \phi(\frac{x}{2^k}, \frac{y}{2^k}) < \infty, \tag{11}$$

$$\|\mathcal{D}_q f(x, y)\|_{\mathcal{Y}} \leqslant \phi(x, y), \tag{12}$$

for all $x, y \in \mathcal{X}$. Then there exists a unique quintic mapping $\mathcal{Q} : \mathcal{X} \longrightarrow \mathcal{Y}$ such that

$$\|f(x) - \mathcal{Q}(x)\|_{\mathcal{Y}} \leqslant \frac{\widetilde{\phi}(x, 0)}{32}, \tag{13}$$

for all $x \in \mathcal{X}$.

Proof. It follows from (11) that $\phi(0, 0) = 0$. Hence from (12) we get $f(0) = 0$. Putting $y = 0$ in (12), we have

$$\|32f(x) - f(2x)\|_{\mathcal{Y}} \leqslant \phi(x, 0),$$

for all $x \in \mathcal{X}$. Interchanging x into $\frac{x}{2}$ in the above inequality, we get

$$\|32f(\frac{x}{2}) - f(x)\|_{\mathcal{Y}} \leqslant \phi(\frac{x}{2}, 0).$$

Using triangular inequality and proceeding this method, we obtain

$$\left\|32^n f(\frac{x}{2^n}) - f(x)\right\|_{\mathcal{Y}} \leqslant \frac{1}{32} \sum_{k=1}^{n} 32^k \phi(\frac{x}{2^k}, 0), \tag{14}$$

for all $x \in \mathcal{X}$. If we show that the sequence $\{32^n f(\frac{x}{2^n})\}$ is Cauchy, then it will be convergent by the completeness of $\mathcal{Y}$. For this, we replace x by $\frac{x}{2^m}$ in (14) and then multiply both side by 32^m, we get

$$\begin{aligned}\Big\|32^{m+n} f(\frac{x}{2^{m+n}}) - 32^m f(\frac{x}{2^m})\Big\|_{\mathcal{Y}} &\leqslant \frac{1}{32}\sum_{k=1}^{n} 32^{k+m}\phi(\frac{x}{2^{k+m}},0)\\ &= \frac{1}{32}\sum_{k=m+1}^{m+n} 32^k \phi(\frac{x}{2^k},0),\end{aligned}$$

for all $x \in \mathcal{X}$, and $n > m > 0$. Thus the above sequence converges to the mapping $\mathcal{Q}$. In fact,

$$\mathcal{Q}(x) := \lim_{n\to\infty} 32^n f(\frac{x}{2^n}).$$

Now, similar to the proof of Theorem 3.1, we can complete the rest of the proof. □

Corollary 2.4. *Let β, γ, r and s be non-negative real numbers such that $r, s > 5$. Suppose that $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is a mapping fulfilling*

$$\|\mathcal{D}_q f(x,y)\|_{\mathcal{Y}} \leqslant \beta\|x\|^r_{\mathcal{X}} + \gamma\|y\|^s_{\mathcal{X}}, \tag{15}$$

for all $x, y \in \mathcal{X}$. Then there exists a unique quintic mapping $\mathcal{Q} : \mathcal{X} \longrightarrow \mathcal{Y}$ such that

$$\|f(x) - \mathcal{Q}(x)\|_{\mathcal{Y}} \leqslant \frac{\beta}{2^r - 32}\|x\|^r_{\mathcal{X}}, \tag{16}$$

for all $x \in \mathcal{X}$, and all $x \in \mathcal{X}\backslash\{0\}$ if $r < -5$. Also, if for each fixed $x \in \mathcal{X}$ the mappings $t \mapsto f(tx)$ from $\mathbb{R}$ to $\mathcal{Y}$ is continuous, then $\mathcal{Q}(tx) = t^5\mathcal{Q}(x)$ for all $x \in \mathcal{X}$ and all $t \in \mathbb{R}$.

Proof. First, we note that if we put $x = y = 0$ in (15), we have $f(0) = 0$. Defining $\phi(x,y) = \beta\|x\|^r_{\mathcal{X}} + \gamma\|y\|^s_{\mathcal{X}}$ in Theorem 3.3, we can obtain (16). The rest of the proof is obvious by the proof of Corollary 3.4. □

The quintic functional equation (1) can be superstable under some conditions. It is shown in the next result. Recall that a functional equation is called *superstable* if every approximately solution is an exact solution of it.

Corollary 2.5. *Let r, s and α be a non-negative real numbers such that $r + s \neq 5$. If $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is a mapping such that*

$$\|\mathcal{D}_q f(x,y)\| \leqslant \alpha\|y\|^s \text{ (or } \alpha\|x\|^r\|y\|^s), \tag{17}$$

for all $x, y \in \mathcal{X}$, then the mapping f is quintic. Also, if for each fixed $x \in \mathcal{X}$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to $\mathcal{Y}$ is continuous, then $f(tx) = t^5 f(x)$ for all $x \in \mathcal{X}$ and all $t \in \mathbb{R}$.

Proof. The inequality (17) shows that $f(0) = 0$. Putting $y = 0$ in (17), we get $f(2x) = 32f(x)\,(x \in \mathcal{X})$, and so $f(x) = \frac{f(2^n x)}{32^n}$ for all $x \in \mathcal{X}$ and $n \in \mathbb{N}$. Letting $\phi(x, y) = \alpha\|y\|^s$ (or $\phi(x, y) = \alpha\|x\|^r\|y\|^s$) in Theorems 2.1 and 2.3, we can see that $\mathcal{Q} = f$ is a quintic mapping. □

3. Stability of (1) in Non-Archimedean Normed Spaces

We first recall some definitions and basic facts in the non-Archimedean normed spaces setting.

By a non-Archimedean field we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathbb{K}$ into $[0, \infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r||s|$, and $|r + s| \leqslant \max\{|r|, |s|\}$ for all $r, s \in \mathbb{K}$. Clearly $|1| = |-1| = 1$ and $|n| \leqslant 1$ for all $n \in \mathbb{N}$.

Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\| : X \longrightarrow \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|rx\| = |r|\|x\|, \quad (x \in X, r \in \mathbb{K})$;

(iii) the strong triangle inequality (ultrametric); namely,

$$\|x + y\| \leqslant \max\{\|x\|, \|y\|\}, \qquad (x, y \in X).$$

Then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space*. Due to the fact that

$$\|x_n - x_m\| \leqslant \max\{\|x_{j+1} - x_j\|; m \leqslant j \leqslant n - 1\}, \quad (n \geqslant m),$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space X. By a complete non-Archimedean normed space we mean one in which every Cauchy sequence is convergent.

In [14], Hensel discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. The most interesting example of non-Archimedean normed spaces is p-adic numbers. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: for all $x, y > 0$, there exists an integer n such that $x < ny$.

Let p be a prime number. For any non-zero rational number $x = p^r \frac{m}{n}$ in which m and n are coprime to the prime number p. Consider the p-adic absolute value $|x|_p = p^{-r}$ on $\mathbb{Q}$. It is easy to check that $|\cdot|$ is a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $|\cdot|$ which is denoted by $\mathbb{Q}_p$ is said to be the p-adic number field. One should remember that if $p > 2$, then $|2^n|$ $= 1$ in for all integers n. In [12], the stability of some functional equations in non-Archimedean normed spaces are investigated (see also [9] and [21]).
From now on, we assume that X is a real vector space and Y is a complete non-Archimedean normed space unless otherwise stated explicitly. In the upcoming theorem, we prove the stability of the functional equation (1) in non-Archimedean normed spaces.

Theorem 3.1. *Let $\phi : X \times X \longrightarrow [0,\infty)$ be a function such that*

$$\lim_{k\to\infty} \frac{1}{|32|^k}\varphi(2^k x, 2^k y) = 0, \tag{18}$$

for all $x, y \in X$. Suppose that $f : X \longrightarrow Y$ is a mapping satisfying the inequality

$$\|\mathcal{D}_q f(x,y)\| \leqslant \varphi(x,y), \tag{19}$$

for all $x, y \in X$. Then there exists a unique quintic mapping $Q : X \longrightarrow Y$ such that

$$\|f(x) - Q(x)\| \leqslant \frac{1}{|32|}\widetilde{\varphi}(x), \tag{20}$$

for all $x \in X$ where $\widetilde{\varphi}(x) = \sup\left\{\frac{\varphi(2^j x,0)}{|32|^j} : j \in \mathbb{N}\cup\{0\}\right\}$.

Proof. Putting $y = 0$ in (19), we have

$$\left\|f(x) - \frac{f(2x)}{32}\right\| \leqslant \frac{1}{|32|}\varphi(x,0), \tag{21}$$

for all $x \in X$. Replacing x by $2^n x$ in (21) and then dividing both sides by $|32|^{n+1}$, we get

$$\left\|\frac{1}{32^n} f(2^n x) - \frac{1}{32^{n+1}} f(2^{n+1}x)\right\| \leqslant \frac{1}{|32|^{n+1}}\varphi(2^n x, 0), \tag{22}$$

for all $x \in X$ and all non-negative integers n. Thus the sequence $\left\{\frac{f(2^n x)}{32^n}\right\}$ is Cauchy by (18) and (22). Completeness of non-Archimedean space Y allows us to assume that there exists a mapping Q so that

$$\lim_{n\to\infty} \frac{f(2^n x)}{32^n} = Q(x). \tag{23}$$

For each $x \in X$ and non-negative integers n, we have

$$\begin{aligned} \left\| f(x) - \frac{f(2^n x)}{32^n} \right\| &= \left\| \sum_{j=0}^{n-1} \frac{f(2^j x)}{32^j} - \frac{f(2^{j+1} x)}{32^{j+1}} \right\| \\ &\leqslant \max \left\{ \left\| \frac{f(2^j x)}{32^j} - \frac{f(2^{j+1} x)}{32^{j+1}} \right\| : 0 \leqslant j < n \right\} \\ &\leqslant \frac{1}{|32|} \max \left\{ \frac{\varphi(2^j x, 0)}{|32|^j} : 0 \leqslant j < n \right\}. \end{aligned} \tag{24}$$

Taking n tends to approach infinity in (24) and applying (23), we can see that the inequality (20) holds. It follows from (18), (19) and (23) that for all $x, y \in \mathcal{X}$,

$$\|\mathcal{D}_q Q(x,y)\| = \lim_{n\to\infty} \frac{1}{|32|^n} \|\mathcal{D}_q f(2^n x, 2^n y)\| \leqslant \lim_{n\to\infty} \frac{1}{|32|^n} \varphi(2^n x, 2^n y) = 0,$$

Hence, the mapping Q satisfies (1) and thus it is quintic. Now, let $\mathcal{Q} : X \longrightarrow Y$ be another quintic mapping satisfying (20). Then we have

$$\begin{aligned} \|Q(x) - \mathcal{Q}(x)\| &= \lim_{k\to\infty} \frac{1}{|32|^k} \|Q(2^k x) - \mathcal{Q}(2^k x)\| \\ &\leqslant \lim_{k\to\infty} \frac{1}{|32|^k} \max\{\|Q(2^k x) - f(2^k x)\|, \\ &\qquad\qquad \|f(2^k x) - \mathcal{Q}(2^k x)\|\} \\ &\leqslant \frac{1}{|32|} \lim_{k\to\infty} \lim_{n\to\infty} \max \left\{ \frac{\varphi(2^j x, 0)}{|32|^j} : k \leqslant j < n + k \right\} \\ &= \frac{1}{|32|} \lim_{k\to\infty} \sup \left\{ \frac{\varphi(2^j x, 0)}{|32|^j} : k \leqslant j < \infty \right\} = 0. \end{aligned}$$

for all $x \in X$. This shows the uniqueness of Q. $\square$

Corollary 3.2. *Let $\alpha > 0$, X be a non-Archimedean normed space and let $f : X \longrightarrow Y$ be a mapping satisfying the inequality*

$$\|\mathcal{D}_q f(x,y)\| \leqslant \alpha \left(\Gamma(\|x\|) + \Gamma(\|y\|) \right), \tag{25}$$

for all $x, y \in \mathcal{X}$. If $\Gamma : [0,\infty) \longrightarrow [0,\infty)$ is a function satisfying $\Gamma(|r|s) \leqslant \Gamma(|r|)\Gamma(s)$ for all $r, s \in [0,\infty)$ for which $\Gamma(|2|) < |32|$, then there exists a unique quintic mapping $Q : X \longrightarrow Y$ such that

$$\|f(x) - Q(x)\| \leqslant \frac{\alpha \Gamma(\|x\|)}{|32|},$$

for all $x \in X$.

Proof. Defining $\varphi : X \times X \longrightarrow [0,\infty)$ via $\phi(x,y) = \alpha\left(\Gamma(\|x\|) + \Gamma(\|y\|)\right)$, we get

$$\lim_{n\to\infty} \frac{1}{|32|^n}\phi(2^n x, 2^n y) \leqslant \lim_{n\to\infty} \left(\frac{\Gamma(|2|)}{|32|}\right)^n \varphi(x,y) = 0.$$

for all $x, y \in X$. We also have

$$\widetilde{\varphi}(x) = \sup\left\{\frac{\varphi(2^j x, 0)}{|32|^j} : 0 \leqslant j < \infty\right\} = \varphi(x,0) = \alpha\left(\Gamma(\|x\|)\right).$$

for all $x \in X$. Now, Theorem 3.1 implies the desired result. □

We have the following result which is analogous to Theorem 3.1 for the functional equations (1). We bring the proof for the sake of completeness.

Theorem 3.3. *Let* $\varphi : X \times X \longrightarrow [0,\infty)$ *be a function such that*

$$\lim_{k\to\infty} |32|^k \phi(\frac{x}{2^k}, \frac{y}{2^k}) = 0, \tag{26}$$

for all $x, y \in X$. *Suppose that* $f : X \longrightarrow Y$ *is a mapping satisfying the inequality*

$$\|\mathcal{D}_q f(x,y)\| \leqslant \varphi(x,y), \tag{27}$$

for all $x, y \in X$. *Then there exists a unique quintic mapping* $Q : X \longrightarrow Y$ *such that*

$$\|f(x) - Q(x)\| \leqslant \widetilde{\varphi}(x), \tag{28}$$

for all $x \in X$ *where* $\widetilde{\varphi}(x) = \sup\left\{|32|^j \varphi(\frac{x}{2^{j+1}}, 0) : j \in \mathbb{N} \cup \{0\}\right\}$.

Proof. Similar to the proof of Theorem 3.1, we have

$$\|32f(x) - f(2x)\| \leqslant \varphi(x,0), \tag{29}$$

for all $x \in X$. If we replace x by $\frac{x}{2^{n+1}}$ in the inequality (29) and then multiply both sides of the result to $|32|^n$, we get

$$\|32^{n+1} f(\frac{x}{2^{n+1}}) - 32^n f(\frac{x}{2^n})\| \leqslant |32|^n \varphi(\frac{x}{2^{n+1}}, 0), \tag{30}$$

for all $x \in X$ and all non-negative integers n. Thus, we conclude from (26) and (30) that the sequence $\left\{32^n f(\frac{x}{2^n})\right\}$ is Cauchy. Since Y is a non-Archimedean Banach space, this sequence converges in Y to the mapping Q. Indeed,

$$Q(x) = \lim_{n\to\infty} 32^n f(\frac{x}{2^n}), \qquad (x \in X). \tag{31}$$

Using induction and (29), one can show that

$$\Big\|32^n f(\frac{x}{2^n}) - f(x)\Big\| \leqslant \max\Big\{|32|^j \varphi(\frac{x}{2^{j+1}}, 0) : 0 \leqslant j < n\Big\}, \tag{32}$$

for all $x \in X$ and non-negative integers n. Since the right hand side of the inequality (32) tends to 0 as n to approach infinity, by applying (31), we deduce the inequality (28). Now, similar to the proof of Theorem 3.1, we can complete the rest of the proof. □

Corollary 3.4. *Let α, r and s be positive real numbers such that $r, s \neq 5$ and $|2| < 1$. Suppose that X is a non-Archimedean normed space and $f : X \longrightarrow Y$ is a mapping fulfilling*

$$\|\mathcal{D}_q f(x, y)\| \leqslant \alpha(\|x\|^r + \|y\|^s),$$

for all $x, y \in X$. Then there exists a unique quintic mapping $Q : X \longrightarrow Y$ such that

$$\|f(x) - Q(x)\| \leqslant \begin{cases} \frac{\alpha\|x\|^r}{|32|} & r, s > 5 \\ \\ \frac{\alpha\|x\|^r}{|2|^r} & r, s < 5 \end{cases}$$

for all $x \in X$.

Proof. The result follows from Theorems 3.1 and 3.3 by letting $\varphi(x, y) = \alpha(\|x\|^r + \|y\|^s)$. □

In the next result, we prove the superstability of the functional equations (1) under some conditions.

Corollary 3.5. *Let α, r and s be positive real numbers such that $r + s \neq 5$ and $|2| < 1$. Suppose that $\mathcal{X}$ is a non-Archimedean normed space and $f : X \longrightarrow Y$ is a mapping fulfilling*

$$\|\mathcal{D}_q f(x, y)\| \leqslant \alpha\|x\|^r\|y\|^s,$$

for all $x, y \in X$. Then f is a quintice mapping.

Proof. Taking $\varphi(x, y) = \alpha\|x\|^r\|y\|^s$ in Theorems 3.1 and 3.3, we can obtain the required result. □

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