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Maps Preserving the ϵ-Pseudo Spectrum of Some Product of Operators

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Abstract. Let $B(H)$ be the algebra of all bounded linear operators on an infinite dimensional complex Hilbert space H. In this paper, we characterize all bijective maps φ on $B(H)$ satisfying

$$\sigma_\epsilon(T_1 \bullet_* T_2 \circ_* T_3) = \sigma_\epsilon(\varphi(T_1) \bullet_* \varphi(T_2) \circ_* \varphi(T_3)),$$

for all $T_1, T_2, T_3 \in B(H)$, where $T_1 \bullet_* T_2 = T_1T_2 + T_2T_1^*$ and $T_1 \circ_* T_2 = T_1T_2 - T_2T_1^*$, and $\sigma_\varepsilon(T)$ denote the ϵ-pseudo spectrum of $T \in B(H)$. We also describe bijective maps φ on $B(H)$ that satisfy

$$\sigma_\epsilon(T_1 \Diamond T_2 \diamond_* T_3) = \sigma_\epsilon(\varphi(T_1) \Diamond \varphi(T_2) \diamond_* \varphi(T_3)),$$

for all $T_1, T_2, T_3 \in B(H)$, where $T_1 \Diamond T_2 = T_1T_2^* + T_2^*T_1$ and $T_1 \diamond_* T_2 = T_1T_2^* - T_2T_1$.

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1 Introduction and Preliminaries

Throughout the paper, suppose $B(H)$ is the space of all bounded linear operators on an infinite dimensional complex Hilbert space H and I be the identity operator. Let $B_s(H)$, $B_a(H)$ and $P(H)$ be the set of all self-adjoint operators, the set of anti-self-adjoint operators and the set of all projection operators in $B(H)$, respectively. The trace of a finite rank operator T will be denoted by TrT and we write $Z(B(H))$ for the center of $B(H)$. For an operator $T \in B(H)$, the spectrum, the adjoint and the transpose of T relative to an arbitrary but fixed orthogonal basis of H are denoted by $\sigma(T)$, T^* and T^t, respectively. For $T, S \in B(H)$ denote by $T \bullet_* S = TS + ST^*$ and $T \circ_* S = TS - ST^*$ the Jordan $*$-product and the skew Lie product of T and S, respectively. For a fixed positive real number $\epsilon > 0$, the ϵ-pseudo spectrum of T, $\sigma_\epsilon(T)$, is the set

$$\{\lambda \in \mathbb{C} : \|(\lambda I - T)^{-1}\| \geq \epsilon^{-1}\}$$

with the convention that $\|(\lambda I - T)^{-1}\| = \infty$ if $\lambda \in \sigma(T)$. The upper-semi continuity of the spectrum implies that,

$$\sigma(T) = \bigcap_{\epsilon > 0} \sigma_\epsilon(T).$$

For more information about these notions, one can see [11].

Several authors described maps on matrices or operators that preserve the ε-pseudo spectral radius and the ϵ-pseudo spectrum of different kinds of products; see for instance [1, 4, 5, 6, 7, 8, 9] and the references therein. Recently, nonlinear maps preserving the products of a mixture of the (skew) Lie product and the Jordan $*$-product have receiveed a fair a moount of attention, see [2] and its references.

In this paper, we will investigate the structure of the nonlinear maps preserving the ϵ-pseudo spectrum of different kinds of mixture product of operators on $B(H)$.

In the first lemma, we collect some preliminary results of the ϵ-pseudo spectrum which will be used to prove of the main results. For each $z \in \mathbb{C}$ and $\delta > 0$, suppose $D_\delta(z)$ is the open disk of the complex plane $\mathbb{C}$ centered at z and of radius δ.

Lemma 1.1. *(See [8, 11]) For an operator $T \in B(H)$ and $\epsilon > 0$, the following statements hold.*
(i) $\sigma(T) + D_\epsilon(0) \subseteq \sigma_\epsilon(T)$.
(ii) If T is normal, then $\sigma_\epsilon(T) = \sigma(T) + D_\epsilon(0)$.
(iii) For every $z \in \mathbb{C}, \sigma_\epsilon(T + zI) = z + \sigma_\epsilon(T)$.
(iv) For every nonzero $z \in \mathbb{C}, \sigma_\epsilon(zT) = z\sigma_{\frac{\epsilon}{|z|}}(T)$.
(v) For every $z \in \mathbb{C}$, we have $\sigma_\epsilon(T) = D_\epsilon(z)$ if and only if $T = zI$.
(vi) $\sigma_\epsilon(T^t) = \sigma_\epsilon(T)$, where T^t is the transpose of T relative to a fixed orthonormal basis of H.
(vii) For every unitary operator $U \in B(H)$, we have $\sigma_\varepsilon(UTU^*) = \sigma_\varepsilon(T)$.
(viii) For every conjugate unitary operator U, we have $\sigma_\varepsilon(UTU^*) = \sigma_\varepsilon(T^*)$.

For two nonzero vectors $x, y \in H$, let $x \otimes y$ stands for the operator of rank at most one defined by

$$(x \otimes y)z = \langle z, y\rangle\, x, \qquad \forall\ z \in H.$$

The following lemma discusse the spectrum of the skew Lie product $(y \otimes y) \bullet_* S$ for every nonzero vector $y \in H$ and $S \in B(H)$.

Lemma 1.2. *(See [3, Corollary 2.1]) Let $S \in B(H)$ and $y \in H$ be a nonzero vector. Then*

$$\sigma(S(y \otimes y) + (y \otimes y)S) = \{0, \langle Sy, y\rangle \pm \sqrt{\langle S^2y, y\rangle}\ \}.$$

The third lemma gives necessary and sufficient conditions for two operators to be equal in term of the spectrum.

Lemma 1.3. *(See [3, Lemma 2.2]) Let T and S be in $B(H)$. Then the following statements are equivalent.*
(i) $T = S$.
(ii) $\sigma(AT - TA^*) = \sigma(AS - SA^*)$ for each operator $A \in B(H)$.
(iii) $\sigma(AT - TA^*) = \sigma(AS - SA^*)$ for each operator $A \in B_a(H)$.

We will use of the following theorem in the proof of Theorem 2.2.

Theorem 1.4. *(See [8, Theorem 3.3]) A surjective map φ from $B_s(H)$ into itself satisfies*

$$\sigma_\epsilon(TS+ST)=\sigma_\epsilon(\varphi(T)\varphi(S)+\varphi(S)\varphi(T))\quad (T,S\in B_s(H))$$

if and only if there exists a unitary operator $U\in B(H)$ such that either $\varphi(T)=\mu UTU^*$ or $\varphi(T)=\mu UT^tU^*$ for all $T\in B_s(H)$, where $\mu\in\{-1,1\}$.

2 Main Results

The following theorem is one of the purposes of the paper.

Theorem 2.1. *Let φ be a bijective map on $B(H)$ satisfying*

$$\sigma_\epsilon(T_1\bullet_* T_2\circ_* T_3)=\sigma_\epsilon(\varphi(T_1)\bullet_*\varphi(T_2)\circ_*\varphi(T_3)),\quad (T_1,T_2,T_3\in B(H)).$$

Then there exist an invertible operator $S\in B(H)$ and a unitary operator $U\in B(H)$ such that $\varphi(T)=SUTU^*$ or $\varphi(T)=SUT^tU^*$ for every $T\in B(H)$.

Proof. We break the proof into several claims.

Claim 1. $\varphi(iI)^*=-\varphi(iI)\in Z(B(H))$.

By the surjectivity of φ there exists $S\in B(H)$ such that $\varphi(S)=\frac{iI}{2}$. Then

$$\begin{aligned}D_\epsilon(0)&=\sigma_\epsilon((i\varphi^{-1}(\frac{iI}{2})-i\varphi^{-1}(\frac{iI}{2}))\circ_* S)=\sigma_\epsilon(iI\bullet_*\varphi^{-1}(\frac{iI}{2})\circ_* S)\\&=\sigma_\epsilon(\varphi(iI)\bullet_*\frac{iI}{2}\circ_*\frac{iI}{2})=\sigma_\varepsilon(\frac{-1}{2}(\varphi(iI)+\varphi(iI)^*)).\end{aligned}$$

Lemma 1.1 implies that, $\varphi(iI)^*=-\varphi(iI)$.

Now let $T\in B(H)$ is arbitrary. Then

$$\begin{aligned}D_\epsilon(0)&=\sigma_\epsilon((iT-iT)\circ_* S)=(iI\bullet_* T\circ_* S)=\sigma_\epsilon(\varphi(iI)\bullet_*\varphi(T)\circ_*\varphi(S))\\&=\sigma_\epsilon((\varphi(iI)\varphi(T)+\varphi(T)\varphi(iI)^*)\circ_*\frac{iI}{2})\\&=\sigma_\epsilon(\frac{iI}{2}(\varphi(iI)(\varphi(T)-\varphi(T)^*)-(\varphi(T)-\varphi(T)^*)\varphi(iI))).\end{aligned}$$

By Lemma 1.1(v), we have $\varphi(iI)(\varphi(T)-\varphi(T)^*)-(\varphi(T)-\varphi(T)^*)\varphi(iI) = 0$. The surjectivity of φ implies that, $\varphi(iI)B = B\varphi(iI)$ for every $B \in B_a(H)$ and hence $\varphi(iI)B = B\varphi(iI)$ for every $B \in B_s(H)$. Since for every $A \in B(H)$, we have $A = A_1 + iA_2$, where A_1 and A_2 are self-adjoint elements. Hence $\varphi(iI)A = A\varphi(iI)$ holds true for all $A \in B(H)$, then $\varphi(iI) \in Z(B(H))$.

Claim 2. φ preserves the self-adjoint and anti-self-adjoint elements in both direction.

Let $T = T^*$ and $\varphi(S) = \frac{I}{2}$ for some $S \in B(H)$. We have

$$\begin{aligned} D_\epsilon(0) &= \sigma_\epsilon(S \bullet_* T \circ_* \varphi^{-1}(iI)) = \sigma_\epsilon(\frac{I}{2} \bullet_* \varphi(T) \circ_* iI) \\ &= \sigma_\epsilon(i(\varphi(T) - \varphi(T)^*)). \end{aligned}$$

It follows from Lemma 1.1 that, $\varphi(T)-\varphi(T)^* = 0$, and so $\varphi(T) = \varphi(T)^*$. Similarly, if $\varphi(T) = \varphi(T)^*$, then

$$\begin{aligned} D_\epsilon(0) &= \sigma_\epsilon(\varphi(\frac{I}{2}) \bullet_* \varphi(T) \circ_* \varphi(iI)) = \sigma_\epsilon(\frac{I}{2} \bullet_* T \circ_* iI) \\ &= \sigma_\epsilon(i(T - T^*)), \end{aligned}$$

so $T = T^*$. For the second part of this claim, let $T \in B_a(H)$ and $\varphi(S) = I$ for some $S \in B(H)$, we have

$$\begin{aligned} D_\epsilon(0) &= \sigma_\epsilon(T \bullet_* \varphi^{-1}(iI) \circ_* S) = \sigma_\epsilon(\varphi(T) \bullet_* iI \circ_* \varphi(S)) \\ &= \sigma_\epsilon(2i(\varphi(T) + \varphi(T)^*)). \end{aligned}$$

Again by Lemma 1.1, we see that $\varphi(T)^* = -\varphi(T)$ for every $T \in B_a(H)$. Conversely, let $\varphi(T)^* = -\varphi(T)$, then

$$\begin{aligned} D_\epsilon(0) &= \sigma_\epsilon(\varphi(T) \bullet_* \varphi(iI) \circ_* \varphi(I)) = \sigma_\epsilon(T \bullet_* iI \circ_* I) \\ &= \sigma_\epsilon(2i(T + T^*)), \end{aligned}$$

so $T^* + T = 0$ and $T \in B_a(H)$.

Claim 3. $\varphi^2(I)\varphi(iI) = iI$ and $\varphi^2(iI)\varphi(I) = -I$. Hence $\varphi(I)$ and $\varphi(iI)$ are invertible.

We have

$$\begin{aligned} D_\epsilon(4i) &= \sigma_\epsilon(4iI) = \sigma_\epsilon(I \bullet_* iI \circ_* I) = \sigma_\epsilon(\varphi(I) \bullet_* \varphi(iI) \circ_* \varphi(I)) \\ &= \sigma_\varepsilon((\varphi(I)\varphi(iI) + \varphi(iI)\varphi(I)^*) \circ_* \varphi(I)) = \sigma_\epsilon(4\varphi^2(I)\varphi(iI)). \end{aligned}$$

By Lemma 1.1, $\varphi^2(I)\varphi(iI) = iI$. Similarly, we have

$$\begin{aligned} D_\epsilon(-4) &= \sigma_\epsilon(-4I) = \sigma_\epsilon(I \bullet_* iI \circ_* iI) = \sigma_\varepsilon(\varphi(I) \bullet_* \varphi(iI) \circ_* \varphi(iI)) \\ &= \sigma_\varepsilon((\varphi(I)\varphi(iI) + \varphi(iI)\varphi(I)^*) \circ_* \varphi(iI)) = \sigma_\epsilon(4\varphi^2(iI)\varphi(I)). \end{aligned}$$

It follows that, again by lemma 1.1 $\varphi^2(iI)\varphi(I) = -I$.

Now, we define the map ψ of $B(H)$ into itself with $\psi(T) = -i\varphi(I)\varphi(iI)\varphi(T)$ for any $T \in B(H)$. It is clear that ψ is a bijective map which $\psi(I) = I$ and $\psi(iI) = iI$, and also satisfies $\sigma_\epsilon(T_1 \bullet_* T_2 \circ_* T_3) = \sigma_\epsilon(\psi(T_1) \bullet_* \psi(T_2) \circ_* \psi(T_3))$ for all $T_1, T_2, T_3 \in B(H)$. Furthermore, it is clear that ψ preserves the self-adjoint elements in both direction.

Claim 4. We have the following statments:
(i) $\sigma_{\frac{\epsilon}{2}}(T \circ_* S) = \sigma_{\frac{\epsilon}{2}}(\psi(T) \circ_* \psi(S))$ for every $T, S \in B(H)$.
(ii) $\psi(\frac{iI}{2}) = \frac{iI}{2}$.
(iii) $\sigma_{\frac{\epsilon}{2}}(T) = \sigma_{\frac{\epsilon}{2}}(\psi(T))$ for every $T \in B(H)$.
(iv) $\psi(iT) = i\psi(T)$ for all $T \in B_s(H)$.

(i) For every $T, S \in B(H)$, we have

$$\begin{aligned} \sigma_\epsilon(2(TS - ST^*)) &= \sigma_\epsilon(I \bullet_* T \circ_* S) = \sigma_\epsilon(\psi(I) \bullet_* \psi(T) \circ_* \psi(S)) \\ &= \sigma_\epsilon(2(\psi(T)\psi(S) - \psi(S)\psi(T)^*)). \end{aligned}$$

It follows that $\sigma_{\frac{\epsilon}{2}}(T \circ_* S) = \sigma_{\frac{\epsilon}{2}}(\psi(T) \circ_* \psi(S))$ for every $T, S \in B(H)$.

(ii) We have

$$\begin{aligned} D_\epsilon(-2) &= \sigma_\epsilon(-2I) = \sigma_\epsilon(I \bullet_* iI \circ_* \frac{iI}{2}) \\ &= \sigma_\epsilon(\psi(I) \bullet_* \psi(iI) \circ_* \psi(\frac{iI}{2}))) = \sigma_\epsilon(4i\psi(\frac{iI}{2})). \end{aligned}$$

It follows that, by lemma 1.1 $\psi(\frac{iI}{2})=\frac{iI}{2}$.

(iii) For all $T\in B(H)$, by (ii) we have

$$\begin{aligned}\sigma_{\frac{\epsilon}{2}}(iT)&=\sigma_{\frac{\epsilon}{2}}(\frac{iI}{2}T+T\frac{iI}{2})=\sigma_{\frac{\epsilon}{2}}(\frac{iI}{2}T-T(\frac{iI}{2})^*)\\&=\sigma_{\frac{\epsilon}{2}}(\psi(\frac{iI}{2})\psi(T)-\psi(T)\psi(\frac{iI}{2})^*)\\&=\sigma_{\frac{\epsilon}{2}}(\psi(\frac{iI}{2})\psi(T)+\psi(T)\psi(\frac{iI}{2}))\\&=\sigma_{\frac{\epsilon}{2}}(\frac{iI}{2}\psi(T)+\psi(T)\frac{iI}{2})=\sigma_{\frac{\epsilon}{2}}(i\psi(T)).\end{aligned}$$

this implies that, $\sigma_{\frac{\epsilon}{2}}(T)=\sigma_{\frac{\epsilon}{2}}(\psi(T))$ for every $T\in B(H)$.

(iv) Note that $S(iT)-(iT)S^*$ is normal, where $T\in B_s(H)$ and $S\in B(H)$, so from this and Lemma 1.1(ii) we get

$$\begin{aligned}\sigma(\psi(S)\psi(iT)-\psi(iT)\psi(S)^*)&=\sigma(S(iT)-(iT)S^*)=i\sigma(ST-TS^*)\\&=i\sigma(\psi(S)\psi(T)-\psi(T)\psi(S)^*)\\&=\sigma(\psi(S)(i\psi(T))-(i\psi(T))\psi(S)^*).\end{aligned}$$

By surjectivity of ψ and lemma 1.3, we have $\psi(iT)=i\psi(T)$ for every $T\in B_s(H)$.

Claim 5. There exists a unitary operator U on H such that $\psi(T)=UTU^*$ or $\psi(T)=UT^tU^*$ for every $T\in B_s(H)$.

Since ψ preserves the self-adjoint operators in both direction, Claim 4(iii) together Lemma 1.1(ii), implies that $\sigma(\psi(P))=\sigma(P)$, for every $P\in P(H)$. On the other hand, a self adjoint operator is a projection if and only if its spectrum is a subset of $\{0,1\}$. This implies that $P\in P(H)$ if and only if $\psi(P)\in P(H)$. Let $P,Q\in P(H)$ such that $PQ=QP=0$. It follows from claim 4(iv) that

$$\begin{aligned}D_{\frac{\epsilon}{2}}(0)&=\sigma_{\frac{\epsilon}{2}}(iP\circ_* Q)=\sigma_{\frac{\epsilon}{2}}(\psi(iP)\circ_*\psi(Q))\\&=\sigma_{\frac{\epsilon}{2}}(i(\psi(P)\psi(Q)+\psi(Q)\psi(P))),\end{aligned}$$

and consequently, $\psi(P)\psi(Q) + \psi(Q)\psi(P) = 0$. Since $\psi(P)$ and $\psi(Q)$ are projection, then $\psi(P)\psi(Q) = \psi(Q)\psi(P) = 0$. Conversely, if $\psi(P)$ and $\psi(Q)$ are projections such that $\psi(P)\psi(Q) = \psi(Q)\psi(P) = 0$, then a similar discussion implies that $PQ = QP = 0$. So, by [10, Corollary 1.5], there exists a unitary or conjugate unitary operator U on H such that $\psi(P) = UPU^*$ for every $P \in P(H)$.
Now let $T \in B_s(H)$ and y be an unit vector in H. First assume that U is unitary. It follows from Lemma 1.1(ii) and claim 4(iv) that

$$\begin{aligned} D_{\frac{\epsilon}{2}}(0) + \sigma(iT(y\otimes y) + (y\otimes y)iT) &= \sigma_{\frac{\epsilon}{2}}(iT(y\otimes y) + (y\otimes y)iT) \\ &= \sigma_{\frac{\epsilon}{2}}(iT(y\otimes y) - (y\otimes y)(iT)^*) \\ &= \sigma_{\frac{\epsilon}{2}}(\psi(iT)\psi(y\otimes y) - \psi(y\otimes y)\psi(iT)^*) \\ &= \sigma_{\frac{\epsilon}{2}}(i\psi(T)U(y\otimes y)U^* + U(y\otimes y)U^* i\psi(T)) \\ &= D_{\frac{\epsilon}{2}}(0) + \sigma(i\psi(T)U(y\otimes y)U^* + U(y\otimes y)U^* i\psi(T)). \end{aligned}$$

So $\sigma(T(y\otimes y) + (y\otimes y)T) = \sigma(\psi(T)U(y\otimes y)U^* + U(y\otimes y)U^*\psi(T))$. Since $Tr(T(y\otimes y)) = \langle Ty, y\rangle$ and the trace is a linear functional over the space of trace-class operators, we get

$$\begin{aligned} 2\langle Ty, y\rangle &= Tr(T(y\otimes y) + (y\otimes y)T) \\ &= Tr(\psi(T)U(y\otimes y)U^* + U(y\otimes y)U^*\psi(T)) \\ &= 2\langle U^*\psi(T)U, y\rangle . \end{aligned}$$

It follows that $\psi(T) = UTU^*$ for every $T \in B_s(H)$.

Now assume that U is conjugate unitary. We define the map $J : H \to H$ by $J(\sum_{i\in\Lambda} \lambda_i e_i) = \sum_{i\in\Lambda} \overline{\lambda_i} e_i$, where $\{e_i\}_{i\in\Lambda}$ is an orthonormal basis of H. It is easy to see that J is conjugate unitary and $JT^*J = T^t$. Let $U = VJ$, then V is unitary, and $\psi(T) = VJTJV^* = VT^tV^*$ for every $T \in B(H)$.

It is easy to see that maps $T \to T^t$ and $T \to U^*TU$ preserve the ϵ-pseudo spectrum of skew Lie product, so we might as well assume that $\psi(T) = T$ for every $T \in B_s(H)$.

Claim 6. $\psi(iT) = iT$ for every $T \in B_s(H)$.

Let $y \in H$ be an arbitrary nonzero vector and $S = iT$, where $T \in B_s(H)$. Lemma 1.1(ii) implies that

$$\begin{aligned} D_{\frac{\epsilon}{2}}(0) + \sigma(S(y \otimes y) + (y \otimes y)S) &= \sigma_{\frac{\epsilon}{2}}(S(y \otimes y) + (y \otimes y)S) \\ &= \sigma_{\frac{\epsilon}{2}}(S(y \otimes y) - (y \otimes y)S^*) \\ &= \sigma_{\frac{\epsilon}{2}}(\psi(S)\psi(y \otimes y) - \psi(y \otimes y)\psi(S)^*) \\ &= \sigma_{\frac{\epsilon}{2}}(\psi(S)(y \otimes y) + (y \otimes y)\psi(S)) \\ &= D_{\frac{\epsilon}{2}}(0) + \sigma(\psi(S)(y \otimes y) + (y \otimes y)\psi(S)). \end{aligned}$$

Hence $\sigma(S(y\otimes y)+(y\otimes y)S) = \sigma(\psi(S)(y\otimes y)+(y\otimes y)\psi(S))$. By Lemma 1.2,

$$\{0, \langle Sy, y\rangle \pm \sqrt{\langle S^2y, y\rangle}\} = \{0, \langle \psi(S)y, y\rangle \pm \sqrt{\langle \psi(S)^2y, y\rangle}\}.$$

Therefore, either

$$\langle Sy, y\rangle + \sqrt{\langle S^2y, y\rangle} = \langle \psi(S)y, y\rangle + \sqrt{\langle \psi(S)^2y, y\rangle}$$

and

$$\langle Sy, y\rangle - \sqrt{\langle S^2y, y\rangle} = \langle \psi(S)y, y\rangle - \sqrt{\langle \psi(S)^2y, y\rangle},$$

or

$$\langle Sy, y\rangle + \sqrt{\langle S^2y, y\rangle} = \langle \psi(S)y, y\rangle - \sqrt{\langle \psi(S)^2y, y\rangle}$$

and

$$\langle Sy, y\rangle - \sqrt{\langle S^2y, y\rangle} = \langle \psi(S)y, y\rangle + \sqrt{\langle \psi(S)^2y, y\rangle}.$$

We easily get that $\langle Sy, y\rangle = \langle \psi(S)y, y\rangle$ and so $\psi(iT) = iT$ for every $T \in B_s(H)$.

Claim 7. φ takes the desired form.
Let $T \in B(H)$ be arbitrary. For any nonzero vector $y \in H$ and $\alpha > 0$,

we have

$$\begin{aligned} i\alpha\sigma_{\frac{\delta}{\alpha}}((y\otimes y)T+T(y\otimes y)) &= \sigma_\delta((i\alpha y\otimes y)T-T(i\alpha y\otimes y)^*) \\ &= \sigma_\delta(\psi(i\alpha y\otimes y)\psi(T)-\psi(T)\psi(i\alpha y\otimes y)^*) \\ &= \sigma_\delta((i\alpha x\otimes x)\psi(T)+\psi(T)(i\alpha y\otimes y)) \\ &= i\alpha\sigma_{\frac{\delta}{\alpha}}((y\otimes y)\psi(T)+\psi(T)(y\otimes y)), \end{aligned}$$

where $\delta=\frac{\epsilon}{2}$. On the other hand

$$\begin{aligned} \sigma((y\otimes y)T+T(y\otimes y)) &= \bigcap_{\alpha>0}\sigma_{\frac{\delta}{\alpha}}((y\otimes y)T+T(y\otimes y)) \\ &= \bigcap_{\alpha>0}\sigma_{\frac{\delta}{\alpha}}((y\otimes y)\psi(T)+\psi(T)(y\otimes y)) \\ &= \sigma((y\otimes y)\psi(T)+\psi(T)(y\otimes y)). \end{aligned}$$

Thus $\sigma((y\otimes y)T+T(y\otimes y))=\sigma((y\otimes y)\psi(T)+\psi(T)(y\otimes y))$. Therefore, following the same argument as the one in the proof of Claim 6, one concludes that $\langle Ty,y\rangle=\langle\psi(T)y,y\rangle$ for any nonzero vector $y\in H$. Hence $\psi(T)=T$, and therefore $\varphi(T)=SUTU^*$ or $\varphi(T)=SUT^tU^*$ for every $T\in B(H)$, where $S=\varphi(I)$.
□

We closed this paper with the following theorem which characterizes bijective maps that satisfy

$$\sigma_\epsilon(T_1\lozenge T_2\diamond_* T_3)=\sigma_\epsilon(\varphi(T_1)\lozenge\varphi(T_2)\diamond_*\varphi(T_3)),\quad (T_1,T_2,T_3\in B(H)),$$

where $T_1\lozenge T_2=T_1T_2^*+T_2^*T_1$ and $T_1\diamond_* T_2=T_1T_2^*-T_2T_1$.

Theorem 2.2. *Let φ is a bijective map on $B(H)$ satisfying*

$$\sigma_\epsilon(T_1\lozenge T_2\diamond_* T_3)=\sigma_\epsilon(\varphi(T_1)\lozenge\varphi(T_2)\diamond_*\varphi(T_3)),\quad (T_1,T_2,T_3\in B(H)).$$

If $\varphi(iI)$ be anti-selfadjoint, then $\varphi^2(I)$ is invertible and there exist a unitary operator $U\in B(H)$ such that $\varphi(T)=\lambda(\varphi^2(I))^{-1}UTU^*$ or $\varphi(T)=\lambda(\varphi^2(I))^{-1}UT^tU^*$ for every $T\in B(H)$, where $\lambda\in\{-1,1\}$.

Proof. We shall prove this theorem in five steps .

Step 1. $\varphi(I)^* = \varphi(I) \in Z(B(H))$.

By the surjectivity of φ, there exist $S \in B(H)$ such that $\varphi(S) = I$. For every $T \in B(H)$, we have

$$\begin{aligned} D_\epsilon(0) &= \sigma_\epsilon(T \Diamond S \diamond_* I) = \sigma_\epsilon(\varphi(T) \Diamond \varphi(S) \diamond_* \varphi(I)) \\ &= \sigma_\epsilon(2\varphi(T)\varphi(I)^* - 2\varphi(I)\varphi(T)). \end{aligned}$$

Let $T = S$, by Lemma 1.1 we can conclude that $\varphi(I)^* = \varphi(I)$. The surjectivity of φ implies that $\varphi(I) \in Z(B(H))$.

Step 2. φ preserves the self-adjoint elements in both direction.
Let $T = T^*$. We have

$$\begin{aligned} D_\epsilon(0) &= \sigma_\epsilon(I \Diamond I \diamond_* T) = \sigma_\epsilon(\varphi(I) \Diamond \varphi(I) \diamond_* \varphi(T)) \\ &= \sigma_\epsilon(2\varphi(I)^2(\varphi(T)^* - \varphi(T))). \end{aligned}$$

This implies that $\varphi(T) = \varphi(T)^*$. Similarly, if $\varphi(T) = \varphi(T)^*$, then $T = T^*$.

Step 3. $\varphi^2(I)\varphi(iI) = iI$, that is $\varphi^2(I)$ is invertible.

We have

$$\begin{aligned} D_\epsilon(-4i) &= \sigma_\epsilon(-4iI) = \sigma_\epsilon(I \Diamond I \diamond_* iI) = \sigma_\epsilon(\varphi(I) \Diamond \varphi(I) \diamond_* \varphi(iI)) \\ &= \sigma_\epsilon(-4\varphi^2(I)\varphi(iI)). \end{aligned}$$

It follows that, by lemma 1.1 $\varphi^2(I)\varphi(iI) = iI$.

Now, defining a map ψ on $B(H)$ by $\psi(T) = \varphi^2(I)\varphi(T)$ for any $T \in B(H)$. It is clear to show that ψ is a bijection with $\psi(iI) = iI$, and satisfies $\sigma_\epsilon(T_1 \Diamond T_2 \diamond_* T_3) = \sigma_\epsilon(\psi(T_1) \Diamond \psi(T_2) \diamond_* \psi(T_3))$ for all $T_1, T_2, T_3 \in B(H)$. Furthermore, for every $T, S \in B(H)$, we have

$$\begin{aligned} \sigma_\epsilon(-2i(TS^* + S^*T)) &= \sigma_\epsilon(T \Diamond S \diamond_* iI) = \sigma_\epsilon(\psi(T) \Diamond \psi(S) \diamond_* \psi(iI)) \\ &= \sigma_\epsilon(-2i(\psi(T)\psi(S)^* + \psi(S)^*\psi(T))). \end{aligned}$$

It follows that, $\sigma_{\frac{\epsilon}{2}}(TS^* + S^*T) = \sigma_{\frac{\epsilon}{2}}(\psi(T)\psi(S)^* + \psi(S)^*\psi(T))$ for every $T, S \in B(H)$.

Step 4. There exists a unitary operator U on H such that $\psi(T) = \lambda UTU^*$ or $\psi(T) = \lambda UT^tU^*$ for every $T \in B_s(H)$, where $\lambda \in \{-1, 1\}$.

It is clear that ψ preserves the self-adjoint elements in both direction, so $\psi|_{B_s(H)} : B_s(H) \to B_s(H)$ is a bijective map which satisfies $\sigma_{\frac{\epsilon}{2}}(TS + ST) = \sigma_{\frac{\epsilon}{2}}(\psi(T)\psi(S) + \psi(S)\psi(T))$ for every $T, S \in B_s(H)$. So, by Theorem 1.4, there exists a unitary operator U on H such that $\psi(T) = \lambda UTU^*$ or $\psi(T) = \lambda UT^tU^*$ for every $T \in B_s(H)$, where $\lambda \in \{-1, 1\}$.

Since the maps $T \to T^t$ and $T \to U^*TU$ preserve the pseudo spectrum of $TS^* + S^*T$, we might as well assume that $\psi(T) = T$ for all $T \in B_s(H)$.

Step 5. $\psi(T) = T$ for all $T \in B(H)$.

Let $T \in B(H)$ be arbitrary. For any vector $y \in H$ and $\alpha > 0$, we have

$$\begin{aligned}
\alpha\sigma_{\frac{\delta}{\alpha}}(T(y \otimes y) + (y \otimes y)T) &= \sigma_\delta(T(\alpha y \otimes y) + (\alpha y \otimes y)T) \\
&= \sigma_\delta(\psi(T)\psi(\alpha y \otimes y) + \psi(\alpha y \otimes y)\psi(T)) \\
&= \sigma_\delta(\psi(T)(\alpha y \otimes y) + (\alpha y \otimes y)\psi(T)) \\
&= \alpha\sigma_{\frac{\delta}{\alpha}}(\psi(T)(y \otimes y) + (y \otimes y)\psi(T)),
\end{aligned}$$

where $\delta = \frac{\epsilon}{2}$. On the other hand

$$\begin{aligned}
\sigma(T(y \otimes y) + (y \otimes y)T) &= \bigcap_{\alpha>0} \sigma_{\frac{\delta}{\alpha}}(T(y \otimes y) + (y \otimes y)T) \\
&= \bigcap_{\alpha>0} \sigma_{\frac{\delta}{\alpha}}(\psi(T)(y \otimes y) + (y \otimes y)\psi(T)) \\
&= \sigma(\psi(T)(y \otimes y) + (y \otimes y)\psi(T)).
\end{aligned}$$

Thus $\sigma(T(y \otimes y) + (y \otimes y)T) = \sigma(\psi(T)(y \otimes y) + (y \otimes y)\psi(T))$. By the same argument of proof Claim 6 in Theorem 2.1, we conclude that

$\langle Ty, y\rangle = \langle \psi(T)y, y\rangle$ for any nonzero vector $y \in H$. As a result, $\psi(T) = T$, and therefore $\varphi(T) = \lambda(\varphi^2(I))^{-1}UTU^*$ or $\varphi(T) = \lambda(\varphi^2(I))^{-1}UT^tU^*$ for every $T \in B(H)$. $\square$

3 Conclusion

In this paper, we will investigate the structure of the nonlinear maps preserving the ϵ-pseudo spectrum of different kinds of mixture product of operators on $B(H)$.

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