Journal of Mathematical Extension
Vol. 18, No. 4, (2024) (5)1-28
URL: https://doi.org/10.30495/JME.2024.2971
ISSN: 1735-8299
Original Research Paper

Existence Results of a Nonlinear Fractional Integral Equation of Two Variables in the Space of Regulated Functions and Numerical Methods

S. Alipour Fatideh
Mashhad Branch, Islamic Azad University

M. Khanehgir*
Mashhad Branch, Islamic Azad University

R. Allahyari
Mashhad Branch, Islamic Azad University

M. Mehrabinezhad
Mashhad Branch, Islamic Azad University

H. Amiri Kayvanloo
Mashhad Branch, Islamic Azad University

Abstract. In the present work, we first characterize relatively compact subsets of the space of regulated functions defined on Davison spaces. Then, we introduce a measure of noncompactness on this kind of spaces. By using the technique of the measure of noncompactness and a fixed point theorem of Darbo type, we study the existence of solutions of a nonlinear fractional integral equation of two variables in the space of regulated functions. Also, we add some illustrative examples in verification of our existence results. Eventually, we apply the Adomian

Received: December 2023; Accepted: July 2024
*Corresponding Author

decomposition method to find solution of the problem with high accuracy.

AMS Subject Classification: 47H10; 47H08; 26A33; 46A99
Keywords and Phrases: Darbo's fixed point theorem, Davison space, Fractional integral equation, Measure of noncompactness, Regulated function

1 Introduction

The fractional calculus, for the sake of its high accuracy, is a significant topic that is considered as a powerful tool for studying natural phenomena in many fields including biology, biophysics, electrical networks, rheology and dynamical systems coupled with lots of mathematical and physical modelings formulated by a vast variety of fractional operators (see [3, 5, 7, 8, 20, 30, 34, 35, 36]). Recently, developments in the theory of fractional integral equations have flourished in the various fields of physics, chemistry, engineering, and economics (see [16, 19, 24, 25, 26, 29, 31, 32, 37]). An efficient approach for solving fractional differential and integral equations consists in applying new measures of noncompactness for some Banach spaces. In this way, extensions of some fixed point theorems such as extension of Darbo's fixed point theorem via measure of noncompactness can be very useful (see [4, 11, 13, 23, 33]). The measure of noncompactness theory (MNC for short) [10] is a remarkable branch of the nonlinear functional analysis. It permits us to select an important class of operators as generalizations of compact operators.

The notion of a regulated function was presented in the middle of the twentieth century [6]. Subsequently, some mathematicians introduced this notion from different points of view and obtained some of its applications [17, 18, 21, 22]. Particularly the approach offered in [17] seems to be appropriate and transparent. In 2018, Banaś and Zajac [12] formulated a criterion for relative compactness in the space of regulated functions on a bounded interval $J = [a, b]$, so-called regulated functions, and proved this characterization is equivalent to a known criterion obtained earlier by Frankova. Next, in 2019, Olszowy [28] constructed arithmetically convenient measures of noncompactness in the spaces of regulated functions $R(J, E)$ (E is a Banach space) and $R(J)$. In 1979, Davison generalized regulated functions on a topological space [14] and proved some remarkable facts concerning these new types of functions. However, up to now, it has not been defined a measure of noncompactness in the space of regulated functions $S(Y, Z)$ for a Davison space (Y, τ, σ). In this work, we are going to construct such measure of noncompactness. On the other hand, fractional integral equations appear in various real-life problems and enormous areas of science, and bioengineering (see [1, 15]). Some scholars focus on such type of equations with the help of fixed point theory. Rabbani et al. [31] studied the existence of solution of a fractional integral equation in two variables, by using the technique of measure of noncompactness. In the present manuscript, we are motivated to discuss the solvability of the following nonlinear fractional integral equation of two variables

$$x(t,s) = G(t,s) + F\Big(t, s, \int_0^t \int_0^s \frac{u(t,s,v,w,x(v,w))}{(s-v)^{1-\alpha}(t-w)^{1-\beta}} dvdw, x(t,s)\Big) \tag{1}$$

in the space of regulated functions defined on a Davison space, where $\alpha, \beta \in (0,1)$ and $t, s \in [0,T]$. This type of integral equations may be occurred in the mathematical physics modeling, engineering, and so on (see [1] and its monograph).

We organize this paper as follows. In Section 2, we collect some auxiliary facts which will be needed further on. In Section 3, we aim to formulate a standard approach to characterize relatively compact sets in the space of regulated functions defined on a Davison space, and then we construct a measure of noncompactness on this space. In Section 4, using a fixed point theorem of Darbo type, we review the existence of solutions of the fractional integral equation in two variables (1) in a regulated space. We also provide some illustrative examples to show applicability of our existence theorem. Eventually, in Section 5 we use the Adomian decomposition method to find solution of the problem.

2 Preliminaries

Here, we preliminarily give some auxiliary facts which will be needed in our further considerations.

Definition 2.1. [12] A function $x : [a, b] \to \mathbb{R}$ is said to be a regulated function if for any $t \in [a, b)$ the right-sided limit $x(t^+) := \lim_{s \to t^+} x(s)$ exists and for every $t \in (a, b]$ the left-sided limit $x(t^-) := \lim_{s \to t^-} x(s)$ exists.

For instance, the greatest integer function, step functions, càdlàag functions, functions of bounded variation, and continuous functions are regulated functions on an interval $[\imath, \jmath]$ (see [12]). The set of all regulated functions is denoted by $R([a, b])$.

Theorem 2.2. *[12] Let X be a bounded subset of the space $R([a, b])$. The set X is relatively compact in $R([a, b])$ if and only if X is equiregulated on the interval $[a, b]$ i.e., the following two conditions are satisfied:*
(a) For all $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x \in X$, $t \in (a, b]$ and $u, v \in (t - \delta, t) \cap [a, b]$ we have $|x(u) - x(v)| \leq \varepsilon$.
(b) For all $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x \in X$, $t \in [a, b)$ and $u, v \in (t, t + \delta) \cap [a, b]$ we have $|x(u) - x(v)| \leq \varepsilon$.

In 1979, Davison introduced the concept of Davison space and generalized regulated functions in this new setting [14].

Definition 2.3. [14] A triple (Y, τ, σ) is called a Davison space if (Y, τ) is a topological space and σ is an algebra of sets on Y, so that $\tau \bigcap \sigma$ is a base for τ.

Example 2.4. [27] (i) Suppose that τ is the usual topology on $\mathbb{R}$ and ϖ is the algebra generated by the intervals of the form $(\imath, \jmath)$, $-\infty \leq \imath < \jmath \leq +\infty$. Then $(\mathbb{R}, \tau, \varpi)$ is a Davison space.
(ii) Assume that $\Omega \subseteq \mathbb{R}^2$ is a compact set. Also, assume that τ is the usual topology on $\mathbb{R}^2$ and ω is the algebra of sets with finite perimeter. Then this is a Davison space.

Definition 2.5. [14] Let (Y, τ, σ) be a Davison space and $(Z, \|.\|_Z)$ be a normed space. Then $h : Y \to Z$ is called σ-regulated at $y \in Y$ if for each $\varepsilon > 0$, there exist $B_1, B_2, \ldots, B_n \in \sigma$ such that $y \in \text{int}(\bigcup_{i=1}^{n} B_i)$ and $\|h(y_1) - h(y_2)\|_Z < \varepsilon$ for each $y_1, y_2 \in B_i$ $(i = 1, 2, \ldots, n)$.
If h is σ-regulated (regulated for short) at each point of Y, then h is σ-regulated on Y.

Constant functions, continuous functions, σ-atoms and σ-molecules are examples of regulated functions (see also [27] for more examples). Now, suppose that (Y, τ, σ) is a Davison space and $(Z, \|.\|_Z)$ is a normed space. Also, suppose that $B(Y, Z)$ is the set of all bounded functions, i.e., the set of functions $f : Y \to Z$ in which

$$\|f\|_{sup} = \sup\{\|f(y)\|_Z : \ y \in Y\} \leq M$$

for some $M > 0$. The space of regulated functions from Y to Z is denoted by $S(Y, Z)$. It is known that if Y is compact, then $S(Y, Z) = \overline{S(Y, Z)} \subseteq B(Y, Z)$ [14].
We end this section by recalling the well-known Darbo type fixed point theorem.

Theorem 2.6. *(Darbo type fixed point theorem) [9] Suppose that Ω is a nonempty, bounded, closed and convex subset of a Banach space Υ and also suppose that $H : \Omega \to \Omega$ is a continuous mapping. Assume that there exists a constant $k < 1$ such that*

$$\eta(HX) \leq k\eta(X)$$

for any nonempty subset X of Ω, where η is a measure of noncompactness defined in Υ. Then H has a fixed point in the set Ω.

3 On the Space of Regulated Functions

In this section, we are going to characterize relative compact sets in the regulated function space $S(Y, Z)$, where Y is a Davison space. Then we introduce a measure of noncompactness on this space.

Definition 3.1. Suppose that $\mathfrak{S} \subseteq S(Y, Z)$. We say that $\mathfrak{S}$ is equiregulated if for each $\varepsilon > 0$ and $y \in Y$ there exist $B_1, B_2, \ldots, B_m \in \sigma$ such that $y \in \text{int}(\bigcup_{i=1}^{m} B_i)$ and for all $s_1, s_2 \in B_i$, $(i = 1, 2, \ldots, m)$, and for all $u \in \mathfrak{S}$, we have $\|u(s_1) - u(s_2)\|_Z < \varepsilon$.

Lemma 3.2. *Suppose that $\mathfrak{S} \subseteq S(Y, Z)$ is relatively compact. Then $\mathfrak{S}$ is equiregulated.*

Proof. Let $\varepsilon > 0$ be given. Since $\overline{\mathfrak{S}}$ is compact, then there exist $u_1, u_2, \ldots, u_m \in \mathfrak{S}$ such that $\mathfrak{S} \subseteq \bigcup_{i=1}^{m} B_{\frac{\varepsilon}{3}}(u_i)$, where $B_{\frac{\varepsilon}{3}}(u_i)$ is the ball with center (u_i) and radius $\frac{\varepsilon}{3}$. Since for each $1 \leq i \leq m$, $u_i \in S(Y, Z)$, then for each arbitrary but fix $y \in Y$, there exist $A_{i,1}, A_{i,2}, \ldots, A_{i,k_i} \in \sigma$ (for some $k_i \in \mathbb{N}$) such that $y \in \mathrm{int}(\bigcup_{j=1}^{k_i} A_{i,j})$ and for each $s_1, s_2 \in A_{i,j}$ $(j = 1, 2, \ldots, k_i)$ we have

$$\|u_i(s_1) - u_i(s_2)\|_Z < \frac{\varepsilon}{3}. \tag{2}$$

Now, for each $1 \leq i \leq m$ we ignore the sets $A_{i,j}$s' which $y \notin A_{i,j}$ and we rename the remaining members and name them as $A'_{i,j}$. Hence for each $1 \leq i \leq m$ we obtain the sets $A'_{i,1}, \ldots, A'_{i,l_i}$ in which $y \in A'_{i,j}$ $(1 \leq j \leq l_i)$ and for each $s_1, s_2 \in A'_{i,j}$,

$$\|u_i(s_1) - u_i(s_2)\|_Z < \frac{\varepsilon}{3}.$$

Put $l = \min\{l_1, \ldots, l_m\}$. By applying choice principle, for each $1 \leq i \leq m$ we can select l members of each family $\{A'_{i,1}, \ldots, A'_{i,l_i}\}$, say $\{B_{i,1}, \ldots, B_{i,l}\}$. Take $N_j = \bigcap_{i=1}^{m} B_{i,j}$ $(1 \leq j \leq l)$. For each $s_1, s_2 \in N_j$ and each $u \in \mathfrak{S}$, we have

$$\|u(s_1)-u(s_2)\|_Z \leq \|u(s_1)-u_i(s_1)\|_Z+\|u_i(s_1)-u_i(s_2)\|_Z+\|u_i(s_2)-u(s_2)\|_Z$$

$$\leq 2\|u - u_i\|_{sup} + \tfrac{\varepsilon}{3} \leq \varepsilon. \qquad \square$$

Under additional conditions, the reverse of the above lemma is also true. In what follows, we intend to deal with this subject. We propose a slightly modification in the [14, Lemma 3] gives us the following result.

Lemma 3.3. *Assume that Y is compact and $\mathfrak{S} \subseteq S(Y, Z)$ is equiregulated. Also, assume that $\varepsilon > 0$ is given. Then there exist $Y_1, Y_2, \ldots, Y_m \in \sigma$ such that $Y = Y_1 \bigcup Y_2 \bigcup \ldots \bigcup Y_m$ and if $y, y' \in Y_i$ $(i = 1, 2, \ldots, m)$, then $\|u(y) - u(y')\|_Z \leq \varepsilon$ for each $u \in \mathfrak{S}$.*

Lemma 3.4. *Assume that Y is a compact Davison space, Z is a normed space with the Heine-Borel property, and $\mathfrak{S} \subseteq S(Y, Z)$ is equiregulated.*

Also, assume that $\varepsilon > 0$ *is given. If* $Y = Y_1 \bigcup Y_2 \bigcup \dots Y_m$ *is as Lemma 3.3 and for each* $1 \le i \le m$ *there exists* $a_i \in Y_i$ *in which* $\{\|u(a_i)\|_Z : u \in \mathfrak{S}\} \le k$ *for some* $k > 0$*, then* $\mathfrak{S}$ *is relatively compact.*

Proof. According to Lemma 3.3, if $y, y' \in Y_i$, $1 \le i \le m$, then $\|u(y) - u(y')\|_Z \le \varepsilon$ for each $u \in \mathfrak{S}$. Let $y \in Y$, and so $y \in Y_{i_0}$ for some $1 \le i_0 \le m$. Using the assumptions we obtain

$$\|u(y)\|_Z \le \|u(y) - u(a_{i_0})\|_Z + \|u(a_{i_0})\|_Z \le \varepsilon + k \text{ (say } \delta).$$

Since $B_\delta(0) = \{z \in Z : \|z\|_Z < \delta\}$ is a relatively compact set in Z, so there exist $z_1, z_2, \dots, z_{m'} \in Z$ such that $B_\delta(0) \subseteq \bigcup_{i=1}^{m'} B_{\frac{\varepsilon}{2}}(z_i)$, We consider the set

$$W = \{u : Y \to Z : \ \forall 1 \le i \le m, \ u(Y_i) = \{z_j\} \text{ for some } j \in \{1, 2, \dots, m'\}\}.$$

Obviously $W \subseteq S(Y, Z)$. We are going to show that W is a finite ε-net for $\mathfrak{S}$, i.e., for every $u \in \mathfrak{S}$ there is $\varphi \in W$ satisfying $\|u - \varphi\|_{sup} \le \varepsilon$. For, suppose that $b_i \in Y_i$ $(1 \le i \le m)$, then $u(b_i) \in B_\delta(0)$ and thus there exists $1 \le j_i \le m'$ such that $u(b_i) \in B_{\frac{\varepsilon}{2}}(z_{j_i})$. Let us define the function φ by $\varphi(y) = z_{j_i}$ (for each $y \in Y_i$ and $1 \le i \le m$). Evidently, $\varphi \in W$ and moreover for each $y'_l \in Y_l$ $(l = 1, \dots, m)$ we have

$$\|u(y'_l) - \varphi(y'_l)\|_Z \le \|u(y'_l) - u(b_l)\|_Z + \|u(b_l) - \varphi(y'_l)\|_Z \le \varepsilon.$$

Hence $\|u - \varphi\|_{sup} \le \varepsilon$ and the proof is complete. □

From now on, we assume that Y is a compact Davison space and Z is a normed space with the Heine-Borel property. Also, assume that $\mathfrak{M}_{S(Y,Z)}$ is the collection of bounded sets of regulated functions from Y into Z and $\mathfrak{N}_{S(Y,Z)}$ is its subfamily consisting of nonempty relatively compact subsets of $S(Y, Z)$. For $\emptyset \ne \Omega \subseteq S(Y, Z)$, the symbols $\overline{\Omega}$ and $\mathrm{Conv}(\Omega)$ will denote the closure and closed convex hull of Ω, respectively.

As a consequences of Lemma 3.4 we have the following result.

Proposition 3.5. *Assume that* $\mathfrak{S} \in \mathfrak{M}_{S(Y,Z)}$ *is equiregulated. Then* $\mathfrak{S}$ *is relatively compact.*

Definition 3.6. A mapping $\eta : \mathfrak{M}_{S(Y,Z)} \to \mathbb{R}_+ = [0, +\infty)$ is said to be a measure of noncompactness (MNC for short) in $S(Y, Z)$ if for any $\mathcal{W}, \mathcal{W}_1, \mathcal{W}_2 \in \mathfrak{M}_{S(Y,Z)}$ it fulfills the following conditions:

$(M1)$ $\emptyset \neq \ker(\eta) = \{\mathcal{W} \in \mathfrak{M}_{S(Y,Z)} : \eta(\mathcal{W}) = 0\} \subseteq \mathfrak{N}_{S(Y,Z)}$.

$(M2)$ $\mathcal{W}_1 \subseteq \mathcal{W}_2$ implies that $\eta(\mathcal{W}_1) \leq \eta(\mathcal{W}_2)$.

$(M3)$ $\eta(\overline{\mathcal{W}}) = \eta(\mathcal{W})$.

$(M4)$ $\eta(\mathrm{Conv}(\mathcal{W})) = \eta(\mathcal{W})$.

$(M5)$ $\eta(\gamma\mathcal{W}_1 + (1-\gamma)\mathcal{W}_2) \leq \gamma\eta(\mathcal{W}_1) + (1-\gamma)\eta(\mathcal{W}_2)$ for any $\gamma \in [0, 1]$.

$(M6)$ If $\{\mathcal{W}_j\}$ is a sequence of closed chains in $\mathfrak{M}_{S(Y,Z)}$ such that $\mathcal{W}_{j+1} \subset \mathcal{W}_j$ for $j = 1, 2, \ldots$ and $\lim_{j\to\infty} \eta(\mathcal{W}_j) = 0$, then $\mathcal{W}_\infty = \bigcap_{j=1}^{\infty} \mathcal{W}_j \neq \emptyset$.

Now, we are in a position to describe a MNC in the space $S(Y, Z)$. For $\mathcal{B} \in \mathfrak{M}_{S(Y,Z)}$, $u \in \mathcal{B}$, $x \in Y$ and $\varepsilon > 0$, let us put

$$\begin{aligned}\eta(u, x, \varepsilon) \quad = \quad & \sup\big\{\|u(t) - u(s)\|_Z : \ \exists K_1, \ldots, K_n \in \sigma; \\ & x \in \mathrm{int}(\bigcup_{i=1}^{n} K_i), \ \mathrm{diam}(K_i) \leq \varepsilon, \ s, t \in K_i\big\},\end{aligned}$$

where $\mathrm{diam}(K_i) = \sup\{d(\varsigma, \nu) : \varsigma, \nu \in K_i\}$. Then we define

$$\eta(B, x, \varepsilon) = \sup\{\eta(u, x, \varepsilon), \ u \in B\},$$

and

$$\eta(B, \varepsilon) = \sup\{\eta(B, x, \varepsilon), \ x \in Y\}.$$

Eventually, let us define the quantity

$$\eta(B) = \lim_{\varepsilon\to 0} \eta(B, \varepsilon). \tag{3}$$

Theorem 3.7. *The function $\eta : \mathfrak{M}_{S(Y,Z)} \to \mathbb{R}_+$ given by (3), defines a MNC in $S(Y, Z)$.*

Proof. Let $\mathcal{W} \in \ker(\eta)$, we are going to show that $\mathcal{W}$ is relatively compact. Suppose that $\delta > 0$ is given. Due to definition of η, there is a sufficiently small positive number ε in which $\eta(\mathcal{W}, \varepsilon) < \delta$. Let $x \in Y$ be arbitrary and fix. Then, $\eta(\mathcal{W}, x, \varepsilon) < \delta$. Therefore there are the sets $A_{1,x}, A_{2,x}, \ldots, A_{n,x}$ in σ such that $x \in \mathrm{int}(\bigcup_{i=1}^{n_x} A_{i,x})$ (say $N(x)$), and for each $t, s \in A_{i,x}$, $u \in \mathcal{W}$ we have $\|u(t)-u(s)\|_Z < \delta$. Since $\{N(x) : x \in Y\}$ is an open cover for Y, hence

$$Y \subseteq N(x_1) \bigcup N(x_2) \bigcup \ldots \bigcup N(x_k)$$

for some $x_1, x_2, \ldots, x_k \in Y$, where $N(x_i) = \operatorname{int}(\bigcup_{j=1}^{n_i} A_{i,j})$, $i = 1, 2, \ldots, k$ and for each $u \in \mathcal{W}$, and $t, s \in A_{i,j}$, $\|u(t) - u(s)\|_Z < \delta$. Now, $x \in Y$ implies that $x \in \operatorname{int}(A_{1,1} \cup \ldots \cup A_{k,n_k})$ and also $t, s \in A_{i,j}$ implies that $\|u(t) - u(s)\|_Z < \varepsilon$ for each $u \in \mathcal{W}$. It follows that $\mathcal{W}$ is equiregulated. In view of Proposition 3.5 $\mathcal{W}$ is relatively compact.
$(M2)$ is obvious by the definition of η. Now, from $(M2)$ $\eta(\mathcal{W}) \leq \eta(\overline{\mathcal{W}})$. Suppose that $\eta(\mathcal{W}) \leq \xi$ for some positive number ξ. Also, suppose that $u \in \overline{\mathcal{W}}$. Hence there exists the sequence $\{u_m\}$ in $\mathcal{W}$ in which $\lim_{m\to\infty} u_m = u$. Hence $\|u_m - u\|_{sup} < \frac{\xi}{3}$ for m large enough. Therefore for each $x \in Y$, and $\varepsilon > 0$ there exists $K_1, \ldots, K_n \in \sigma$ such that $x \in \operatorname{int}(\bigcup_{i=1}^{n} K_i)$, $\operatorname{diam}(K_i) \leq \varepsilon$, and for each $t, s \in K_i$, $\|u_m(t) - u_m(s)\|_Z < \xi$. We get

$$\|u(t) - u(s)\|_Z \leq \|u(t) - u_m(t)\|_Z + \|u_m(t) - u_m(s)\|_Z + \|u_m(s) - u(s)\|_Z < \xi$$

which implies that $\eta(\overline{\mathcal{W}}) \leq \xi$. This proves $(M3)$.
Now, we prove $(M4)$. In view of $(M2)$ $\eta(\mathcal{W}) \leq \eta(\operatorname{Conv}(\mathcal{W}))$. Let $\varepsilon > 0$ be arbitrary, $n \in \mathbb{N}$, $x \in Y$, $0 \leq \beta_i \leq 1$, and $v_i \in \mathcal{W}$ $(i = 1, 2, \ldots, n)$, then we have

$$\begin{aligned}
\eta(\sum_{i=1}^{n} \beta_i v_i, x, \varepsilon) &= \sup\Big\{\|\sum_{i=1}^{n} \beta_i v_i(t) - \sum_{i=1}^{n} \beta_i v_i(s)\|_Z : \exists A_1, \ldots, A_n; \\
&\qquad x \in \operatorname{int}(\bigcup_{i=1}^{n} A_i),\ s, t \in A_i,\ \operatorname{diam}(A_i) \leq \varepsilon\Big\} \\
&\leq \sup\Big\{(\sum_{i=1}^{n} \beta_i) \sup_{1\leq i\leq n} \|v_i(t) - v_i(s)\|_Z : \exists A_1, \ldots, A_n; \\
&\qquad x \in \operatorname{int}(\bigcup_{i=1}^{n} A_i),\ s, t \in A_i,\ \operatorname{diam}(A_i) \leq \varepsilon\Big\} \\
&\leq \eta(\mathcal{W}).
\end{aligned}$$

It follows that $\eta(\mathrm{Conv}(\mathcal{W})) \leq \eta(\mathcal{W})$, hence $\eta(\mathrm{Conv}(\mathcal{W})) = \eta(\mathcal{W})$. For proving $(M5)$ let $\mathcal{W}_1, \mathcal{W}_2 \in \mathfrak{M}_{S(Y,Z)}$, $u_1 \in \mathcal{W}_1$, $u_2 \in \mathcal{W}_2$, $x \in Y$, $0 \leq \gamma \leq 1$ and $\varepsilon > 0$ be arbitrary. Then

$$\begin{aligned}\eta(u_1, x, \varepsilon) \;&=\; \sup\Big\{\|u_1(t) - u_1(s)\|_Z :\ \exists A_1, \ldots, A_n;\\ &\qquad x \in \mathrm{int}(\bigcup_{i=1}^{n} A_i),\ s,t \in A_i,\ \mathrm{diam}(A_i) \leq \varepsilon,\ 1 \leq i \leq n\Big\},\end{aligned}$$

and

$$\begin{aligned}\eta(u_2, x, \varepsilon) \;&=\; \sup\Big\{\|u_2(t) - u_2(s)\|_Z :\ \exists B_1, \ldots, B_m;\\ &\qquad x \in \mathrm{int}(\bigcup_{j=1}^{m} B_j),\ s,t \in B_j,\ \mathrm{diam}(B_j) \leq \varepsilon,\ 1 \leq j \leq m\Big\}.\end{aligned}$$

Take $C_{ij} = A_i \bigcap B_j$, $1 \leq i \leq n$, $1 \leq j \leq m$. Obviously, $\mathrm{diam}(C_{ij}) \leq \varepsilon$. We have

$$\begin{aligned}&\sup\{\|(\gamma u_1 + (1-\gamma)u_2)(t) - (\gamma u_1 + (1-\gamma)u_2)(s)\|_Z :\ \exists C_{11}, \ldots, C_{nm};\\ &\qquad x \in \mathrm{int}(\bigcup_{1\leq i\leq n, 1\leq j\leq m} C_{ij}),\ s,t \in C_{ij},\ \mathrm{diam}(C_{ij}) \leq \varepsilon\}.\\ &\leq \gamma \sup\{\|u_1(t) - u_1(s)\|_Z :\ x \in \mathrm{int}(\bigcup_{i=1}^{n} A_i),\ s,t \in A_i\}\\ &+(1-\gamma) \sup\{\|u_2(t) - u_2(s)\|_Z :\ x \in \mathrm{int}(\bigcup_{j=1}^{m} B_j),\ s,t \in B_j\}\\ &\leq \gamma\eta(\mathcal{W}_1) + (1-\gamma)\eta(\mathcal{W}_2).\end{aligned}$$

Taking the supremum over $u_1 \in \mathcal{W}_1$, $u_2 \in \mathcal{W}_2$, $t, s \in C_{ij}$ and taking limit as $\varepsilon \to 0$, we obtain the relation $(M5)$.

To prove $(M6)$, let $B_n \in \mathfrak{M}_{S(Y,Z)}$, $B_n = \overline{B_n}$, $B_{n+1} \subset B_n$ for $n = 1, 2, \ldots$ and $\lim_{n\to\infty} \eta(B_n) = 0$. For any $n \in \mathbb{N}$, take $u_n \in B_n$. We claim that $G = \overline{\{u_n\}}$ is compact. For, let ζ, ε be positive numbers and $x \in Y$ be given. Then there exists $n_1 \in \mathbb{N}$ such that for each $n \geq n_1$, $\eta(u_n, x, \varepsilon) < \zeta$. Thus there exist the sets $A_{1,x}, A_{2,x}, \ldots, A_{n_x,x}$ in $\mathfrak{M}_{S(Y,Z)}$ such that

$\mathrm{diam}(A_{i,x}) \leq \varepsilon$ $(i = 1, \ldots, n_x)$, $x \in \mathrm{int}(\bigcup_{i=1}^{n_x} A_{i,x})$ and for each $t, s \in A_{i,x}$, $\|u_n(t) - u_n(s)\|_Z < \zeta$. Hence $Y \subseteq \bigcup_{x \in Y} \mathrm{int}(\bigcup_{i=1}^{n_x} A_{i,x})$. Since Y is compact, then $Y \subseteq \bigcup_{j=1}^{k} \mathrm{int}(\bigcup_{i=1}^{n_{x_j}} A_{i,x_j})$ for some $x_1, \ldots, x_k \in Y$. By relabeling the members of $\{A_{i,x_j} : 1 \leq j \leq k, 1 \leq i \leq n_{x_j}\}$ we obtain

$$\Big\{A_{i,x_j} : 1 \leq j \leq k, 1 \leq i \leq n_{x_j}\Big\} = \Big\{C_\gamma : 1 \leq \gamma \leq k'\Big\}.$$

We get $\mathrm{diam}(C_\gamma) \leq \varepsilon$ and for each $n \geq n_1$ and for each $t, s \in C_\gamma$, $\|u_n(t) - u_n(s)\|_Z < \zeta$. Therefore, $G_1 = \{u_n : \ n \geq n_1\}$ is equiregulated. On the other hand, take $y_i \in Y_i$ $(i = 1, 2, \ldots, n)$ arbitrary but fixed. Then, we get

$$\|u_n(a_i)\|_Z \leq \varepsilon + \sup_{1 \leq i \leq n} \|u_n(y_i)\|_Z,$$

where $a_i \in Y_i$, $i = 1, 2, \ldots, n$. Now, by Lemma 3.4 G_1 is compact. Hence G_1 has a convergent subsequence $\{u_{n_j}\}$ with $\lim_{j \to \infty} u_{n_j} = u_0$. Since $u_n \in B_n$, $B_{n+1} \subseteq B_n$ and B_n is closed for all n we yield

$$u_0 \in \bigcap_{n=1}^{\infty} B_n = B_\infty,$$

that completes the proof of $(M6)$. □

4 Solvability of a Nonlinear Fractional Integral Equation of Two Variables in the Space of Regulated Functions

In this section, we investigate the solvability of the nonlinear fractional integral equation (1) in a space of regulated functions. We also provide an illustrative example to show the applicability of our result. Consider the following hypotheses.

(1) $G : [0, T] \times [0, T] \to \mathbb{R}$ is a continuous function.

(2) $F : [0,T] \times [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function and there exist $k_1 > 0$ and $0 < k_2 < 1$ such that

$$|F(t,s,p_1,q) - F(t,s,p_2,q)| \leq k_1|p_1 - p_2|,$$

and

$$|F(t,s,p,q_1) - F(t,s,p,q_2)| \leq k_2|q_1 - q_2|$$

for each $t, s \in [0,T]$ and $p, q, p_1, p_2, q_1, q_2 \in \mathbb{R}$.
(3) $|F(t,s,0,0)| \leq \varphi(t,s)$, where $\varphi : [0,T] \times [0,T] \to \mathbb{R}$ is a bounded function.
(4) The function $u : [0,T] \times [0,T] \times [0,T] \times [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous. Furthermore, there exists a continuous mapping $\theta : [0,T] \times [0,T] \to \mathbb{R}$ such that

$$|u(t,s,v,w,x) - u(t,s,v,w,x')| \leq \theta(t,s)|x - x'|$$

for all $t, s, v, w \in [0,T]$ such that $v \leq t$, $w \leq s$ and for all $x, x' \in \mathbb{R}$.
(5) If $u_0(t,s) = \sup\{|u(t,s,v,w,0)| : \ 0 \leq v \leq t, \ 0 \leq w \leq s\}$, $\eta_1(t,s) = \theta(t,s)s^\alpha t^\beta$, $\eta_2(t,s) = u_0(t,s)s^\alpha t^\beta$, $\overline{\eta_1} = \sup\{\eta_1(t,s) : \ t,s \in [0,T]\}$, and $\overline{\eta_2} = \sup\{\eta_2(t,s) : \ t,s \in [0,T]\}$, then there exists a positive solution r_0 for the inequality

$$\|G\|_{sup} + \|\varphi\|_{sup} + k_2 r_0 + \frac{k_1}{\alpha\beta}(\overline{\eta_1} r_0 + \overline{\eta_2}) \leq r_0.$$

(6) For each x in $\mathbb{R}$, $F(.,.,.,x)$ is continuous on $[0,T] \times [0,T] \times [-r_0, r_0]$.

Theorem 4.1. *Under the assumptions (1)-(6), the nonlinear fractional integral equation (1) has at least one solution in the space* $S(([0,T] \times [0,T], \tau, \sigma), \mathbb{R})$*, where* $([0,T] \times [0,T], \tau, \sigma)$ *is the Davison space as Example 2.4 (ii).*

Proof. First, we define the operator $\Lambda : S(([0,T] \times [0,T], \tau, \sigma), \mathbb{R}) \to S(([0,T] \times [0,T], \tau, \sigma), \mathbb{R})$ by

$$\Lambda x(t,s) = G(t,s) + F\Big(t, s, \int_0^t \int_0^s \frac{u(t,s,v,w,x(v,w))}{(s-v)^{1-\alpha}(t-w)^{1-\beta}} dv dw, x(t,s)\Big), \tag{4}$$

for $\alpha, \beta \in (0,1)$ and $t, s \in [0,T]$. Then using the conditions (1)-(4) we have

$$\begin{aligned}
\|\Lambda x\| &= \sup_{(t,s)\in[0,T]\times[0,T]} |\Lambda x(t,s)| \\
&= \sup_{(t,s)\in[0,T]\times[0,T]} |G(t,s) + F(t,s,Ju(t,s,x),x(t,s))| \\
&\le \sup_{(t,s)\in[0,T]\times[0,T]} |G(t,s)| \\
&\quad + \sup_{(t,s)\in[0,T]\times[0,T]} |F(t,s,Ju(t,s,x),x(t,s)) - F(t,s,0,x(t,s))| \\
&\quad + \sup_{(t,s)\in[0,T]\times[0,T]} |F(t,s,0,x(t,s)) - F(t,s,0,0)| \\
&\quad + \sup_{(t,s)\in[0,T]\times[0,T]} |F(t,s,0,0)| \\
&\le \|G\|_{sup} + k_1 \sup_{(t,s)\in[0,T]\times[0,T]} |Ju(t,s,x)| \\
&\quad + k_2 \sup_{(t,s)\in[0,T]\times[0,T]} |x(t,s)| + \sup_{(t,s)\in[0,T]\times[0,T]} |\varphi(t,s)| \\
&\le \|G\|_{sup} + k_1 \sup_{(t,s)\in[0,T]\times[0,T]} \int_0^t \int_0^s \frac{u(t,s,v,w,x(v,w)) - u(t,s,v,w,0)}{(s-v)^{1-\alpha}(t-w)^{1-\beta}} dvdw \\
&\quad + k_1 \sup_{(t,s)\in[0,T]\times[0,T]} \int_0^t \int_0^s \frac{u(t,s,v,w,0)}{(s-v)^{1-\alpha}(t-w)^{1-\beta}} dvdw + k_2\|x\| + \|\varphi\|_{sup} \\
&\le \|G\| + k_1 \sup_{(t,s)\in[0,T]\times[0,T]} \int_0^t \int_0^s \frac{\theta(t,s)|x(v,w)|}{(s-v)^{1-\alpha}(t-w)^{1-\beta}} dvdw
\end{aligned}$$

$$\begin{aligned}
&\quad + k_1 \sup_{(t,s)\in[0,T]\times[0,T]} \int_0^t \int_0^s \frac{u_0(t,s)}{(s-v)^{1-\alpha}(t-w)^{1-\beta}} dvdw + k_2\|x\| + \|\varphi\|_{sup} \\
&\le \|G\|_{sup} + k_1 \big(\theta(t,s)\|x\| + u_0(t,s)\big) \frac{s^\alpha t^\beta}{\alpha\beta} + k_2\|x\| + \|\varphi\|_{sup} \\
&\le \|G\|_{sup} + \frac{k_1}{\alpha\beta}(\eta_1(t,s)\|x\| + \eta_2(t,s)) + k_2\|x\| + \|\varphi\|_{sup} \\
&\le \|G\|_{sup} + \frac{k_1}{\alpha\beta}(\overline{\eta_1}\|x\| + \overline{\eta_2}) + k_2\|x\| + \|\varphi\|_{sup},
\end{aligned}$$

where $Ju(t,s,x) = \int_0^t \int_0^s \frac{u(t,s,v,w,x(v,w))}{(s-v)^{1-\alpha}(t-w)^{1-\beta}} dvdw$. Owing to the assumption (5) there exists $r_0 > 0$ such that Λ maps the ball B_{r_0} into itself, where B_{r_0} is a closed ball with center at zero and radius r_0 in $S(([0,T] \times [0,T], \tau, \sigma), \mathbb{R})$. Now, we show that for each $x \in S([0,T] \times [0,T], \mathbb{R})$ Λx is regulated. Let $\varepsilon > 0$ be given. Since for each x in $S([0,T] \times [0,T], \mathbb{R})$ the mapping $F(.,.,.,x)$ is uniformly continuous on the compact subset

$[0,T]\times[0,T]\times[-r_0,r_0]$, then for sufficiently small δ_1 we have

$$\begin{aligned}\varpi^T(F,\delta_1) &= \sup\{|F(t',s',p',x)-F(t,s,p,x)|:\ t,t',s,s'\in[0,T];\ |t'-t|\le\delta_1,\\ &\quad |s'-s|\le\delta_1, x\in[-r_0,r_0],|p|,|p'|\le\frac{1}{\alpha\beta}(\overline{\eta_1}r_0+\overline{\eta_2})\}<\frac{\varepsilon}{3}.\end{aligned}$$

Also, since the mapping G is uniformly continuous on the compact subset $[0,T]\times[0,T]$, then for sufficiently small δ_2 we have

$$\begin{aligned}\varpi^T(G,\delta_2) &= \sup\{|G(t,s)-G(t',s')|:\ t,t',s,s'\in[0,T];\ |t'-t|\le\delta_2,\\ &\quad |s'-s|\le\delta_2, x\in[-r_0,r_0],|p|,|p'|\le\frac{1}{\alpha\beta}(\overline{\eta_1}r_0+\overline{\eta_2})\}<\frac{\varepsilon}{3}.\end{aligned}$$

Put $\delta=\min\{\delta_1,\delta_2\}$, hence $\varpi^T(F,\delta)<\frac{\varepsilon}{3}$ and $\varpi^T(G,\delta)<\frac{\varepsilon}{3}$. Next, let $(t_1,s_1)\in[0,T]\times[0,T]$ be given. Since x is regulated, then there exist the sets $D_1,\dots,D_n$ in σ such that $(t_1,s_1)\in\operatorname{int}(\bigcup_{l=1}^{n}D_l)$ and for each $(t,s),(t',s')\in D_l$, $l=1,\dots,n$ we have

$$|x(t,s)-x(t',s')|<\frac{\varepsilon}{3k_2}.$$

Now, suppose that $t<t'$, $s<s'$ we obtain

$$\begin{aligned}|\Lambda x(t,s)-\Lambda x(t',s')| &\le |G(t,s)-G(t',s')|\\ &\quad +\Big|F\big(t,s,\int_0^t\int_0^s\frac{u(t,s,v,w,x(v,w))}{(s-v)^{1-\alpha}(t-w)^{1-\beta}}dvdw,x(t,s)\big)\\ &\quad -F\big(t',s',\int_0^{t'}\int_0^{s'}\frac{u(t',s',v,w,x(v,w))}{(s'-v)^{1-\alpha}(t'-w)^{1-\beta}}dvdw,x(t,s)\big)\Big|\\ &\quad +\Big|F\big(t',s',\int_0^{t'}\int_0^{s'}\frac{u(t',s',v,w,x(v,w))}{(s'-v)^{1-\alpha}(t'-w)^{1-\beta}}dvdw,x(t,s)\big)\\ &\quad -F\big(t',s',\int_0^{t'}\int_0^{s'}\frac{u(t',s',v,w,x(v,w))}{(s'-v)^{1-\alpha}(t'-w)^{1-\beta}}dvdw,x(t',s')\big)\Big|\\ &\le \varpi^T(G,\delta)+\varpi^T(F,\delta)+k_2|x(t,s)-x(t',s')|.\end{aligned}$$

Put $\mathfrak{F}=\{D_i:\ 1\le i\le n\}$. By eliminating the sets with diagonal equal or greater than δ from $\mathfrak{F}$ (if it is necessary we can choose a subset of D_i with the diagonal less than δ) and relabeling this subset again we obtain the subfamily $\mathfrak{F}'$ as follows:

$$\mathfrak{F}'=\{D_{l'}:1\le l'\le m',\ D_{l'}\neq\emptyset \text{ and } \operatorname{diam}(D_{l'})<\delta\}.$$

From this we obtain $(t_1, s_1) \in \mathrm{int}(\bigcup_{l'=1}^{m'} D_{l'})$ and for each $(t,s),(t',s') \in D_{l'}$, $l' = 1,\dots,m'$, $|\Lambda x(t,s) - \Lambda x(t',s')| < \varepsilon$.
Next, we show that Λ is continuous. Suppose that $\varepsilon > 0$ is given, $y \in B_{r_0}$ and $t, s \in [0,T]$. We can write

$$\begin{aligned}
|\Lambda x(t,s) - \Lambda y(t,s)| &\leq \Big|F\big(t,s,\int_0^t\int_0^s \frac{u(t,s,v,w,x(v,w))}{(s-v)^{1-\alpha}(t-w)^{1-\beta}}dvdw, x(t,s)\big) \\
&\quad -F\big(t,s,\int_0^t\int_0^s \frac{u(t,s,v,w,x(v,w))}{(s-v)^{1-\alpha}(t-w)^{1-\beta}}dvdw, y(t,s)\big)\Big| \\
&\quad +\Big|F\big(t,s,\int_0^t\int_0^s \frac{u(t,s,v,w,x(v,w))}{(s-v)^{1-\alpha}(t-w)^{1-\beta}}dvdw, y(t,s)\big) \\
&\quad -F\big(t,s,\int_0^t\int_0^s \frac{u(t,s,v,w,y(v,w))}{(s-v)^{1-\alpha}(t-w)^{1-\beta}}dvdw, y(t,s)\big)\Big| \\
&\leq k_2|x(t,s) - y(t,s)| \\
&\quad +k_1\int_0^t\int_0^s \frac{|u(t,s,v,w,x(v,w)) - u(t,s,v,w,y(v,w))|}{(s-v)^{1-\alpha}(t-w)^{1-\beta}}dvdw \\
&\leq k_2\|x-y\| + k_1\int_0^t\int_0^s \frac{|\theta(t,s)|x(v,w) - y(v,w)|}{(s-v)^{1-\alpha}(t-w)^{1-\beta}}dvdw \\
&\leq k_2\delta + k_1\theta(t,s)\|x-y\|\frac{s^\alpha t^\beta}{\alpha\beta} \\
&\leq k_2\delta + k_1\frac{\|x-y\|\eta_1(t,s)}{\alpha\beta} \\
&\leq \delta(k_2 + \frac{k_1\overline{\eta_1}}{\alpha\beta}).
\end{aligned}$$

Hence Λ is continuous. Finally, let $M \subseteq B_{r_0}$. We prove that $\eta(\Lambda(M)) \leq k_2\eta(M)$. For, let $x \in M$. We have

$$|\Lambda x(t,s) - \Lambda x(t',s')| \leq \varpi^T(G,\varepsilon) + \varpi^T(F,\varepsilon) + k_2|x(t,s) - x(t',s')|, \quad (5)$$

where $\varpi^T(G,\varepsilon) = \sup\{|G(t',s') - G(t,s)| : \ t,t',s,s' \in [0,T]; \ |t'-t| \leq \varepsilon, \ |s'-s| \leq \varepsilon\}$. Since G is continuous, then $\varpi^T(G,\varepsilon)$ goes to zero as ε tends to zero. Hence the inequality (5) implies that $\eta(\Lambda(M)) \leq k_2\eta(M)$. Thus, all conditions of Theorem 2.6 are satisfied and therefore Λ has a fixed point in the space $S(([0,T]\times[0,T],\tau,\sigma),\mathbb{R})$. It follows that Eq. (1) has at least one solution in this space. $\square$

Example 4.2. Consider the following equation

$$x(t,s) = \frac{1}{2}e^{-t^3s^3} + \frac{1}{14}e^{-(t^2+s^2)}\sin\big(\int_0^t\int_0^s \frac{4^{-\frac{ts}{3}}|x(v,w)|}{(s-v)^{\frac{1}{2}}(t-w)^{\frac{1}{2}}}dvdw + \frac{x(t,s)}{8}\big) \quad (6)$$

for $t, s \in [0, 1.1]$. Put $\alpha = \beta = \frac{1}{2}$, $G(t, s) = \frac{1}{2}e^{-t^3s^3}$, and

$$F(t, s, p, q) = \frac{1}{2}e^{-t^3s^3} + \frac{1}{14}e^{-(t^2+s^2)}\sin(p + \frac{q}{8}).$$

Clearly, F is continuous. Since
$|F(t, s, p_1, q) - F(t, s, p_2, q)|$

$$\begin{aligned} &= \frac{1}{14}e^{-(t^2+s^2)}\left|\sin(p_1 + \frac{q}{8}) - \sin(p_2 + \frac{q}{8})\right| \\ &\leq \frac{1}{14}|p_1 - p_2|, \end{aligned}$$

and
$|F(t, s, p, q_1) - F(t, s, p, q_2)|$

$$\begin{aligned} &= \frac{1}{14}e^{-(t^2+s^2)}\left|\sin(p + \frac{q_1}{8}) - \sin(p + \frac{q_2}{8})\right| \\ &\leq \frac{1}{112}|q_1 - q_2|, \end{aligned}$$

then condition (2) of Theorem 4.1 holds. Also, $F(t, s, 0, 0) = \frac{1}{2}$, $\varphi(t, s) = \frac{1}{2}$ is bounded with $\|\varphi\|_{sup} = \frac{1}{2}$, $u(t, s, v, w, x) = 4^{-\frac{ts}{3}}|x(v, w)|$, and

$$|u(t, s, v, w, x) - u(t, s, v, w, x')| \leq 4^{-\frac{ts}{3}}|x(v, w) - x'(v, w)|.$$

Hence $\theta(t, s) = 4^{-\frac{ts}{3}}$ and so it is continuous. Furthermore, $u_0(t, s) = \sup\{|u(t, s, v, w, 0)| : \ 0 \leq v \leq t, \ 0 \leq w \leq s\} = 0$, $\eta_1(t, s) = 4^{-\frac{ts}{3}}s^{\frac{1}{2}}t^{\frac{1}{2}}$, and $\eta_2(t, s) = 0$. The function η_1 admits its maximum value at point $(1, 1.0820212807)$, $\overline{\eta_1} = 0.23210055546$ and $\overline{\eta_2} = 0$. Finally, the inequality in condition (5) of Theorem 4.1 is as follows:

$$1 + \frac{1}{112}r_0 + \frac{0.4642011092r_0}{7} \leq r_0,$$

which has positive solution $r_0 = 2$. Thus all assumptions of Theorem 4.1 hold and therefore Eq. (6) has a solution in the regulated space $S(([0, 1.1] \times [0, 1.1], \tau, \sigma), \mathbb{R})$.

Example 4.3. Consider the following equation

$$x(t, s) = \sec(\sqrt{t^2 + s^2}) + \frac{1}{2^{ts}}\arctan\Big(\frac{1}{3^{t^4+s^4}} + \frac{e^{-(t+s)}}{60}\int_0^t\int_0^s \frac{7^{-\frac{ts+vw}{10}}|x(v, w)|}{(s-v)^{\frac{2}{3}}(t-w)^{\frac{2}{3}}}dvdw$$

$$+\frac{x(t,s)}{t^4+s^4+50}) \tag{7}$$

for $t,s\in[0,\frac{1}{2}]$. Here we have $\alpha=\beta=\frac{1}{3}$, $G(t,s)=\sec(\sqrt{t^2+s^2})$, and

$$F(t,s,p,q)=\frac{1}{2^{ts}}\arctan\big(\frac{1}{3^{t^4+s^4}}+\frac{e^{-(t+s)}}{60}p+\frac{q}{t^4+s^4+50}\big).$$

Clearly, F is continuous. Since
$|F(t,s,p_1,q)-F(t,s,p_2,q)|$

$$\begin{aligned} &= \Big|\frac{1}{2^{ts}}\arctan\big(\frac{1}{3^{t^4+s^4}}+\frac{e^{-(t+s)}}{60}p_1+\frac{q}{t^4+s^4+50}\big) \\ &\quad -\frac{1}{2^{ts}}\arctan\big(\frac{1}{3^{t^4+s^4}}+\frac{e^{-(t+s)}}{60}p_2+\frac{q}{t^4+s^4+50}\big)\Big| \\ &\le \frac{1}{2^{ts}}\frac{e^{-(t+s)}}{60}|p_1-p_2| \\ &\le \frac{1}{60}|p_1-p_2|, \end{aligned}$$

and
$|F(t,s,p,q_1)-F(t,s,p,q_2)|$

$$\begin{aligned} &= \Big|\frac{1}{2^{ts}}\arctan\big(\frac{1}{3^{t^4+s^4}}+\frac{e^{-(t+s)}}{60}p+\frac{q_1}{t^4+s^4+50}\big) \\ &\quad -\frac{1}{2^{ts}}\arctan\big(\frac{1}{3^{t^4+s^4}}+\frac{e^{-(t+s)}}{60}p+\frac{q_2}{t^4+s^4+50}\big)\Big| \\ &\le \frac{1}{2^{ts}(t^4+s^4+50)}|q_1-q_2| \\ &\le \frac{1}{50}|q_1-q_2|, \end{aligned}$$

then condition (2) of Theorem 4.1 holds. Also $F(t,s,0,0)=\frac{1}{2^{ts}}\arctan\big(\frac{1}{3^{t^4+s^4}}\big)$ and $\varphi(t,s)=\frac{1}{2^{ts}}\arctan\big(\frac{1}{3^{t^4+s^4}}\big)$ is bounded and $\|\varphi\|_{sup}=\frac{\pi}{2}$. Furthermore, $u(t,s,v,w,x)=7^{-\frac{ts+vw}{10}}|x(v,w)|$ and

$$|u(t,s,v,w,x)-u(t,s,v,w,x')|\le 7^{-\frac{ts+vw}{10}}|x-x'|\le 7^{-\frac{ts}{10}}|x-x'|.$$

Hence $\theta(t,s)=7^{-\frac{ts+vw}{10}}$ is continuous. $u_0(t,s)=\sup\{|u(t,s,v,w,0)|:0\le v\le t,\ 0\le w\le s\}=0$, $\eta_1(t,s)=\theta(t,s)s^{\alpha}t^{\beta}=7^{-\frac{ts}{10}}s^{\frac{1}{3}}t^{\frac{1}{3}}$ and

$\eta_2(t,s) = 0$. The function η_1 admits its maximum value at point $(\frac{1}{\sqrt[5]{27(\ln 7)^3}}, \frac{10}{\sqrt[5]{27(\ln 7)^3}})$, $\overline{\eta_1} = 0.102239147$ and $\overline{\eta_2} = 0$. Finally, the inequality in condition (5) of Theorem 4.1 is as follows:

$$1 + \frac{\pi}{2} + \frac{1}{50}r_0 + \frac{9 \times 0.102239147 r_0}{60} \leq r_0,$$

which has positive solution $r_0 = 3$. Thus all assumptions of Theorem 4.1 hold and therefore Eq. (7) has a solution in the regulated space $S(([0, \frac{1}{2}] \times [0, \frac{1}{2}], \tau, \sigma), \mathbb{R})$.

Example 4.4. Consider the following equation

$$x(t,s) = e^{\frac{ts}{2}} + \frac{1}{12}e^{-ts}\int_0^s\int_0^t \frac{e^{\frac{-vw}{2}}x(v,w)+1}{\sqrt{t-v}\sqrt{s-w}}dvdw, \tag{8}$$

for $t, s \in [0,1]$. Here we have $\alpha = \beta = \frac{1}{2}$, $G(t,s) = e^{\frac{ts}{2}}$, and

$$F(t,s,p,q) = \frac{1}{12}e^{-ts}p.$$

Clearly, F is continuous. Also, we have

$$|F(t,s,p_1,q) - F(t,s,p_2,q)| = \frac{1}{12}e^{-ts}|p_1 - p_2|, \tag{9}$$

and $|F(t,s,p,q_1) - F(t,s,p,q_2)| = 0 \leq k_2|q_1 - q_2|$. Put $k_1 = \frac{1}{12}$ and $k_2 = \frac{1}{2} < 1$.
Then condition (2) of Theorem 4.1 holds. Also $F(t,s,0,0) = 0 \leq \varphi(t,s) = \frac{1}{2}$ and φ is bounded and $\|\varphi\|_{sup} = \frac{1}{2}$. Furthermore, $u(t,s,v,w,x) = e^{\frac{-vw}{2}}x + 1$ and

$$|u(t,s,v,w,x) - u(t,s,v,w,x')| = e^{\frac{-vw}{2}}|x - x'| \leq |x - x'|.$$

Take $\theta(t,s) = 1$. Then $u_0(t,s) = \sup\{|u(t,s,v,w,0)| : \ 0 \leq v \leq t, \ 0 \leq w \leq s\} = 1$, and $\eta_1(t,s) = \eta_2(t,s) = \sqrt{ts}$. The function η_1 admits its maximum value at point $(1,1)$, and $\overline{\eta_1} = \overline{\eta_2} = 1$. Finally, the inequality in condition (5) of Theorem 4.1 is as follows:

$$e^{\frac{1}{2}} + \frac{1}{2} + \frac{1}{2}r_0 + \frac{1}{3}(r_0 + 1) \leq r_0,$$

which has positive solution $r_0 = 15$. Thus all assumptions of Theorem 4.1 hold and therefore Eq. (8) has a solution in the regulated space $S(([0,1] \times [0,1], \tau, \sigma), \mathbb{R})$.

5 Numerical Results

In the Adomian decomposition method [2], the unknown function $u(x)$ of any equation decomposes into a sum of an infinite number of components defined by

$$u(x) = \sum_{n=0}^{\infty} u_n(x) = u_0(x) + u_1(x) + u_2(x) + \dots, \tag{10}$$

where the components $u_n(x)$, $n \geq 0$ can be determined in a recursive manner. Indeed, all components $u_0, u_1, u_2, \dots$ can be found individually. The determination of these components can be obtained easily with the help of a recurrence relation that involves simple integrals. To obtain the recurrence relation, we replace (10) in a Volterra integral equation and we get

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_0^x k(x,t)(\sum_{n=0}^{\infty} u_n(t))dt, \tag{11}$$

or

$$u_0(x) + u_1(x) + \dots = f(x) + \lambda \int_0^x k(x,t)(u_0(t) + u_1(t) + \dots)dt.$$

One can consider the component $u_0(x)$ by all terms that are not included under the integral sign. Thus, we have $u_0(x) = f(x)$, and

$$u_{n+1}(x) = \lambda \int_0^x k(x,t)u_n(t)dt, \ n \geq 0.$$

For instance, the first three components are $u_0(x) = f(x)$, $u_1(x) = \lambda \int_0^x k(x,t)u_0(t)dt$, and

$$u_2(x) = \lambda \int_0^x k(x,t)u_1(t)dt. \tag{12}$$

In a similar way and using (12), the other components are determined. So, the solution $u(x) = \sum_{n=0}^{\infty} u_n(x)$ of the Volterra integral equation (11) is readily deduced. It is obviously observed that the decomposition method converts the integral equation into an interesting determination of computable components. Also, it has been proved that if a problem has an exact solution, then the obtained series converges very rapidly to the solution. However, if a closed form solution is not attainable, then a truncated number of terms is used for numerical results. The more components we apply the higher accuracy we obtain.

Now, we are going to use the Adomian decomposition method to solve the integral equation (8) in Example 4.4. Due to Adomian procedure, let $x_0(t,s) = e^{\frac{ts}{2}}$, so

$$\begin{aligned} x_1(t,s) &= \frac{1}{12}e^{-ts}\int_0^s\int_0^t \frac{e^{\frac{-vw}{2}}x_0(v,w)+1}{\sqrt{t-v}\sqrt{s-w}}dvdw \\ &= \frac{1}{12}e^{-ts}\int_0^s\int_0^t \frac{2}{\sqrt{t-v}\sqrt{s-w}}dvdw = \frac{2}{3}\sqrt{ts}e^{-ts}. \end{aligned}$$

Then

$$x_2(t,s) = \frac{1}{12}e^{-ts}\int_0^s\int_0^t \frac{e^{\frac{-vw}{2}}x_1(v,w)+1}{\sqrt{t-v}\sqrt{s-w}}dvdw \tag{13}$$

One can see that we have a closed form for the approximations $x_0(t,s)$ and $x_1(t,s)$. But $x_2(t,s)$, cannot be obtained analytically, hence we use the two-dimensional two points Gauss quadrature rule to evaluate the integral part in $x_2(t,s)$ numerically. First of all we remind the two points Gauss quadrature rule as below:

$$\int_{-1}^1 f(x)dx \simeq f(-\frac{1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}}).$$

To approximate the integral

$$\int_0^t f(x)dx \tag{14}$$

by Gauss quadrature rule, we have to shift the interval $[0,t]$ to $[-1,1]$. This can be simply done by taking $x = \frac{t}{2}(u+1)$. Thus, integral (14) can be approximated as follows:

$$\int_0^t f(x)dx = \int_{-1}^1 \frac{t}{2}f(\frac{t}{2}(u+1))du \simeq \frac{t}{2}[f(\frac{t}{2}(-\frac{1}{\sqrt{3}}+1)) + f(\frac{t}{2}(\frac{1}{\sqrt{3}}+1))].$$

The two dimensional two points Gauss quadrature rule can be deduced as follows:

$$\int_0^s \int_0^t f(x,y)dxdy \simeq \frac{t}{2}\frac{s}{2}[f(\frac{t}{2}(1-\alpha),\frac{s}{2}(1-\alpha)) + f(\frac{t}{2}(1-\alpha),\frac{s}{2}(1+\alpha))$$
$$+f(\frac{t}{2}(1+\alpha),\frac{s}{2}(1-\alpha)) + f(\frac{t}{2}(1+\alpha),\frac{s}{2}(1+\alpha))],$$

where $\alpha = \frac{1}{\sqrt{3}}$. If we evaluate $x_2(t,s)$ in a point (t_i,s_j), where t_i and s_j belong to $[0,1]$, then

$$x_2(t_i,s_j) = \frac{1}{12}e^{t_i s_j}\int_0^{s_j}\int_0^{t_i} F(t_i,s_j,v,w)dvdw, \tag{15}$$

where

$$F(t_i,s_j,v,w) = \frac{e^{\frac{-vw}{2}}x_1(v,w)+1}{\sqrt{t_i - v}\sqrt{s_j - w}}. \tag{16}$$

By applying Formula (16) in (15) we have

$$\begin{aligned} x_2(t_i,s_j) \;\simeq\; & \frac{1}{12}e^{t_i s_j}\frac{t_i s_j}{2}[F(t_i,s_j,\frac{t_i}{2}(1-\alpha),\frac{s_j}{2}(1-\alpha)) \\ & +F(t_i,s_j,\frac{t_i}{2}(1-\alpha),\frac{s_j}{2}(1+\alpha)) \\ & +F(t_i,s_j,\frac{t_i}{2}(1+\alpha),\frac{s_j}{2}(1-\alpha)) \\ & +F(t_i,s_j,\frac{t_i}{2}(1+\alpha),\frac{s_j}{2}(1+\alpha))]. \end{aligned} \tag{17}$$

Thus (8) can be approximated by considering three terms in the Equation (10), i.e.

$$x(t,s) \simeq x_0(t,s) + x_1(t,s) + x_2(t,s). \tag{18}$$

In order to check how good this approximation is, we substitute equation (18) in both sides of the equation (8). Now, we find the absolute error in different points in the 2-cell $[0,1]\times[0,1]$ and gather them in the following table. In Figures 1 and 2 we have plotted the approximate solution (18).

Table 1: Absolute error at point (t_i, s_j) related to the Example 4.4

	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0	0	0	0	0	0	0	0	0	0	0
0.1	0	0.2020	0.0278	0.0334	0.0377	0.0413	0.0445	0.0473	0.0497	0.0518	0.0538
0.2	0	0.0278	0.0377	0.0445	0.0497	0.0537	0.0571	0.0599	0.0622	0.0641	0.0658
0.3	0	0.0334	0.0445	0.0518	0.0571	0.0611	0.0641	0.0665	0.0684	0.0698	0.071
0.4	0	0.0377	0.0497	0.0571	0.0622	0.0659	0.0684	0.0703	0.0716	0.0724	0.0728
0.5	0	0.0413	0.0537	0.0611	0.0659	0.069	0.071	0.0722	0.0728	0.073	0.0727
0.6	0	0.0445	0.0571	0.0641	0.0684	0.071	0.0724	0.0729	0.0729	0.0724	0.0715
0.7	0	0.0473	0.0599	0.0665	0.0703	0.0722	0.0729	0.0728	0.0722	0.071	0.0696
0.8	0	0.0497	0.0622	0.0684	0.0716	0.0728	0.0729	0.0722	0.0709	0.0691	0.0671
0.9	0	0.0518	0.0641	0.0698	0.0724	0.0729	0.0724	0.071	0.0691	0.0668	0.0643
1	0	0.0538	0.0658	0.071	0.0728	0.0727	0.0715	0.0696	0.0671	0.0643	0.0615

6 Conclusion

Davison [14] generalized regulated functions on a topological space and proved some remarkable facts concerning them. Also, a two dimensional nonlinear fractional integral equation was studied by Rabani et al. [31]. Now, in this work, we first characterize relatively compact subsets of the space of regulated functions defined on Davison spaces. Then, we introduce a measure of noncompactness on this kind of spaces. We also discuss the existence of solutions of this equation in the regulated space $S(([0,T] \times [0,T], \tau, \sigma), \mathbb{R})$ via measure of noncompactness and with the help of a fixed point theorem of Darbo type. Eventually, we use the Adomian decomposition method to find solution of the problem with high accuracy. As a future project, one can investigate the solvability of the equation with the help of other fixed point theorems.

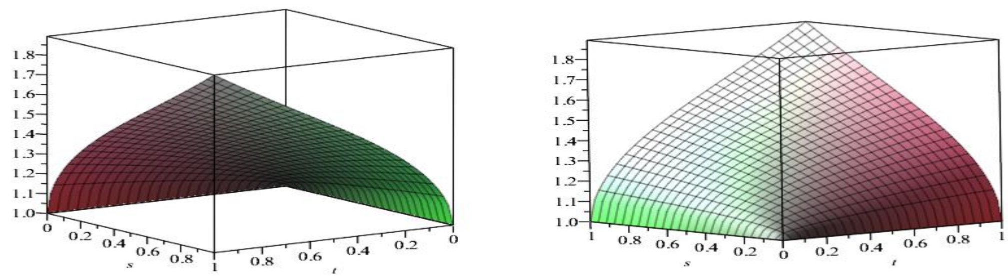

Figure 1: Approximate solution of the Example 4.4

Acknowledgements

We are very grateful to the referees for their comments and useful advice on our manuscript.

References

[1] S. Abbas, M. Benchohra and J. Henderson, On global asymptotic stability of solutions of nonlinear quadratic Volterra integral equations of fractional order, *Commun. Appl. Nonlinear Anal.*, 19 (2012), 79–89.

[2] W. Abdul-Majid, Linear and Nonlinear Integral Equations: Methods and Applications, Springer, (2011).

[3] Y. Adjabi, M.E. Samei, M.M. Matar and J. Alzabut, Langevin differential equation in frame of ordinary and Hadamard fractional derivatives under three point boundary conditions, *Aims Math.*, 6(3) (2021), 2796–2843.

[4] R. Allahyari, The behaviour of measures of noncompactness in $L^\infty(\mathbb{R}^n)$ with application to the solvability of functional integral equations, *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matemaiticas*, 112(2) (2018), 561–573.

[5] H. Amiri Kayvanloo, M. Khanehgir and R. Allahyari, A family of measures of noncompactness in the Hölder space $C^{n,\gamma}(\mathbb{R}_+)$ and its application to some fractional differential equations and numerical methods, *J. Comput. Appl. Math.*, 363 (2020), 256–272.

[6] G. Aumann, *Reelle Funktionen, Grundlehren Math. Wiss.*, 68, Springer, Berlin (1954).

[7] S.M. Aydogan, A. Hussain, F.M. Sakar, On a nonlinear fractional-order model of covid-19 under AB-fractional derivative, *J. Math. Ext.* 15, SI-NTFCA, (11) (2021), 1–31.

[8] D. Baleanu, S. Etemad and S. Rezapour, A hybrid Caputo fractional modeling for thermostat with hybrid boundary value conditions, *Bound. Value Probl.*, 2020, 64 (2020), 16 pages.

[9] J. Banaś and K. Goebel, *Measures of Noncompactness in Banach Spaces, In: Lecture Notes In Pure And Applied Mathematics*, 60, Dekker, New York (1980).

[10] J. Banaś and M. Mursaleen, *Sequence Spaces and Measures of Noncompactness with Applications to Differential and Integral Equations*, Springer New Delhi Heidelberg New York Dordrecht London, (2014).

[11] J. Banaś and R. Nalepa, On a measure of noncompactness in the space of functions with tempered increments, *J. Math. Anal. Appl.*, 435(2) (2016), 1634–1651.

[12] J. Banaś and T. Zajac, On a measure of noncompactness in the space of regulated functions and its applications, *Adv. Nonlinear Anal.*, 8(1) (2018), 1099–1110.

[13] L. Benhamouche and S. Djebali, Solvability of functional integral equations in the Fréchet space $C(\Omega)$, *Mediterr. J. Math.*, 13(6) (2016), 4805–4817.

[14] T.M.K. Davison, A generalization of regulated functions, *The American Mathematical Monthly*, 86, No. 3 (Mar., 1979), 202–204

[15] M.A. Darwish, On quadratic integral equation of fractional orders, *J. Math. Anal. Appl.*, 311 (2005), 112–119.

[16] K. Diethelm and A.D. Freed, On the solution of nonlinear fractional-order differential equations used in the modeling of viscoplasticity, *In Scientific Computing in Chemical Engineering II*, 217–224, Springer, Berlin, Heidelberg (1999).

[17] J. Dieudonne, *Foundations of Modern Analysis*, Academic Press, New York (1969).

[18] D. Frankova, *Regulated functions*, Math. Bohem., 116(1) (1991), 20–59.

[19] L. Gaul, P. Klein and S. Kemple, Damping description involving fractional operators, *Mech. Syst. Signal Process.*, 5(2) (1991), 81–88.

[20] M. Ghasemi, M. Khanehgir, R. Allahyari and H.A. Kayvanloo, Positive solutions of infinite coupled system of fractional differential equations in the sequence space of weighted means, *Aims Math.*, 7(2) (2021), 2680–2694.

[21] C.S. Honig, *Volterra Stieltjes-Integral Equations*, North-Holland, Amsterdam (1975).

[22] C.S. Honig, Equations Integrales Generalisées et Applications, *Publ. Math. Orsay*, 83–01, Expose No. 5. (1983).

[23] H.A. Kayvanloo, M. Khanehgir and R. Allahyari, A family of measures of noncompactness in the Hölder space $C^{n,\gamma}(\mathbb{R}_+)$ and its application to some fractional differential equations and numerical methods, *J. Comput. Appl. Math.*, 363 (2020), 256–272.

[24] A.A. Kilbas and J.J. Trujillo, Differential equations of fractional order: methods, results and problems-I. *Applicable Anal.*, 78(1-2) (2001), 153–192.

[25] V. Lakshmikantham, Theory of fractional functional differential equations, *Nonlinear Anal. Theory, Methods, Appl.*, 69(10) (2008), 3337–3343.

[26] R. Metzler, W. Schick, H.G. Kilian and T.F. Nonnenmacher, Relaxation in filled polymers: A fractional calculus approach, *The J. Chemical Phys.*, 103(16) (1995), 7180–7186.

[27] J. O'Donovan, regulated functions on topological spaces, *Real Anal. Exch.*, 33(2) (2007/2008), 405–416

[28] L. Olszowy, Measures of noncompactness in the space of regulated functions, *J. Math. Anal. Appl.*, 476 (2019), 860–874.

[29] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and Some of Their Applications*, Elsevier, 198 Amsterdam, The Netherlands (1998).

[30] A. Pratap, R. Raja, J. Alzabut, J. Dianavinnarasi, J. Cao and G. Rajchakit, Finite-time Mittag-Leffler stability of fractional order quaternion-valued memristive neural networks with impulses, *Neural Process. Lett.*, 51 (2020), 1485–1526.

[31] M. Rabbani, A. Das, B. Hazarika and R. Arab, Existence of solution for two dimensional nonlinear fractional integral equation by measure of noncompactness and iterative algorithm to solve it, *J. Comput. Applied Math.*, 370(1) (2020), 112654.

[32] J. Sabatier, O.P. Agrawal and J.T. Machado, *Advances in Fractional Calculus*, Dordrecht: 4(9), Springer (2007).

[33] S. Saiedinezhad, On a measure of noncompactness in the Holder space $Ck, \gamma(\Omega)$ and its application, *J. Comput. Applied Math.*, **346** (2019), 566–571.

[34] F.M. Sakar, N. Tasar and B. Seker, A study on various subclasses of uniformly harmonic starlike mappings by Pascal distribution series, *J. Math. Ext.*, 17(9) (2023), 1–17.

[35] S.T.M. Thabet, S. Etemad and S. Rezapour, On a new structure of the pantograph inclusion problem in the Caputo conformable setting, *Bound. Value Probl.*, 2020, 171 (2020), 21 pages.

[36] N.H. Tuan, H. Mohammadi and S. Rezapour, A mathematical model for COVID-19 transmission by using the Caputo fractional derivative, *Chaos, Solitons, Fract.*, 140 (2020), 110107, 11 pages.

[37] Y. Zhao, S. Sun, Z. Han and M. Zhang, Positive solutions for boundary value problems of nonlinear fractional differential equations, *Applied Math. Comput.*, 217(16) (2011), 6950–6958.

Sodabeh Alipour Fatideh
Department of Mathematics
Assistant Professor of Mathematics
Department of Mathematics, Mashhad Branch, Islamic Azad University
Mashhad, Iran
E-mail: alipoor@iau.ac.ir

Mahnaz Khanehgir
Department of Mathematics
Associate Professor of Mathematics
Department of Mathematics, Mashhad Branch, Islamic Azad University
Mashhad, Iran
E-mail: khanehgir@mshdiau.ac.ir

Reza Allahyari
Department of Mathematics
Associate Professor of Mathematics
Department of Mathematics, Mashhad Branch, Islamic Azad University
Mashhad, Iran
E-mail: rezaallahyari@mshdiau.ac.ir

Mohammad Mehrabinezhad
Department of Mathematics
Assistant Professor of Mathematics
Department of Mathematics, Mashhad Branch, Islamic Azad University
Mashhad, Iran
E-mail: mmehrabi@mshdiau.ac.ir

Hojjatollah Amiri Kayvanloo
Department of Mathematics
Assistant Professor of Mathematics
Department of Mathematics, Mashhad Branch, Islamic Azad University

Mashhad, Iran
E-mail: amiri.hojjat93@mshdiau.ac.ir