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Original Research Paper

Analysis of Higher-Order Fractional Differential Equations with Fractional Boundary Conditions and Stability Insights Involving the Mittag-Leffler Operator

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Abstract. This study explores fractional differential equations with order $2 < \mu < 3$, including fractional boundary conditions, using the Mittag-Leffler operator. By adding ABC-fractional boundary conditions, we expand the conventional framework to include Dirichlet and Neumann types, offering a wider range for applying boundary conditions. We prove the existence and uniqueness of solutions through the Leray-Schauder alternative fixed point theorem and the Banach contraction principle, respectively. Additionally, we examine how solutions

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depend continuously on initial data, providing insight into the stability and robustness of the system when initial conditions change. We back up our theoretical discoveries with examples that show how the results are relevant and applicable in practice. This research greatly improves comprehension of fractional differential equations with fractional boundary conditions, providing new viewpoints and methods for studying complex systems with both regular and irregular behaviors. The impacts of this study reach a wide range of scientific and engineering fields, as using fractional derivatives in modeling offers a more precise representation of actual phenomena.

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1 Introduction

The concept of differentiation, a fundamental pillar of calculus, has been extended beyond its integer-order confines to embrace the complexities of non-local and memory-dependent phenomena. This extension, known as fractional calculus, introduces the notion of fractional derivatives. Unlike traditional derivatives, these fractional counterparts allow for the exploration of intricate behaviors and dynamic patterns that are prevalent in various natural and artificial systems.

The applications of fractional derivatives are wide-ranging, permeating diverse fields of science and engineering [14, 15, 21]. From modeling anomalous diffusion in porous media to describing the viscoelastic properties of materials, fractional derivatives have proven their efficacy in capturing intricate features that elude traditional integer-order derivatives. These applications extend into disciplines like biology, where they find use in explaining neuron firing patterns and drug dispersion in biological tissues, as well as in finance, where they offer insights into complex price dynamics.

Among the various formulations of fractional derivatives, the Atangana-Baleanu fractional derivative in the sense of Caputo (ABC-derivative) emerges as a notable mathematical tool. Its distinct advantage lies in its ability to simultaneously capture both fractal and non-fractal characteristics within a system. Unlike other fractional derivatives, the ABC-derivative excels at modeling processes that exhibit power-law growth rates and self-similar behaviors. This exceptional capability lends itself to the accurate representation of complex real-world systems that exhibit a blend of regularity and irregularity, making the ABC-derivative a potent instrument for dissecting intricate phenomena [2, 12, 16].

Recent advances in the field have further expanded the scope of fractional differential equations, exploring various complex systems and their stability properties. For instance, Ahmad et al. [4] examined the existence and stability of a neutral stochastic fractional differential system, highlighting the intricate behaviors introduced by stochastic elements in fractional systems. Farahi et al. [9] contributed to the understanding of infinite systems of fractional equations in sequence spaces, offering insights into the solvability of such systems. Additionally, the work by George et al. [10] on a coupled system of pantograph problems using positive contraction-type inequalities demonstrates the applicability of fractional calculus in handling coupled differential systems. Other significant contributions include studies on quantum inclusions [11] and hybrid versions of generalized Sturm-Liouville-Langevin equations [13], which underscore the versatility of fractional calculus in addressing diverse mathematical challenges. These studies, along with Salim et al.'s exploration of deformable implicit fractional differential equations in metric spaces [19], collectively enrich the foundation upon which our current research builds.

Building upon the foundation laid by previous research on fractional differential equations utilizing the ABC-derivative, this paper aims to address more complex scenarios by incorporating higher-order fractional differential equations with Mittag-Leffler operators. This transition from prior equations to the current problem is essential for advancing our understanding of systems exhibiting both fractal and non-fractal dynamics.

In this paper, we focus on a fractional differential equation of order $2 < \mu < 3$, incorporating fractional boundary conditions through the utilization of the ABC-fractional derivative:

$$\begin{cases} {}^{\mathcal{ABC}}\mathcal{D}^{\mu}_{0^+}u(t) = \psi\left(t, u(t), {}^{\mathcal{AB}}\mathcal{I}^{\nu}_{0^+}u(t)\right), \quad t \in I = [0,1], \quad 2 < \mu < 3, \quad 0 < \nu < 1, \\ u(0) = u_0, \\ {}^{\mathcal{ABC}}\mathcal{D}^{\mu-1}_{0^+}u(1) = u_1, \\ {}^{\mathcal{ABC}}\mathcal{D}^{\mu-2}_{0^+}u(1) = u_2. \end{cases} \tag{1}$$

where $u_0, u_1, u_2 \in \mathbb{R}$ and $\psi : I \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a real-valued function which is continuous and verifies certain conditions.

This specific form of boundary conditions introduces a novel aspect to the problem, leading to intriguing mathematical challenges and opportunities for analysis.

The primary contribution of this paper lies in its comprehensive investigation of the considered fractional differential equation with ABC-fractional derivative and fractional boundary conditions. While previous studies have explored fractional differential equations and their solutions, our research introduces a unique blend of aspects that enriches the field. Specifically, our work presents the following notable contributions:

- **Novel Boundary Conditions:** By incorporating the ABC-fractional derivative and fractional boundary conditions, our study ventures into uncharted territory, offering a new perspective on the behavior of fractional differential equations. Fractional boundary conditions possess a broader scope and can serve as a means to extend and generalize boundary conditions of the Dirichlet or Neumann types. This contribution is particularly valuable as it extends the applicability of fractional calculus to scenarios with non-standard boundary conditions.

- **Triple Analysis Approach:** To address different facets of the problem, we adopt a three-fold analytical approach. We establish the existence of solutions using the Leray-Schauder alternative fixed point theorem, ensuring the robustness of our results. Moreover, we delve into the uniqueness of solutions using the Banach principle, providing insights into the distinctiveness of solutions within the problem domain. Additionally, we explore the continuous dependence of solutions on initial data, shedding light on the stability of the system under consideration.

- **Practical Relevance:** Theoretical findings are often validated by illustrative examples. A concrete examples that illustrate the application of our results are given. These examples validate the theoretical framework.

In summary, this paper significantly extends the current understanding of fractional differential equations by introducing fractional boundary conditions through the ABC-fractional derivative. The combination of theoretical rigor, a comprehensive analytical approach, and practical relevance distinguishes our work and paves the way for further exploration of this intriguing area of mathematical research.

2 Preliminaries

This section is dedicated to introducing fundamental definitions and lemmas related to fractional calculus, which will be employed in our findings. For a deeper understanding, refer to [1, 6, 15, 17, 20] and the sources cited therein.

In this manuscript, we use the notation $\mathcal{C}(I)$ to represent the set of all real-valued continuous functions on interval I, equipped with the norm $\|u\| = \sup_{t\in I}|u(t)|$, by $\mathcal{AC}(I)$ the set of all real-valued absolutely continuous functions on interval I. Additionally, we denote by

$$\mathcal{AC}^n(I) = \left\{ w : I \longrightarrow \mathbb{R} : w^{(n-1)} \in \mathcal{AC}(I) \right\}, \qquad \text{for} \quad n \in \mathbb{N}^*.$$

.

Definition 2.1. (see [18]) Let δ, $\eta \in \mathbb{C}$. The Mittag-Leffler function is given as

$$\mathcal{E}_{\delta,\eta}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\delta+\eta)} z^n, \quad \text{for} \quad z \in \mathbb{C} \quad \text{and} \quad Re(\delta) > 0.$$

Where Γ represents the Euler Gamma function.

We denote by

$$\mathcal{E}_{\delta}(z) := \mathcal{E}_{\delta,1}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\delta+1)} z^n, \quad \text{for} \quad \delta,\, z \in \mathbb{C} \quad \text{and} \quad Re(\delta) > 0.$$

Definition 2.2. (see [15, 17]) Let $\mu > 0$, the Riemann-Liouville fractional integral of order μ is defined by

$$\mathcal{I}^{\mu}_{0^+}\varphi(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\theta)^{\mu-1}\varphi(\theta)d\theta, \quad \text{for} \quad a.e. \quad t \in I,$$

where $\varphi \in L^1(0,1)$.

Definition 2.3. (see [5, 7, 8]) Let $\varphi \in H^1(0,1)$. The ABC-fractional derivative of φ of order $0<\mu<1$ is given by

$$^{\mathcal{ABC}}\mathcal{D}^{\mu}_{0^+}\varphi(t) = \frac{\mathcal{N}(\mu)}{1-\mu}\int_0^t \mathcal{E}_{\mu}\left[-\frac{\mu}{1-\mu}(t-\theta)^{\mu}\right]\varphi'(\theta)d\theta,$$

where $\mathcal{N}(\mu)>0$ designates a normalization function which follow $\mathcal{N}(0)=\mathcal{N}(1)=1$.
The associated AB-fractional integral of order $0<\mu<1$ is given by

$$^{\mathcal{AB}}\mathcal{I}^{\mu}_{0^+}\varphi(t) = \frac{1-\mu}{\mathcal{N}(\mu)}\varphi(t) + \frac{\mu}{\mathcal{N}(\mu)}\mathcal{I}^{\mu}_{0^+}\varphi(t).$$

Definition 2.4. For $n\in\mathbb{N}^*$ and φ be such that $\varphi^{(n)}\in H^1(0,1)$. The ABC-fractional derivative of order $n<\mu<n+1$ is given by

$$^{\mathcal{ABC}}\mathcal{D}^{\mu}_{0^+}\varphi(t) := {}^{\mathcal{ABC}}\mathcal{D}^{\mu-n}_{0^+}\varphi^{(n)}(t).$$

The associated AB-fractional integral of order $n<\mu<n+1$ is given by

$$^{\mathcal{AB}}\mathcal{I}^{\mu}_{0^+}\varphi(t) := \mathcal{I}^{n}_{0^+}\,{}^{\mathcal{AB}}\mathcal{I}^{\mu-n}_{0^+}\varphi(t).$$

Lemma 2.5. *(see [3]) For $n\in\mathbb{N}^*$ and $n<\mu<n+1$. We have*

$$\left({}^{\mathcal{AB}}\mathcal{I}^{\mu}_{0^+}\,{}^{\mathcal{ABC}}\mathcal{D}^{\mu}_{0^+}u\right)(t) = u(t) + \sum_{i=0}^{n}\sigma_i t^i,$$

$\sigma_i\in\mathbb{R},\ i=0,1,2,...,n,$ *and* $u\in\mathcal{AC}^{n+1}(I)$.

3 Main Results

3.1 Existence and uniqueness results

Within this subsection, we initiate by introducing the solution to our presented problem. This solution is formulated in the subsequent lemma:

Lemma 3.1. *Let $\Psi \in \mathcal{AC}(I)$ and $\Psi(0)=0$, the following problem*

$$\begin{cases} {}^{\mathcal{ABC}}\mathcal{D}_{0^+}^{\mu} u(t) = \Psi(t), \quad t \in I=[0,1], \quad 2<\mu<3, \\ u(0)=u_0, \\ {}^{\mathcal{ABC}}\mathcal{D}_{0^+}^{\mu-1} u(1) = u_1, \\ {}^{\mathcal{ABC}}\mathcal{D}_{0^+}^{\mu-2} u(1) = u_2. \end{cases} \tag{2}$$

has a solution given as follows:

$$\begin{aligned} u(t) &= u_0 + \frac{1}{\omega}\left[u_2 - \int_0^1 (1-\theta)\Psi(\theta)d\theta - \frac{\mathcal{E}_{\mu-2,3}\left[-\frac{\mu-2}{3-\mu}\right]}{\mathcal{E}_{\mu-2,2}\left[-\frac{\mu-2}{3-\mu}\right]}\left(u_1 - \int_0^1 \Psi(\theta)d\theta\right)\right] t \\ &+ \frac{3-\mu}{\mathcal{N}(\mu-2)}\int_0^t (t-\theta)\Psi(\theta)d\theta + \frac{\mu-2}{\mathcal{N}(\mu-2)\Gamma(\mu)}\int_0^t (t-\theta)^{\mu-1}\Psi(\theta)d\theta \\ &+ \frac{1}{2\omega}\left(u_1 - \int_0^1 \Psi(\theta)d\theta\right)t^2. \end{aligned} \tag{3}$$

where

$$\omega = \frac{\mathcal{N}(\mu-2)}{3-\mu}\mathcal{E}_{\mu-2,2}\left[-\frac{\mu-2}{3-\mu}\right].$$

Proof. By applying operator ${}^{\mathcal{AB}}\mathcal{I}_{0^+}^{\mu}$ on both sides of the first equation of problem (2) (lemma 2.5), we obtain:

$$\begin{aligned} u(t) &= \sigma_0 + \sigma_1 t + \sigma_2 t^2 + {}^{\mathcal{AB}}\mathcal{I}_{0^+}^{\mu}\Psi(t) \\ &= \sigma_0 + \sigma_1 t + \sigma_2 t^2 + \mathcal{I}_{0^+}^{2}\, {}^{\mathcal{AB}}\mathcal{I}_{0^+}^{\mu-2}\Psi(t) \\ &= \sigma_0 + \sigma_1 t + \sigma_2 t^2 + \frac{3-\mu}{\mathcal{N}(\mu-2)}\int_0^t (t-\theta)\Psi(\theta)d\theta \\ &\quad + \frac{\mu-2}{\mathcal{N}(\mu-2)\Gamma(\mu)}\int_0^t (t-\theta)^{\mu-1}\Psi(\theta)d\theta. \end{aligned} \tag{4}$$

where σ_0, σ_1 and σ_2 are real numbers to be determined.

According to the previous equation, when $t=0$, we obtain

$$\sigma_0 = u(0) = u_0. \tag{5}$$

Note that, for $x \in \mathcal{AC}^3(I)$, we have

$$\begin{aligned}
&{}^{\mathcal{ABC}}\mathcal{D}_{0^+}^{\mu-1}x(t)\\
&={}^{\mathcal{ABC}}\mathcal{D}_{0^+}^{\mu-2}x'(t)\\
&=\frac{\mathcal{N}(\mu-2)}{3-\mu}\int_0^t \mathcal{E}_{\mu-2}\left[-\frac{\mu-2}{3-\mu}(t-\theta)^{\mu-2}\right]x'(\theta)d\theta\\
&=\frac{\mathcal{N}(\mu-2)}{3-\mu}\int_0^t \sum_{n=0}^{\infty}\frac{1}{\Gamma(n(\mu-2)+1)}\left(-\frac{\mu-2}{3-\mu}(t-\theta)^{\mu-2}\right)^n x'(\theta)d\theta\\
&=\frac{\mathcal{N}(\mu-2)}{3-\mu}\sum_{n=0}^{\infty}\frac{1}{\Gamma(n(\mu-2)+1)}\left(-\frac{\mu-2}{3-\mu}\right)^n\int_0^t (t-\theta)^{n(\mu-2)}x'(\theta)d\theta\\
&=\frac{\mathcal{N}(\mu-2)}{3-\mu}\sum_{n=0}^{\infty}\frac{1}{\Gamma(n(\mu-2)+2)}\left(-\frac{\mu-2}{3-\mu}\right)^n\int_0^t (t-\theta)^{n(\mu-2)+1}x'(\theta)d\theta\\
&=\frac{\mathcal{N}(\mu-2)}{3-\mu}\sum_{n=0}^{\infty}\left(-\frac{\mu-2}{3-\mu}\right)^n \mathcal{I}_{0^+}^{n(\mu-2)+2}x'(t),
\end{aligned}$$

$${}^{\mathcal{ABC}}\mathcal{D}_{0^+}^{\mu-2}x(t)=\frac{\mathcal{N}(\mu-2)}{3-\mu}\sum_{n=0}^{\infty}\left(-\frac{\mu-2}{3-\mu}\right)^n \mathcal{I}_{0^+}^{n(\mu-2)+2}x(t).$$

In particular,

$$\begin{aligned}
&{}^{\mathcal{ABC}}\mathcal{D}_{0^+}^{\mu-1}t=0,\quad {}^{\mathcal{ABC}}\mathcal{D}_{0^+}^{\mu-1}t^2=2\,{}^{\mathcal{ABC}}\mathcal{D}_{0^+}^{\mu-2}t=2\frac{\mathcal{N}(\mu-2)}{3-\mu}\mathcal{E}_{\mu-2,2}\left[-\frac{\mu-2}{3-\mu}t^{\mu-2}\right]t,\\
&{}^{\mathcal{ABC}}\mathcal{D}_{0^+}^{\mu-2}t^2=2\frac{\mathcal{N}(\mu-2)}{3-\mu}\mathcal{E}_{\mu-2,3}\left[-\frac{\mu-2}{3-\mu}t^{\mu-2}\right]t^2.
\end{aligned}$$

By applying operator ${}^{\mathcal{AB}}\mathcal{D}_{0^+}^{\mu-1}$ on both sides of equation (4) and then substituting $t=1$, we get:

$$u_1=2\omega\sigma_2+\int_0^1\Psi(\theta)d\theta.$$

Which leads to

$$\sigma_2=\frac{1}{2\omega}\left(u_1-\int_0^1\Psi(\theta)d\theta\right). \tag{6}$$

Now, by applying operator ${}^{\mathcal{AB}}\mathcal{D}_{0^+}^{\mu-2}$ on both sides of equation (4) and then substituting $t = 1$, we get:

$$
\begin{aligned}
u_2 &= \omega\sigma_1 + 2\sigma_2 \frac{\mathcal{N}(\mu-2)}{3-\mu} \mathcal{E}_{\mu-2,3}\left[-\frac{\mu-2}{3-\mu}\right] + \int_0^1 (1-\theta)\Psi(\theta)d\theta \\
&= \omega\sigma_1 + \frac{\mathcal{E}_{\mu-2,3}\left[-\frac{\mu-2}{3-\mu}\right]}{\mathcal{E}_{\mu-2,2}\left[-\frac{\mu-2}{3-\mu}\right]}\left(u_1 - \int_0^1 \Psi(\theta)d\theta\right) + \int_0^1 (1-\theta)\Psi(\theta)d\theta.
\end{aligned}
$$

Which implies that

$$
\sigma_1 = \frac{1}{\omega}\left[u_2 - \frac{\mathcal{E}_{\mu-2,3}\left[-\frac{\mu-2}{3-\mu}\right]}{\mathcal{E}_{\mu-2,2}\left[-\frac{\mu-2}{3-\mu}\right]}\left(u_1 - \int_0^1 \Psi(\theta)d\theta\right) - \int_0^1 (1-\theta)\Psi(\theta)d\theta\right]. \tag{7}
$$

By substituting the equations (5), (6), and (7) into the equation (4), we derive our outcome. □

Remark 3.2. From equation (3), we have

$$
\begin{aligned}
u''(t) &= \frac{1}{\omega}\left(u_1 - \int_0^1 \Psi(\theta)d\theta\right) + \frac{3-\mu}{\mathcal{N}(\mu-2)}\Psi(t) \\
&\quad + \frac{\mu-2}{\mathcal{N}(\mu-2)\Gamma(\mu-2)} \int_0^t (t-\theta)^{\mu-3}\Psi(\theta)d\theta.
\end{aligned}
$$

So, for $\Psi \in \mathcal{AC}(I)$, $u'' \in \mathcal{AC}(I)$, which means that $u \in \mathcal{AC}^{(3)}(I)$.

Thus, ${}^{\mathcal{ABC}}\mathcal{D}^{\mu}_{0+}u(t)$ is well defined for $2 < \mu < 3$. And by performing a simple computation, we arrive at ${}^{\mathcal{ABC}}\mathcal{D}^{\mu}_{0+}u(t) = \Psi(t)$.

This shows that we have equivalence in the previous lemma.

Now, we are poised to establish the subsequent solution operator:

$$
\begin{aligned}
\Upsilon u(t) = u_0 &+ \frac{1}{\omega}\Bigg[u_2 - \int_0^1 (1-\theta)\psi\left(\theta, u(\theta), {}^{\mathcal{AB}}\mathcal{I}_{0+}^{\nu}u(\theta)\right)d\theta \\
&- \frac{\mathcal{E}_{\mu-2,3}\left[-\frac{\mu-2}{3-\mu}\right]}{\mathcal{E}_{\mu-2,2}\left[-\frac{\mu-2}{3-\mu}\right]}\left(u_1 - \int_0^1 \psi\left(\theta, u(\theta), {}^{\mathcal{AB}}\mathcal{I}_{0+}^{\nu}u(\theta)\right)d\theta\right)\Bigg] t \\
&+ \frac{1}{2\omega}\left(u_1 - \int_0^1 \psi\left(\theta, u(\theta), {}^{\mathcal{AB}}\mathcal{I}_{0+}^{\nu}u(\theta)\right)d\theta\right)t^2 \\
&+ \frac{3-\mu}{\mathcal{N}(\mu-2)}\int_0^t (t-\theta)\psi\left(\theta, u(\theta), {}^{\mathcal{AB}}\mathcal{I}_{0+}^{\nu}u(\theta)\right)d\theta \\
&+ \frac{\mu-2}{\mathcal{N}(\mu-2)\Gamma(\mu)}\int_0^t (t-\theta)^{\mu-1}\psi\left(\theta, u(\theta), {}^{\mathcal{AB}}\mathcal{I}_{0+}^{\nu}u(\theta)\right)d\theta, \qquad (8)
\end{aligned}
$$

for $u \in \mathcal{C}(I)$.

We take into consideration the subsequent assumptions, which will serve as the foundation for establishing the existence of the solution to our presented problem:

($\mathcal{A}_1$) Let $\psi\left(., u(.), {}^{\mathcal{AB}}\mathcal{I}_{0+}^{\nu}u(.)\right) \in \mathcal{AC}(I)$ for $u \in \mathcal{AC}(I)$, and there exists tree functions $\phi_i \in \mathcal{C}(I, \mathbb{R}^+)$, $\quad i = 1, 2, 3$ such that

$$|\psi(t, y, z)| \le \phi_1(t) + \phi_2(t)|y| + \phi_3(t)|z|, \quad \text{for all}\, t \in I,\ y,\ z \in \mathbb{R}.$$

($\mathcal{A}_2$) Let $\psi\left(., u(.), {}^{\mathcal{AB}}\mathcal{I}_{0+}^{\nu}u(.)\right) \in \mathcal{AC}(I)$ for $u \in \mathcal{AC}(I)$, and there exists two functions $\psi_i \in L^1(I, \mathbb{R}^+)$, $\quad i = 1, 2$ such that

$$|\psi(t, y, z) - \psi(t, y', z')| \le \psi_1(t)|y - y'| + \psi_2(t)|z - z'|,$$

for all $\quad t \in I, \quad y,\ z,\ y',\ z' \in \mathbb{R}$.

To simplify the relatively complex formulas, we introduce the subsequent notations:

$$\lambda_1 = \frac{(1-\nu)\Gamma(\nu)+1}{\mathcal{N}(\nu)\Gamma(\nu)},$$

$$\lambda_2 = \|\phi_2\| + \lambda_1\|\phi_3\|,$$

$$\lambda_3 = |u_0| + \frac{1}{\omega}\left[\frac{1}{2}|u_1| + |u_2| + \|\phi_1\| + \frac{\mathcal{E}_{\mu-2,3}\left[-\frac{\mu-2}{3-\mu}\right]}{\mathcal{E}_{\mu-2,2}\left[-\frac{\mu-2}{3-\mu}\right]}(|u_1| + \|\phi_1\|)\right] + \left(\frac{3-\mu}{2\mathcal{N}(\mu-2)} + \frac{\mu-2}{\mathcal{N}(\mu-2)\Gamma(\mu+1)}\right)\|\phi_1\|,$$

$$\overline{\lambda_3} = |u_0| + \frac{1}{\omega}\left(\frac{1}{2}|u_1| + |u_2| + \frac{\mathcal{E}_{\mu-2,3}\left[-\frac{\mu-2}{3-\mu}\right]}{\mathcal{E}_{\mu-2,2}\left[-\frac{\mu-2}{3-\mu}\right]}|u_1|\right),$$

$$\lambda_4 = \frac{1}{\omega}\left(1 + \frac{\mathcal{E}_{\mu-2,3}\left[-\frac{\mu-2}{3-\mu}\right]}{\mathcal{E}_{\mu-2,2}\left[-\frac{\mu-2}{3-\mu}\right]}\right) + \frac{3-\mu}{2\mathcal{N}(\mu-2)} + \frac{\mu-2}{\mathcal{N}(\mu-2)\Gamma(\mu+1)},$$

$$\overline{\lambda_4} = \frac{1}{\omega}\left(\frac{3}{2} + \frac{\mathcal{E}_{\mu-2,3}\left[-\frac{\mu-2}{3-\mu}\right]}{\mathcal{E}_{\mu-2,2}\left[-\frac{\mu-2}{3-\mu}\right]}\right) + \frac{3-\mu}{\mathcal{N}(\mu-2)} + \frac{\mu-2}{\mathcal{N}(\mu-2)\Gamma(\mu)},$$

$$\lambda_5 = \int_0^1 \psi_1(\theta)d\theta + \lambda_1\int_0^1 \psi_2(\theta)d\theta.$$

Remark 3.3. 1. For $u \in \mathcal{AC}(I)$, we have

$$\begin{aligned}\left|{}^{\mathcal{AB}}\mathcal{I}_{0^+}^{\nu}u(t)\right| &= \left|\frac{1-\nu}{\mathcal{N}(\nu)}u(t) + \frac{\nu}{\mathcal{N}(\nu)}\mathcal{I}_{0^+}^{\nu}u(t)\right| \\ &\leq \frac{(1-\nu)\Gamma(\nu)+1}{\mathcal{N}(\nu)\Gamma(\nu)}\|u\| \\ &\leq \lambda_1\|u\|, \qquad \text{for all} \quad t \in I.\end{aligned}$$

2. From $(\mathcal{A}_2)$, we have

$$\begin{aligned}\left|\psi\left(t, u(t), {}^{\mathcal{AB}}\mathcal{I}_{0^+}^{\nu} u(t)\right)\right| &\leq \|\phi_1\| + (\|\phi_2\| + \lambda_1\|\phi_3\|)\|u\| \\ &\leq \|\phi_1\| + \lambda_2\|u\|, \quad \text{for all}\, t \in I,\ u \in \mathcal{AC}(I).\end{aligned}$$

Now, we have all the necessary data to present our first existence result.

Theorem 3.4. *Let ψ be a function satisfying assumption $(\mathcal{A}_1)$ and $\psi(0, u_0, 0) = 0$. Suppose in addition that $\lambda_2\lambda_4 < 1$. Then, the problem (1) possesses at least one solution.*

Proof. Let us consider the closed ball

$$\mathcal{B}_{\varrho} = \{u \in \mathcal{C}(I) : \|u\| \leq \varrho\},$$

where

$$\varrho \geq \frac{\lambda_3}{1 - \lambda_2\lambda_4}.$$

We transform our problem into a fixed point problem associated with the operator Υ, which is defined by (8).

(i) Let us prove that $\Upsilon(\mathcal{B}_{\varrho}) \subseteq \mathcal{B}_{\varrho}$.

Using remark 3.3, for $u \in \mathcal{B}_\varrho$, we have

$$
\begin{aligned}
&|\Upsilon u(t)| \\
&\leq \frac{1}{\omega}\left[|u_2| + \frac{1}{2}\|\phi_1\| + \frac{1}{2}\lambda_2\|u\| + \frac{\mathcal{E}_{\mu-2,3}\left[-\dfrac{\mu-2}{3-\mu}\right]}{\mathcal{E}_{\mu-2,2}\left[-\dfrac{\mu-2}{3-\mu}\right]}\left(|u_1| + \|\phi_1\| + \lambda_2\|u\|\right)\right] \\
&\quad + |u_0| + \frac{1}{2\omega}\left(|u_1| + \|\phi_1\| + \lambda_2\|u\|\right) + \frac{3-\mu}{2\mathcal{N}(\mu-2)}\left(\|\phi_1\| + \lambda_2\|u\|\right) \\
&\quad + \frac{\mu-2}{\mathcal{N}(\mu-2)\Gamma(\mu+1)}\left(\|\phi_1\| + \lambda_2\|u\|\right) \\
&\leq |u_0| + \frac{1}{\omega}\left[\frac{1}{2}|u_1| + |u_2| + \|\phi_1\| + \frac{\mathcal{E}_{\mu-2,3}\left[-\dfrac{\mu-2}{3-\mu}\right]}{\mathcal{E}_{\mu-2,2}\left[-\dfrac{\mu-2}{3-\mu}\right]}\left(|u_1| + \|\phi_1\|\right)\right] \\
&\quad + \left(\frac{3-\mu}{2\mathcal{N}(\mu-2)} + \frac{\mu-2}{\mathcal{N}(\mu-2)\Gamma(\mu+1)}\right)\|\phi_1\| \\
&\quad + \lambda_2\left[\frac{1}{\omega}\left(1 + \frac{\mathcal{E}_{\mu-2,3}\left[-\dfrac{\mu-2}{3-\mu}\right]}{\mathcal{E}_{\mu-2,2}\left[-\dfrac{\mu-2}{3-\mu}\right]}\right) + \frac{3-\mu}{2\mathcal{N}(\mu-2)} + \frac{\mu-2}{\mathcal{N}(\mu-2)\Gamma(\mu+1)}\right]\|u\| \\
&\leq \lambda_3 + \lambda_2\lambda_4\varrho \\
&\leq \varrho, \qquad \forall t \in I
\end{aligned}
$$

which means that Υ maps $\mathcal{B}_\varrho$ into itself.

(ii) Now, we show that Υ is completely continuous:
In (i), we have shown that $\Upsilon(\mathcal{B}_\varrho)$ is bounded, and note that the continuity of ψ ensures that of Υ.

The next step is to show that Υ is equicontinuous:
For $u \in \mathcal{B}_\varrho$, $0 \leq t_1 < t_2 \leq 1$, we have

$$
\begin{aligned}
&|\Upsilon u(t_2) - \Upsilon u(t_1)| \\
&\leq \frac{1}{\omega}\left[|u_2| + \int_0^1 (1-\theta)\left|\psi\left(\theta, u(\theta), {}^{\mathcal{AB}}\mathcal{I}_{0+}^{\nu} u(\theta)\right)\right| d\theta\right. \\
&\quad \left. + \frac{\mathcal{E}_{\mu-2,3}\left[-\dfrac{\mu-2}{3-\mu}\right]}{\mathcal{E}_{\mu-2,2}\left[-\dfrac{\mu-2}{3-\mu}\right]}\left(|u_1| + \int_0^1 \left|\psi\left(\theta, u(\theta), {}^{\mathcal{AB}}\mathcal{I}_{0+}^{\nu} u(\theta)\right)\right| d\theta\right)\right]|t_2 - t_1| \\
&\quad + \frac{1}{2\omega}\left(|u_1| + \int_0^1 \left|\psi\left(\theta, u(\theta), {}^{\mathcal{AB}}\mathcal{I}_{0+}^{\nu} u(\theta)\right)\right| d\theta\right)\left|t_2^2 - t_1^2\right|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{3-\mu}{\mathcal{N}(\mu-2)}\left|t_2-t_1\right| \int_0^{t_1}\left|\psi\left(\theta, u(\theta),{ }^{\mathcal{AB}} \mathcal{I}_{0+}^\nu u(\theta)\right)\right| d\theta \\
&+\frac{3-\mu}{\mathcal{N}(\mu-2)} \int_{t_1}^{t_2}\left(t_2-\theta\right)\left|\psi\left(\theta, u(\theta),{ }^{\mathcal{AB}} \mathcal{I}_{0+}^\nu u(\theta)\right)\right| d\theta \\
&+\frac{\mu-2}{\mathcal{N}(\mu-2)\Gamma(\mu)} \int_0^{t_1}\left[\left(t_2-\theta\right)^{\mu-1}-\left(t_1-\theta\right)^{\mu-1}\right]\left|\psi\left(\theta, u(\theta),{ }^{\mathcal{AB}} \mathcal{I}_{0+}^\nu u(\theta)\right)\right| d\theta \\
&+\frac{\mu-2}{\mathcal{N}(\mu-2)\Gamma(\mu)} \int_{t_1}^{t_2}\left(t_2-\theta\right)^{\mu-1}\left|\psi\left(\theta, u(\theta),{ }^{\mathcal{AB}} \mathcal{I}_{0+}^\nu u(\theta)\right)\right| d\theta \\
\leq &\frac{1}{\omega}\left[\left|u_2\right|+\frac{1}{2}\left\|\phi_1\right\|+\frac{1}{2} \lambda_2 \varrho+\frac{\mathcal{E}_{\mu-2,3}\left[-\dfrac{\mu-2}{3-\mu}\right]}{\mathcal{E}_{\mu-2,2}\left[-\dfrac{\mu-2}{3-\mu}\right]}\left(\left|u_1\right|+\left\|\phi_1\right\|+\lambda_2 \varrho\right)\right]\left|t_2-t_1\right| \\
&+\frac{1}{2\omega}\left(\left|u_1\right|+\left\|\phi_1\right\|+\lambda_2 \varrho\right)\left|t_2^2-t_1^2\right|+\frac{3-\mu}{\mathcal{N}(\mu-2)}\left(\left\|\phi_1\right\|+\lambda_2 \varrho\right)\left|t_2-t_1\right| \\
&+\frac{3-\mu}{2\mathcal{N}(\mu-2)}\left(\left\|\phi_1\right\|+\lambda_2 \varrho\right)\left(t_2-t_1\right)^2 \\
&+\frac{\mu-2}{\mathcal{N}(\mu-2)\Gamma(\mu+1)}\left(\left\|\phi_1\right\|+\lambda_2 \varrho\right)\left|t_2^\mu-t_1^\mu\right| .
\end{aligned}
$$

So,

$$
\left|\Upsilon u(t_2)-\Upsilon u(t_1)\right| \longrightarrow 0 \quad \text{as} \quad t_2 \rightarrow t_1
$$

Then, according to Arzela-Ascoli's Theorem, Υ is relatively compact.
Thus, it is completely continuous.

(iii) The following set

$$
\Theta=\{u \in \mathcal{C}(I): u(t)=\kappa(\Upsilon u)(t) \quad \text{for some} \quad \kappa \in(0,1)\},
$$

is bounded. Indeed:
For $u \in \Theta$ and $t \in I$, we have

$$
\begin{aligned}
|u(t)| &= \kappa|\Upsilon u(t)| \\
&<|\Upsilon u(t)| \\
&<\lambda_3+\lambda_2 \lambda_4\|u\| .
\end{aligned}
$$

Hence, we get

$$
\|u\| \leq \frac{\lambda_3}{1-\lambda_2 \lambda_4}<\infty .
$$

Thanks to Lery-Schauder alternative, our problem has at least one solution. □

The theorem presented below establishes the result of uniqueness.

Theorem 3.5. *Let ψ be a function satisfying assumption $(\mathcal{A}_2)$ and $\psi(0, u_0, 0) = 0$. Suppose in addition that $\overline{\lambda_4}\lambda_5 < 1$. Then problem (1) has a unique solution.*

Proof. We consider the following closed ball:

$$\mathcal{B}_\rho = \{u \in \mathcal{C}(I) : \|u\| \leq \rho\},$$

where

$$\rho \geq \frac{\overline{\lambda_3} + \lambda_4\gamma}{1 - \overline{\lambda_4}\lambda_5} \quad \text{and} \quad \gamma = \sup_{t\in I} |\psi(t, 0, 0)|.$$

Not that, using $(\mathcal{A}_2)$, we have

$$\begin{aligned}\left|\psi\left(t, u(t), {}^{\mathcal{AB}}\mathcal{I}^\nu_{0^+}u(t)\right)\right| &\leq \left|\psi\left(t, u(t), {}^{\mathcal{AB}}\mathcal{I}^\nu_{0^+}u(t)\right) - \psi(t, 0, 0)\right| + |\psi(t, 0, 0)| \\ &\leq \psi_1(t)\,|u(t)| + \psi_2(t)\left|{}^{\mathcal{AB}}\mathcal{I}^\nu_{0^+}u(t)\right| + \gamma \\ &\leq (\psi_1(t) + \lambda_1\psi_2(t))\,\rho + \gamma, \quad \forall t \in I \quad \text{and} \quad u \in \mathcal{B}_\rho.\end{aligned}$$

Similarly, we have

$$\left|\psi\left(t, u(t), {}^{\mathcal{AB}}\mathcal{I}^\nu_{0^+}u(t)\right) - \psi\left(t, v(t), {}^{\mathcal{AB}}\mathcal{I}^\nu_{0^+}v(t)\right)\right| \leq (\psi_1(t) + \lambda_1\psi_2(t))\,\|u - v\|,$$
$$\forall t \in I \quad \text{and} \quad u,\, v \in \mathcal{C}(I)$$

Firstly, we prove that Υ maps $\mathcal{B}_\rho$ into itself.
For $u \in \mathcal{B}_\rho$ and $t \in I$, we have

$$\begin{aligned}|\Upsilon u(t)| \leq |u_0| &+ \frac{1}{\omega}\left[|u_2| + \int_0^1 (1-\theta)\left[(\psi_1(\theta) + \lambda_1\psi_2(\theta))\,\rho + \gamma\right] d\theta\right. \\ &+ \left.\frac{\mathcal{E}_{\mu-2,3}\left[-\dfrac{\mu-2}{3-\mu}\right]}{\mathcal{E}_{\mu-2,2}\left[-\dfrac{\mu-2}{3-\mu}\right]}\left(|u_1| + \int_0^1 ((\psi_1(\theta) + \lambda_1\psi_2(\theta))\,\rho + \gamma)\, d\theta\right)\right] \\ &+ \frac{1}{2\omega}\left(|u_1| + \int_0^1 ((\psi_1(\theta) + \lambda_1\psi_2(\theta))\,\rho + \gamma)\, d\theta\right) \\ &+ \frac{3-\mu}{\mathcal{N}(\mu-2)}\int_0^t (t-\theta)\,((\psi_1(\theta) + \lambda_1\psi_2(\theta))\,\rho + \gamma)\, d\theta \\ &+ \frac{\mu-2}{\mathcal{N}(\mu-2)\Gamma(\mu)}\int_0^t (t-\theta)^{\mu-1}\,((\psi_1(\theta) + \lambda_1\psi_2(\theta))\,\rho + \gamma)\, d\theta\end{aligned}$$

$$\begin{aligned}
&\leq |u_0| + \frac{1}{\omega}\left[|u_2| + \frac{1}{2}\gamma + \rho\int_0^1 \psi_1(\theta)d\theta + \lambda_1\rho\int_0^1 \psi_2(\theta)d\theta\right.\\
&\quad\left. + \frac{\mathcal{E}_{\mu-2,3}\left[-\frac{\mu-2}{3-\mu}\right]}{\mathcal{E}_{\mu-2,2}\left[-\frac{\mu-2}{3-\mu}\right]}\left(|u_1| + \gamma + \rho\int_0^1 \psi_1(\theta)d\theta + \lambda_1\rho\int_0^1 \psi_2(\theta)d\theta\right)\right]\\
&\quad + \frac{1}{2\omega}\left(|u_1| + \gamma + \rho\int_0^1 \psi_1(\theta)d\theta + \lambda_1\rho\int_0^1 \psi_2(\theta)d\theta\right)\\
&\quad + \frac{3-\mu}{2\mathcal{N}(\mu-2)}\left(\gamma + 2\rho\int_0^1 \psi_1(\theta)d\theta + 2\lambda_1\rho\int_0^1 \psi_2(\theta)d\theta\right)\\
&\quad + \frac{\mu-2}{\mathcal{N}(\mu-2)\Gamma(\mu+1)}\left(\gamma + \mu\rho\int_0^1 \psi_1(\theta)d\theta + \mu\lambda_1\rho\int_0^1 \psi_2(\theta)d\theta\right)\\
&\leq \overline{\lambda_3} + \gamma\lambda_4 + \overline{\lambda_4}\lambda_5\rho\\
&\leq \rho
\end{aligned}$$

Now, let us show that Υ is a contraction:
For u, $v \in \mathcal{B}_\rho$ and $t \in I$, we have

$$\begin{aligned}
&|\Upsilon u(t) - \Upsilon v(t)|\\
&\leq \frac{1}{\omega}\int_0^1 (1-\theta)\left|\psi\left(\theta, u(\theta), {}^{\mathcal{AB}}\mathcal{I}_{0^+}^{\nu}u(\theta)\right) - \psi\left(\theta, v(\theta), {}^{\mathcal{AB}}\mathcal{I}_{0^+}^{\nu}v(\theta)\right)\right| d\theta\\
&\quad + \frac{\mathcal{E}_{\mu-2,3}\left[-\frac{\mu-2}{3-\mu}\right]}{\omega\mathcal{E}_{\mu-2,2}\left[-\frac{\mu-2}{3-\mu}\right]}\int_0^1 \left|\psi\left(\theta, u(\theta), {}^{\mathcal{AB}}\mathcal{I}_{0^+}^{\nu}u(\theta)\right) - \psi\left(\theta, v(\theta), {}^{\mathcal{AB}}\mathcal{I}_{0^+}^{\nu}v(\theta)\right)\right| d\theta\\
&\quad + \frac{1}{2\omega}\int_0^1 \left|\psi\left(\theta, u(\theta), {}^{\mathcal{AB}}\mathcal{I}_{0^+}^{\nu}u(\theta)\right) - \psi\left(\theta, v(\theta), {}^{\mathcal{AB}}\mathcal{I}_{0^+}^{\nu}v(\theta)\right)\right| d\theta\\
&\quad + \frac{3-\mu}{\mathcal{N}(\mu-2)}\int_0^t (t-\theta)\left|\psi\left(\theta, u(\theta), {}^{\mathcal{AB}}\mathcal{I}_{0^+}^{\nu}u(\theta)\right) - \psi\left(\theta, v(\theta), {}^{\mathcal{AB}}\mathcal{I}_{0^+}^{\nu}v(\theta)\right)\right| d\theta\\
&\quad + \frac{\mu-2}{\mathcal{N}(\mu-2)\Gamma(\mu)}\int_0^t (t-\theta)^{\mu-1}\left|\psi\left(\theta, u(\theta), {}^{\mathcal{AB}}\mathcal{I}_{0^+}^{\nu}u(\theta)\right) - \psi\left(\theta, v(\theta), {}^{\mathcal{AB}}\mathcal{I}_{0^+}^{\nu}v(\theta)\right)\right| d\theta.
\end{aligned}$$

Using remark 3.3, we obtain then

$$\begin{aligned}
&|\Upsilon u(t) - \Upsilon v(t)|\\
&\leq \lambda_5\left(\frac{3}{2\omega} + \frac{\mathcal{E}_{\mu-2,3}\left[-\frac{\mu-2}{3-\mu}\right]}{\omega\mathcal{E}_{\mu-2,2}\left[-\frac{\mu-2}{3-\mu}\right]} + \frac{3-\mu}{\mathcal{N}(\mu-2)} + \frac{\mu-2}{\mathcal{N}(\mu-2)\Gamma(\mu)}\right)\|u-v\|\\
&\leq \overline{\lambda_4}\lambda_5\|u-v\|.
\end{aligned}$$

Hence, in accordance with the Banach Contraction Principle, our problem (1) admits a unique solution. □

3.2 Continuous dependence of the solution from the initial data

Let us denote by $\overline{\lambda_{3,u_0}} = \overline{\lambda_3}$ and

$$\overline{\lambda_{3,v_0}} = |v_0| + \frac{1}{\omega}\left(\frac{1}{2}|u_1| + |u_2| + \frac{\mathcal{E}_{\mu-2,3}\left[-\dfrac{\mu-2}{3-\mu}\right]}{\mathcal{E}_{\mu-2,2}\left[-\dfrac{\mu-2}{3-\mu}\right]}|u_1|\right),$$

where u_0, $v_0 \in \mathbb{R}$.

Theorem 3.6. *Let ψ be a function satisfying assumption $(\mathcal{A}_2)$ and $\psi(0, u_0, 0) = \psi(0, v_0, 0) = 0$. Suppose in addition that $\overline{\lambda_4}\lambda_5 < 1$.*
Let $u = u(t, u_0)$ and $v = v(t, v_0)$ be solutions of (1) corresponding to $u(0) = u_0$ and $v(0) = v_0$, respectively. Then

$$\|u - v\| \leq \frac{1}{1 - \overline{\lambda_4}\lambda_5}|u_0 - v_0|.$$

Proof. Note that u and v exist and are well-defined according to the previous theorem, where we can define the operator Υ on the following closed ball:

$$\mathcal{B}_r = \{w \in \mathcal{C}(I) : \|w\| \leq r\},$$

where

$$r \geq \frac{max(\overline{\lambda_{3,u_0}}, \overline{\lambda_{3,v_0}}) + \lambda_4\gamma}{1 - \overline{\lambda_4}\lambda_5}.$$

Furthermore, we have

$$\begin{aligned}
&|u(t,u_0)-v(t,v_0)|\\
&\leq |u_0-v_0|+\frac{1}{\omega}\int_0^1(1-\theta)\left|\psi\left(\theta,u(\theta),{}^{\mathcal{AB}}\mathcal{I}^{\nu}_{0^+}u(\theta)\right)-\psi\left(\theta,v(\theta),{}^{\mathcal{AB}}\mathcal{I}^{\nu}_{0^+}v(\theta)\right)\right|d\theta\\
&\quad+\frac{\mathcal{E}_{\mu-2,3}\left[-\frac{\mu-2}{3-\mu}\right]}{\omega\mathcal{E}_{\mu-2,2}\left[-\frac{\mu-2}{3-\mu}\right]}\int_0^1\left|\psi\left(\theta,u(\theta),{}^{\mathcal{AB}}\mathcal{I}^{\nu}_{0^+}u(\theta)\right)-\psi\left(\theta,v(\theta),{}^{\mathcal{AB}}\mathcal{I}^{\nu}_{0^+}v(\theta)\right)\right|d\theta\\
&\quad+\frac{1}{2\omega}\int_0^1\left|\psi\left(\theta,u(\theta),{}^{\mathcal{AB}}\mathcal{I}^{\nu}_{0^+}u(\theta)\right)-\psi\left(\theta,v(\theta),{}^{\mathcal{AB}}\mathcal{I}^{\nu}_{0^+}v(\theta)\right)\right|d\theta\\
&\quad+\frac{3-\mu}{\mathcal{N}(\mu-2)}\int_0^t(t-\theta)\left|\psi\left(\theta,u(\theta),{}^{\mathcal{AB}}\mathcal{I}^{\nu}_{0^+}u(\theta)\right)-\psi\left(\theta,v(\theta),{}^{\mathcal{AB}}\mathcal{I}^{\nu}_{0^+}v(\theta)\right)\right|d\theta\\
&\quad+\frac{\mu-2}{\mathcal{N}(\mu-2)\Gamma(\mu)}\int_0^t(t-\theta)^{\mu-1}\left|\psi\left(\theta,u(\theta),{}^{\mathcal{AB}}\mathcal{I}^{\nu}_{0^+}u(\theta)\right)-\psi\left(\theta,v(\theta),{}^{\mathcal{AB}}\mathcal{I}^{\nu}_{0^+}v(\theta)\right)\right|d\theta\\
&\leq |u_0-v_0|+\overline{\lambda_4}\lambda_5\|u-v\|,\qquad \text{for all}\quad t\in I.
\end{aligned}$$

Hence, we get □

$$\|u-v\|\leq\frac{1}{1-\overline{\lambda_4}\lambda_5}|u_0-v_0|.$$

To better understand obtained results, let us consider the following illustrative example:

Example 3.7. The following problem is under consideration:

$$\begin{cases}
{}^{\mathcal{ABC}}\mathcal{D}^{\frac{5}{2}}_{0^+}u(t)=\dfrac{t}{e^{\frac{1}{2}t^2}+20}\left(1+\dfrac{|u(t)|}{|u(t)|+1}+\dfrac{\left|{}^{\mathcal{AB}}\mathcal{I}^{\frac{1}{4}}_{0^+}u(t)\right|}{\left|{}^{\mathcal{AB}}\mathcal{I}^{\frac{1}{4}}_{0^+}u(t)\right|+1}\right),\ t\in I=[0,1],\\
u(0)=\dfrac{1}{40},\\
{}^{\mathcal{ABC}}\mathcal{D}^{\frac{3}{2}}_{0^+}u(1)=\dfrac{1}{12},\\
{}^{\mathcal{ABC}}\mathcal{D}^{\frac{1}{2}}_{0^+}u(1)=\dfrac{1}{24}.
\end{cases}\tag{9}$$

The problem bellow can be expressed as (1), where

$$\psi\left(t,u(t),{}^{\mathcal{AB}}\mathcal{I}^{\frac{1}{4}}_{0^+}u(t)\right)=\frac{t}{e^{\frac{1}{2}t^2}+20}\left(1+\frac{|u(t)|}{|u(t)|+1}+\frac{\left|{}^{\mathcal{AB}}\mathcal{I}^{\frac{1}{4}}_{0^+}u(t)\right|}{\left|{}^{\mathcal{AB}}\mathcal{I}^{\frac{1}{4}}_{0^+}u(t)\right|+1}\right),$$

$$u_0=\frac{1}{40},\quad u_1=\frac{1}{12}\quad\text{and}\quad u_2=\frac{1}{24}.$$

We consider $\mathcal{N}(x) = 1$, for all $x \in I$.
On can verify that assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ are satisfied, where

$$\phi_1(t) = \phi_2(t) = \phi_3(t) = \psi_1(t) = \psi_2(t) = \frac{t}{e^{\frac{1}{2}t^2} + 20}, \qquad \text{for} \quad t \in I.$$

By performing calculations, we obtain

$$\int_0^1 \psi_j(\theta)d\theta = \frac{1}{40}\left(1 + \ln(441) - 2\ln(20 + \sqrt{e})\right), \quad j = 1, 2,$$
$$\|\phi_i\| = \gamma = \frac{1}{e^{\frac{1}{2}} + 20}, \quad \text{for} \quad i = 1, 2, 3,$$
$$\omega \simeq 1.1118; \quad \lambda_1 \simeq 1.0642; \quad \lambda_2 \simeq 0.0953; , 2.2123; \quad \overline{\lambda_4} \simeq 2.6299,$$
$$\lambda_2\lambda_4 \simeq 0.2108 < 1; \quad \overline{\lambda_4}\lambda_5 \simeq 0.1276 < 1.$$

Therefore, all assumptions of theorems 3.4 and 3.5 are satisfied, and thus problem (9) has a unique solution.

Example 3.8. Consider the following problem:

$$\begin{cases} {}^{\mathcal{ABC}}D^{\mu}_{0+}u(t) = \psi(t, u(t), {}^{\mathcal{AB}}\mathcal{I}^{\nu}_{0+}u(t)), \quad t \in [0, 1], \\ u(0) = 0.01, \\ {}^{\mathcal{ABC}}\mathcal{D}^{\mu-1}_{0+}u(1) = 0.02, \\ {}^{\mathcal{ABC}}\mathcal{D}^{\mu-2}_{0+}u(1) = 0.03, \end{cases} \tag{10}$$

where $2 < \mu < 3$, $0 < \nu < 1$, and

$$\psi(t, u(t), {}^{\mathcal{AB}}\mathcal{I}^{\nu}_{0+}u(t)) = \frac{t^2}{40}\left(1 + \frac{|u(t)|}{1 + |u(t)|}\right) + \frac{1}{16}\left|{}^{\mathcal{AB}}\mathcal{I}^{\nu}_{0+}u(t)\right|.$$

For $\nu = 0.5$, we get:

μ	$\overline{\lambda_3}$	$\overline{\lambda_4}$	λ_5	$\overline{\lambda_4} \cdot \lambda_5$
2.7	0.064	2.247	0.052	0.1168
2.8	0.066	2.292	0.052	0.1192
2.9	0.068	2.337	0.052	0.1215

Theoretically, assumption $(\mathcal{A}_2)$ is satisfied, and furthermore, $\overline{\lambda_4}\lambda_5 < 1$. Therefore, according to Theorem 3.5, this problem has a unique solution.

Numerical Results for $u(t)$ are given in the following table:

t	$u(t)$ for $\mu = 2.7$	$u(t)$ for $\mu = 2.8$	$u(t)$ for $\mu = 2.9$
0.00	0.0100	0.0100	0.0100
0.02	0.01005	0.01004	0.01003
0.04	0.01010	0.01008	0.01006
0.06	0.01015	0.01012	0.01009
0.08	0.01020	0.01016	0.01012
0.10	0.01025	0.01020	0.01015
0.12	0.01030	0.01024	0.01018
0.14	0.01035	0.01028	0.01021
0.16	0.01040	0.01032	0.01024
0.18	0.01045	0.01036	0.01027
0.20	0.01050	0.01040	0.01030
0.22	0.01055	0.01044	0.01033
0.24	0.01060	0.01048	0.01036
0.26	0.01065	0.01052	0.01039
0.28	0.01070	0.01056	0.01042
0.30	0.01075	0.01060	0.01045
0.32	0.01080	0.01064	0.01048
0.34	0.01085	0.01068	0.01051
0.36	0.01090	0.01072	0.01054
0.38	0.01095	0.01076	0.01057
0.40	0.01100	0.01080	0.01060
0.42	0.01105	0.01084	0.01063
0.44	0.01110	0.01088	0.01066
0.46	0.01115	0.01092	0.01069
0.48	0.01120	0.01096	0.01072
0.50	0.01125	0.01100	0.01075
0.52	0.01130	0.01104	0.01078
0.54	0.01135	0.01108	0.01081
0.56	0.01140	0.01112	0.01084
0.58	0.01145	0.01116	0.01087
0.60	0.01150	0.01120	0.01090
0.62	0.01155	0.01124	0.01093
0.64	0.01160	0.01128	0.01096
0.66	0.01165	0.01132	0.01099
0.68	0.01170	0.01136	0.01102
0.70	0.01175	0.01140	0.01105
0.72	0.01180	0.01144	0.01108
0.74	0.01185	0.01148	0.01111
0.76	0.01190	0.01152	0.01114
0.78	0.01195	0.01156	0.01117
0.80	0.01200	0.01160	0.01120
0.82	0.01205	0.01164	0.01123
0.84	0.01210	0.01168	0.01126
0.86	0.01215	0.01172	0.01129
0.88	0.01220	0.01176	0.01132
0.90	0.01225	0.01180	0.01135
0.92	0.01230	0.01184	0.01138
0.94	0.01235	0.01188	0.01141
0.96	0.01240	0.01192	0.01144
0.98	0.01245	0.01196	0.01147
1.00	0.01250	0.01200	0.01150

Table 1: Approximate values of $u(t)$ for different values of μ

The graphical resolution for different values of μ is given in the following figure.

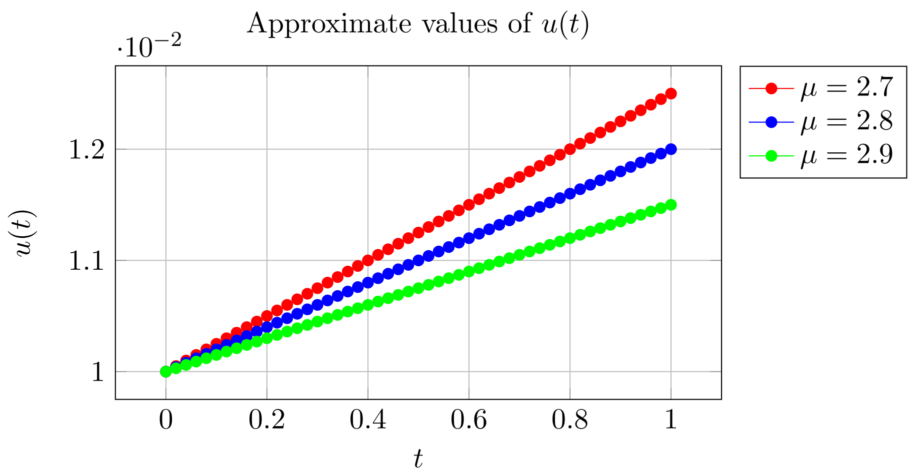

Figure 1: Graphical representation of $u(t)$ for different values of μ

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Conclusion

In this paper, we have expanded the examination of fractional differential equations by including ABC-fractional boundary conditions, providing a wider and more comprehensive perspective than conventional boundary conditions. Our main contributions consist of new boundary conditions, a thorough analytical method, and practical significance demonstrated by examples. Our work stands out due to the blend of theoretical rigor and practical applications, opening up opportunities for additional exploration in this fascinating field of mathematical research.

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