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On Some Kinds of Injectivity of GHS-Acts

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Abstract. In the theory of S-acts, there exists a plethora of compelling findings that establish connections between regularity in a monoid S and injectivity (as well as other notions derived from injectivity) of S-acts. In this paper, we introduce the notion of strongly regular elements in a hypermonoid as a generalization of regular elements in a monoid and extend several classic results to hypermonoids and GHS-acts over hypermonoids. Particularly, we show that a hypermonoid S is strongly regular and injective GHS-act if and only if all right hyperideals of S are C-injective.

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1 Introduction and Preliminaries

Let S be a set. A binary operation takes a pair of elements from S and gives an element in S. Alternatively, a hyperoperation can be defined on S in such a way that associates a pair of elements of S to a non-empty subset of S. The second scenario corresponds to algebraic hyperstructures including hypersemigroups, hypermonoids and hypergroups. In the case of hyperstructures, actions are generalized to hyperactions. Madanshekaf and Ashrafi [3] defined a generalized action of a hypergroup on a set and determined the order of some hypergroups concerning generalized permutations. Hyperactions over hypermonoids were first introduced in [5, 7, 8], while hyperactions over monoids were defined earlier in [4, 6]. The notion of C-injective S-acts was defined and studied in [9], and in [1], C-injectivity was generalized to GHS-acts, and a new concept called semi-injectivity was introduced and established.

There exist classic results that establish connections between injectivity (and related concepts such as W-injectivity and PW-injectivity) and regularity of elements in monoids. This paper introduces a suitable generalization of regularity, called strongly regularity, and extends several classic results to hypermonoids and GHS-acts.

We first define strongly regular elements and strongly regular hypersemigroups. Then we explore the relationship between PW-injectivity and strongly regularity. Specifically, we demonstrate that a right hyperideal $s \circ S$ of a hypermonoid S is PW-injective if and only if s is a strongly regular element of S. Furthermore, we show that S is strongly regular if and only if all GHS-acts are PW-injective. We then examine the relationship between C-injectivity and strongly regularity, showing that S is strongly regular and injective if and only if all right hyperideals of S are C-injective.

In the following, we recall some fundamental concepts related to hypersemigroups and hypermonoids needed in the sequel (see [5, 7]).

Let S be a semigroup. An element $s \in S$ is called *regular* if there exists an element $t \in S$ with $s = sts$. In this case, st is an idempotent. If all elements of S are regular, then S is called a *regular semigroup*. For an arbitrary set A, we denote the powerset of A by $\mathcal{P}(A)$, and as a convention let $\mathcal{P}^*(A) = \mathcal{P}(A) \backslash \{\emptyset\}$. For a non-empty set S, let $\circ : S \times S \to \mathcal{P}^*(S)$ be a map for which $s \circ (t \circ u) = (s \circ t) \circ u$ for any $s, t, u \in S$, where $S' \circ s = \bigcup_{s' \in S'} s' \circ s$ and $s \circ S' = \bigcup_{s' \in S'} s \circ s'$ for any $S' \subseteq S$ and $s \in S$. Then $(S, \circ)$ is called a *hypersemigroup* (or *semihypergroup*). Let S be a hypersemigroup for which there exists $e \in S$ with $s \circ e = e \circ s$ and $s \in s \circ e$ for any $s \in S$. Then S is called a *hypermonoid*, and e is called an *identity element* of S. An identity element e of a hypermonoid S is called a *pure identity* if $s \circ e = \{s\}$ for any $s \in S$. In this paper, we assume that any hypermonoid has a (unique) pure identity element denoted by 1. Let S be a hypermonoid and I be a non-empty subset of S. Then I is called a *right hyperideal* of S if $I \circ S = \bigcup_{s \in S} I \circ s \subseteq I$. The concept of being regular has been generalized for hypersemigroups. Indeed, an element s of a hypersemigroup $(S, \circ)$ is called *regular* if $s \in s \circ S \circ s$, and a hypersemigroup S is called *regular* if all elements of S are regular. Also by an *idempotent* element of a hypersemigroup S we mean $s \in S$ with $s \circ s = \{s\}$.

For the concept of acts over monoids and their properties, we recommend [2]. A generalization of this concept within the context of hyperstructures was introduced and extensively studied in [5, 7], as follows.

Let $(S, \circ)$ be a hypermonoid, X be a non-empty set and $* : X \times S \to \mathcal{P}^*(X)$ have the following properties for any $x \in X$ and $s, t \in S$:

(1) $(x * s) * t = x * (s \circ t)$, and

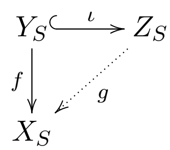
(2) $x \in x * e$ for any $x \in X$,

where $x' * S' = \bigcup_{s' \in S'} x' * s'$ and $X' * s' = \bigcup_{x'' \in X'} x'' * s'$ for any $x' \in X'$, $s' \in S'$, $X' \subseteq X$ and $S' \subseteq S$. Then $(X_S, *)$ is called a *(right) generalized hyper S-act* (or simply a *GHS-act*). Let X_S be a GHS-act and $X' \subseteq X$. If $X' * S \subseteq X'$, then $(X'_S, *)$ is a GHS-act which is called a *GHS-subact* of X_S, where by $X' * S'$ we mean $\bigcup_{x' \in X'} x' * S'$ for any $S' \subseteq S$. Note that the GHS-subacts of S_S are the right hyperideals of S. A GHS-act X_S is called *pure* if $x * \{e\} = \{x\}$ for any $x \in X$. In the sequel, all considered GHS-acts are pure. As a convenience, a singleton set $\{x\}$ may be displayed by just its element x where x is an element of a GHS-act X_S.

Let $\phi : X_S \to Y_S$ be a map between two GHS-acts X_S and Y_S. Then by $\phi(x * s)$ we mean the set $\{\phi(x') : x' \in x * s\}$ for any $x \in X$ and $s \in S$. The map $\phi : X_S \to Y_S$ is called a *GHS-homomorphism* whenever $\phi(x * s) = \phi(x) * s$ for any $x \in X$ and $s \in S$. Let X_S be a GHS-act and

Y_S be a GHS-subact of X_S. A GHS-homomorphism $f : X_S \to Y_S$ is called a *retraction* if $f\iota = \mathrm{id}_{Y_S}$, where $\iota : Y_S \hookrightarrow X_S$ is the inclusion map. If there exists a retraction $f : X_S \to Y_S$, then Y_S is called a *retract* of X_S.

Let X_S be a GHS-act. Then X_S is called *injective* if for any GHS-monomorphism $\iota : Y_S \hookrightarrow Z_S$ and any GHS-homomorphism $f : Y_S \to X_S$, there exists a GHS-homomorphism $g : Z_S \to X_S$ commuting the following diagram:



In the above definition, it suffices to take Y_S as a GHS-subact of Z_S (see [5, Lemma 1]). Considering the above assumptions, X_S is called *F-injective* (*C-injective*) if Y_S is finitely generated (cyclic). Further, X_S is called *PW-injective* if $Y_S = I_S$, where I is a cyclic right hyperideal of S and $Z_S = S_S$. Note that any cyclic right hyperideal of S is of the form $s \circ S$ for some $s \in S$.

2 Main Results

In this section, we explore the relationships between injectivity and its derived concepts in terms of strongly regularity.

In the following, we introduce a new concept which extends the notion of regularity in hypersemigroups.

Definition 2.1. Let $(S, \circ)$ be a hypersemigroup. An element $s \in S$ is called *strongly regular* whenever there exists an idempotent element $z \in s \circ S$ for which $z \circ s = s$, and for any $t \in S$, $z \circ t$ is a singleton set. If all elements of S are strongly regular, then we call S a *strongly regular hypersemigroup*. Moreover, if S is a hypermonoid, then it is said to be a *strongly regular hypermonoid*. Any strongly regular hypersemigroup (hypermonoid) is clearly regular.

For instance, the hypermonoid $S = \{1, s\}$, where 1 is the pure identity element, with $s \circ s = \{1, s\}$ is strongly regular.

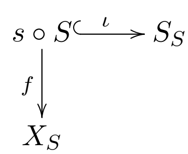
Let S be a semigroup. Define $s \circ t = \{s, t, st\}$ for any $s, t \in S$. Then $(S, \circ)$ is a hypersemigroup. The following example serves to demonstrate that the concepts of regularity and strong regularity are not equivalent in hypersemigroups.

Example 2.2. Consider $\mathbb{N}$ with the above hypersemigroup structure (i.e. $m \circ n = \{m, n, mn\}$ for any $m, n \in \mathbb{N}$). Let $n \in \mathbb{N} \backslash \{1\}$. Then $n \circ 1 \circ n = \{n, 1\} \circ n = (n \circ n) \cup (1 \circ n) = \{n, n^2\} \cup \{1, n\} = \{1, n, n^2\}$ and so $n \in n \circ 1 \circ n \subseteq n \circ \mathbb{N} \circ n$, which means that n is regular. But n is not strongly regular, otherwise, there exists an idempotent $m \in \mathbb{N}$ with $m \in n \circ \mathbb{N}$ and $n = m \circ n = \{m, n, mn\}$. This follows that $mn = n = m$, and hence $n = 1$ which is a contradiction.

Recall from [2, Proposition 3.3.2] that an S-act A_S over a monoid S is principally weakly injective if and only if for any $s \in S$ and any homomorphism $f : sS \to A_S$, there exists $z \in A_S$ with $f(u) = zu$ for any $u \in sS$. In the following, we study this property for GHS-acts where S is a hypermonoid.

Proposition 2.3. *Let S be a hypermonoid. A GHS-act X_S is PW-injective if and only if for any $s \in S$ and any GHS-homomorphism $f : s \circ S \to X_S$, there exists $z \in X_S$ such that $z * t$ is a singleton set for any $t \in S$, and $f(u) = z * u$ for any $u \in s \circ S$.*

Proof. Let X_S be a PW-injective GHS-act. Assume that $s \in S$ and $f : s \circ S \to X_S$ is a GHS-homomorphism. We consider the following diagram:



By PW-injectivity of X_S, there exists a GHS-homomorphism $g : S_S \to X_S$ commuting the diagram. Now let $z = g(1)$. For any $t \in S$, $z * t = g(1) * t = g(1 \circ t) = g(t)$, and hence $z * t$ is a singleton set. Moreover, for any $u \in s \circ S$,

$$f(u) = g(u) = g(1 \circ u) = g(1) * u = z * u.$$

Conversely, let $s \in S$ and consider the above diagram. By the assumption, there exists $z \in X_S$ such that $z * t$ is a singleton set for any $t \in S$ and $f(u) = z * u$ for any $u \in s \circ S$. Define $g : S_S \to X_S$ by setting $g(t) = z * t$ for any $t \in S$. Then g is a GHS-homomorphism commuting the diagram, and hence X_S is PW-injective. □

Corollary 2.4. *Let S be a hypermonoid and $s \in S$. If $s \circ S$ is PW-injective, then s is a strongly regular element of S.*

Proof. Let $s \in S$. Considering $f = \mathrm{id} : s \circ S \to s \circ S$, the identity function, it follows from Proposition 2.3 that there exists an element $z \in s \circ S$ for which $z \circ t$ is a singleton set for any $t \in S$ and $u = f(u) = z \circ u$ for any $u \in s \circ S$. This clearly gives that $s = z \circ s$. It remains to show that z is an idempotent. By the proof of Proposition 2.3, there exists a GHS-homomorphism $g : S_S \to s \circ S$ with $g\iota = f$ and $z = g(1)$, where $\iota : s \circ S \hookrightarrow S_S$ is the inclusion. We have

$$z \circ z = g(1) \circ z = g(1 \circ z) = g(z) = g\iota(z) = f(z) = z.$$

Thus s is strongly regular. □

The following theorem characterizes all hypermonoids S for which any GHS-act is PW-injective.

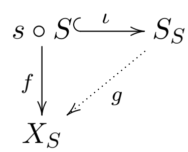
Theorem 2.5. *Let S be a hypermonoid. Then the following are equivalent:*

1. *All GHS-acts are PW-injective.*
2. *All right hyperideals of S are PW-injective.*
3. *All finitely generated right hyperideals of S are PW-injective.*
4. *All principal right hyperideals of S are PW-injective.*
5. *S is strongly regular.*

Proof. The implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are clear.

(4) ⇒ (5) Let $s \in S$. Then $s \circ S$ is PW-injective by the assumption. Then, using Corollary 2.4, s is strongly regular.

(5) ⇒ (1) Let X_S be a GHS-act. We show that X_S is PW-injective. Let $s \in S$ and $f : s \circ S \to X_S$ be a GHS-homomorphism. We show that there exists a GHS-homomorphism $g : S_S \to X_S$ such that $g\iota = f$:



Using the assumption, there exists an idempotent element $z \in s \circ S$ in such a way that $z \circ s = s$ and $z \circ t$ is a singleton set for any $t \in S$. We claim that $s \circ S = z \circ S$. To see this, we note that $z \in s \circ S$, and this implies that $z \circ S \subseteq s \circ S$. For the reverse inclusion, it follows from $z \circ s = s$ that $s \in z \circ S$, which implies that $s \circ S \subseteq z \circ S$. Now define $g : S_S \to X_S$ by setting $g(t) = f(z \circ t)$ for any $t \in S$. It can be easily seen that g is a GHS-homomorphism. For any $z \circ t \in z \circ S = s \circ S$, we have

$$g\iota(z \circ t) = g(z \circ t) = f(z \circ (z \circ t)) = f((z \circ z) \circ t) = f(z \circ t),$$

and consequently, X_S is PW-injective. □

In the following, we show that the injectivity of all principal right hyperideals of a hypermonoid S follows from the injectivity of S if and only if S is strongly regular. An analogous version of this result for S-acts, where S is a monoid, can be found in [2, Theorem 4.5.10].

Corollary 2.6. *Let S be a hypermonoid. Then the following are equivalent:*

1. *All principal right hyperideals of S are injective.*
2. *S is strongly regular and injective.*

Proof. (1) ⇒ (2) Since injectivity implies PW-injectivity, using Theorem 2.5, S is strongly regular. The injectivity of S_S follows from the assumption.

(2) ⇒ (1) Let $s \in S$. We show that $s \circ S$ is injective. Since s is strongly regular by hypothesis, there exists an idempotent element $z \in s \circ S$ such that $z \circ s = s$ and $z \circ t$ is a singleton set for any $t \in S$. As

in the proof of (5) $\Rightarrow$ (1) in Theorem 2.5, we have $s \circ S = z \circ S$. Now we define $f : S_S \to s \circ S$ by setting $f(t) = z \circ t$ for any $t \in S$. Note that

$$f(t_1 \circ t_2) = z \circ (t_1 \circ t_2) = (z \circ t_1) \circ t_2 = f(t_1) \circ t_2$$

for any $t_1, t_2 \in S$. Thus f is a GHS-homomorphism. Now consider the inclusion $\iota : s \circ S \hookrightarrow S_S$. Then

$$f\iota(z \circ t) = f(z \circ t) = z \circ (z \circ t) = (z \circ z) \circ t = z \circ t$$

for any $z \circ t \in z \circ S = s \circ S$. This gives that $s \circ S$ is a retract of S_S. Now it follows from the injectivity of S_S and [5, Theorem 3.1] that $s \circ S$ is injective as well. $\square$

A classification of monoids in terms of the C-injectivity or injectivity of a certain class of their right hyperideals was obtained in [9, Theorem 9]. In the following, we generalize this result to hypermonoids.

Theorem 2.7. *Let S be a hypermonoid. The following statements are equivalent:*

1. *All right hyperideals of S are C-injective.*
2. *All finitely generated right hyperideals of S are C-injective.*
3. *All principal right hyperideals of S are C-injective.*
4. *All principal right hyperideals of S are injective.*
5. *All principal right hyperideals of S are F-injective.*
6. *S is strongly regular and injective.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) $\Rightarrow$ (3) are clear, and by Corollary 2.6, (4) $\Leftrightarrow$ (6).

(3) $\Rightarrow$ (1) Let X_S be a cyclic GHS-subact of a GHS-act Y_S. Let also I be a right hyperideal of S and $f : X_S \to I_S$ be a GHS-homomorphism. We show that f is extended to a GHS-homomorphism from Y_S to I_S. Since X_S is cyclic, $J := f(X)$ is a principal right hyperideal of I. Consider the GHS-homomorphism $\tilde{f} = f : X_S \to J_S$. By the assumption, J_S is C-injective, and so $\tilde{f}$ is extended to a GHS-homomorphism

$\tilde{g} : Y_S \to J_S$. Now we define $g : Y_S \to I_S$ by setting $g(y) = \tilde{g}(y)$ for any $y \in Y_S$. Then g is a GHS-homomorphism extending f. Hence, I_S is C-injective.

(3) $\Rightarrow$ (4) This follows from [1, Theorem 2.3]. □

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