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A Note on ϕ-Morphisms of Hilbert H^*-Modules

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Abstract. In this paper, we demonstrate notion of ϕ-morphism of Hilbert H^*-modules and describe some properties of these module maps. Moreover, we show that if $\phi : A \to B$ is an injective morphism of simple H^*-algebras, the range of $\phi|_{\tau(A)}$ is τ_B-closed, $\{e_i\}_{i\in I}$ is a maximal family of doubly orthogonal minimal projections for A, $\Phi : E \to F$ is a surjective ϕ-morphism of Hilbert H^*-modules, $\{u_{\lambda,i}\}_{\lambda\in\Lambda}$ is an orthonormal basis for E in which for each $\lambda \in \Lambda$, $[u_{\lambda,i}|u_{\lambda,i}] = e_i$ $(i \in I)$ and F is full, then $\{\phi(e_i)\}_{i\in I}$ and $\{\Phi(u_{\lambda,i})\}_{\lambda\in\Lambda}$ are maximal family of doubly orthogonal minimal projections for B and orthonormal basis for F respectively.

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1. Introduction

The notion of ϕ-homomorphism of Hilbert C^*-modules first was introduced by Bakic in [2], then Joita [7] described it in the framework of Hilbert modules over locally C^*-algebras. Authors of [12] and [6] studied ϕ-homomorphisms of Finsler modules over C^*-algebras and Finsler modules over H^*-algebras respectively. Some properties of ϕ-homomorphisms are stable under Hilbert H^*-modules [3,5]. In this paper we use these properties to discover new ones for

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ϕ-homomorphisms of Hilbert H^*-modules. An H^*-algebra, was introduced by Ambrose [1] in the associative case, is a Banach algebra A satisfying the following conditions:
(i) A is itself a Hilbert space under an inner product $\langle .,. \rangle$;
(ii) For each a in A, there is an element a^* in A, the so-called adjoint of a, such that we have both $\langle ab, c\rangle = \langle b, a^*c\rangle$ and $\langle ab, c\rangle = \langle a, cb^*\rangle$ for all $b, c \in A$.

Example 1.1. The Hilbert space $\mathbb{C}^n$, consists of all n-tuples $\{a_i\}_{i=1}^n$ of complex numbers, is an H^*-algebra where for each $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ in $\mathbb{C}^n$, $\{a_i\}_{i=1}^n\{b_i\}_{i=1}^n = \{a_ib_i\}_{i=1}^n$ and $(\{a_i\}_{i=1}^n)^* = \{\overline{a_i}\}_{i=1}^n$.
Obviously any Hilbert space is an H^*-algebra where the product each pair of elements is zero. Of course in this case the adjoint a^* of a need not be unique, in fact every element is an adjoint of every element. Recall that $A_0 = \{a \in A : aA = \{0\}\} = \{a \in A : Aa = \{0\}\}$ (see[1, Lemma 2.1]) is called the annihilator ideal of A. A proper H^*-algebra is an H^*-algebra with zero annihilator ideal. Ambrose [1] proved that an H^*-algebra is proper if and only if every element has a unique adjoint.
The trace class of A is the set $\tau(A) = \{ab : a, b \in A\}$. As in the proof of [10, Lemma 3 one can show that $\tau(A)$ ia linear subspace of A. Further $\tau(A)$ is an ideal of A which is a Banach $*$-algebra under a suitable norm $\tau_A(.)$. The norm τ_A is related to the given norm $\|.\|$ on A by $\tau_A(a^*a) = \|a\|^2$ and $\|b\| \leqslant \tau_A(b)$ for each $a \in A,\ b \in \tau(A)$ ([3]). If A is proper, then $\tau(A)$ is dense in A ([1, Lemma 2.7]). The trace functional tr on $\tau(A)$ is defined by $tr(ab) = \langle a, b^*\rangle = \langle b, a^*\rangle = tr(ba)$ for each $a, b \in A$, in particular $tr(aa^*) = tr(a^*a) = \|a\|^2$. A projection is a self adjoint idempotent $e \in A$, e is called minimal if $e \neq 0$ and $eAe = \mathbb{C}e$. Each simple H^*-algebra (that is an H^*-algebra without nontrivial closed two-sided ideals) contains minimal projections ([3]). Two idempotents e and e' are doubly orthogonal if $\langle e, e'\rangle = 0$ and $ee' = e'e = 0$. A positive member of A is an element $a \in A$ such that $\langle ax, x\rangle \geqslant 0$ for each $x \in A$. It is known from [9] that for each $a \in A$, there exists a unique positive member $[a]$ of A such that $a^*a = [a]^2$. We also recall that if a is a nonzero element in A, then there exists a sequence $\{e_n\}$ of doubly orthogonal projections and a sequence $\{\lambda_n\}$ of positive numbers such that $a^*a = \sum_n \lambda_n e_n$. In this case, $[a] = \sum_n \lambda_n^{\frac{1}{2}} e_n$ and if a is in $\tau(A)$, then $\tau_A(a) = tr([a])$.
The notion of Hilbert H^*-module first was introduced by Saworotnow in [8] under the name of generalized Hilbert space, then many mathematicians such as Cabrera, Martinez, Rodriguez, Bakic and Guljas developed it in several directions.

Definition 1.2. *Let A be a proper H^*-algebra. A Hilbert H^*-module is a left module E over A with a mapping $[\cdot|\cdot] : E \times E \to \tau(A)$ which satisfies the following conditions:*

(i) $[\alpha x|y] = \alpha[x|y]$,

(ii) $[x + y|z] = [x|z] + [y|z]$,

(iii) $[ax|y] = a[x|y]$,

(iv) $[x|y]^* = [y|x]$,

(v) For each nonzero element x in E there is a nonzero element c in A such that $[x|x] = c^*c$,

(vi) E is a Hilbert space with the inner product $(x, y) = tr([x|y])$,

for each $\alpha \in \mathbb{C}$, $x, y, z \in E$, $a \in A$. We denote norm of E by $\|.\|_E$, whence $\|x\|_E = tr([x|x])^{\frac{1}{2}}$. It is an immediate consequence of the above definition that $\|ax\|_E \leqslant \|a\|\|x\|_E$ for all $a \in A$ and $x \in E$.

For, let $x \in E$ then $[x|x] = c^*c$ for some $c \in A$ and $\|x\|_E = tr([x|x])^{\frac{1}{2}} = tr(c^*c)^{\frac{1}{2}} = \|c\|$. So $\|ax\|_E^2 = tr([ax|ax]) = tr(a[x|x]a^*) = tr(ac^*ca^*) = \|ca^*\|^2 \leqslant \|c\|^2\|a\|^2 = \|x\|_E^2\|a\|^2$. We also have $\|ax\|_E \leqslant \tau_A(a)\|x\|_E$ for each $a \in \tau(A)$ and $x \in E$.

As an example of Hilbert H^*-module, let A be a proper H^*-algebra, then it becomes a Hilbert A-module via $[x|y] = xy^*$.

For Hilbert A-module E, the $*$-ideal of A generated by $\{[x|y] : x, y \in E\}$ is denoted by $[E|E]$. We say that E is full if $[E|E]$ is τ_A-dense in $\tau(A)$. An element $u \in E$ is said to be a basic element if there exists a minimal projections $e \in A$ such that $[u|u] = e$. An orthonormal system in E is a family of basic elements $\{u_\lambda\}_{\lambda\in\Lambda}$ satisfying $[u_\lambda|u_\mu] = 0$ for all $\lambda, \mu \in \Lambda$, $\lambda \neq \mu$. An orthonormal basis in E is an orthonormal system generating a dense submodule of E.

We recall from [5], that each Hilbert H^*-module contains basic orthonormal bases. For more details on the Hilbert H^*-modules we refer the reader to [3,5,11].

The notions of ϕ-homomorphism and unitary operators were studied by many mathematicians such as Bakic, Guljas, Joita and Taghavi. In this paper, inspiring of these concepts we introduce ϕ-morphism of Hilbert H^*-modules and unitary operator and then describe some results concerned with these ones. Throughout this note all H^*-algebras are assumed proper and also by a morphism we always mean a $*$-homomorphism of H^*-algebras.

2. Main Results

Here, we give an example including both full and non full Hilbert H^*-modules which is interesting in its own right.

Example 2.1. It is straightforward to see that the H^*-algebra $A = \mathbb{C}^n$ is proper and $\tau(A) = A$ (since A is unital). Clearly, $\{e_1, ..., e_n\}$ (e_i, has 1 as i-th position and 0 elsewhere) is a maximal family of doubly orthogonal projections for A. If $\{a_i\}_{i=1}^n \in A$, then

$$(\{a_i\}_{i=1}^n)^* \{a_i\}_{i=1}^n = \{|a_i|^2\}_{i=1}^n = \sum_{i=1}^n |a_i|^2 e_i,$$

$$[\{a_i\}_{i=1}^n] = \sum_{i=1}^n |a_i| e_i \textit{ and } \tau_A(\{a_i\}_{i=1}^n) = tr([\{a_i\}_{i=1}^n]) =$$

$$tr(\sum_{i=1}^n |a_i| e_i) = \sum_{i=1}^n |a_i| \tag{1}$$

Since for $i = 1, ..., n$, $tr(e_i) = tr(e_i^2) = \langle e_i, e_i \rangle = 1$ where $\langle ., . \rangle$ denotes the usual inner product on $\mathbb{C}^n$.
Let $E = A$ and $[\{a_i\}_{i=1}^n | \{b_i\}_{i=1}^n] = \{a_i \overline{b_i}\}_{i=1}^n$. Then E is a full Hilbert H^*-module over A. For fullness of E, it is enough to substitute $\{b_i\}_{i=1}^n$ with unit of $\mathbb{C}^n$ (we mean by unit of $\mathbb{C}^n$ the element $\{t_i\}_{i=1}^n$ which $t_i = 1$ for each $i = 1, ..., n$). On the other hand, let $F = \{\{a_i\}_{i=1}^n \in \mathbb{C}^n : a_1 = 0\}$. Then F is a Hilbert H^*-module over A with $[\{a_i\}_{i=1}^n | \{b_i\}_{i=1}^n] = \{a_i \overline{b_i}\}_{i=1}^n$ which is not full. For this, let $\{a_i\}_{i=1}^n \in \tau(A)$ $(= A)$ in which a_1 be nonzero. If on the contrary $\overline{[F|F]}^{\tau_A} = \tau(A)$, then there exist $\lambda_j \in \mathbb{C}$, $\{b_{i,j}\}_{i=1}^n$ and $\{c_{i,j}\}_{i=1}^n$ in F $(j = 1, ..., k)$ in which

$$\tau_A(\sum_{j=1}^k \lambda_j [\{b_{i,j}\}_{i=1}^n | \{c_{i,j}\}_{i=1}^n] - \{a_i\}_{i=1}^n) < \epsilon. \tag{2}$$

Put $\{d_i\}_{i=1}^n = \{\sum_{j=1}^k \lambda_j b_{i,j} \overline{c_{i,j}} - a_i\}_{i=1}^n$. Then by (1) the left side of (2) is equal to $\sum_{i=1}^n |d_i|$. Hence $|a_1| = |d_1| \leqslant \sum_{i=1}^n |d_i| < \epsilon$ by (2) and since this is valid for each $\epsilon > 0$ so $a_1 = 0$ which is a contradiction. Therefore F is not full.
The proof of the following lemma is similar to the one in [6, Lemma 2.4] and so it is omitted.

Lemma 2.2. *Let E be a full Hilbert A-module and $a \in A$. Then $ax = 0$ for all $x \in E$ if and only if $a = 0$.*

Remark 2.3. *If $\phi : A \to B$ is an isometric morphism of H^*-algebras, then for each $a \in A$, $\|\phi(a)\|^2 = \|a\|^2$ and so $\langle \phi(a), \phi(a) \rangle = \langle a, a \rangle$. Whence $tr(\phi(aa^*)) = tr(aa^*)$.*

Definition 2.4. *Let E and F be Hilbert modules over H^*-algebras A and B respectively and $\phi : \tau(A) \to \tau(B)$ be a norm continuous morphism. A map $\Phi : E \to F$ is said to be a ϕ-morphism if $[\Phi(x)|\Phi(y)] = \phi([x|y])$ for all x, y in E.*

We can extend ϕ to a continuous morphism $\bar{\phi} : A \to B$. Obviously, Φ is a $\bar{\phi}$-morphism, i.e. $[\Phi(x)|\Phi(y)] = \bar{\phi}([x|y])$ for each x, y in E. From now on we mean by a ϕ-morphism, a $\bar{\phi}$-morphism. Using polarization identity, one conclude that Φ is a ϕ-morphism if and only if $[\Phi(x)|\Phi(x)] = \phi([x|x])$ for each x in E. It is easy to see that each ϕ-morphism is necessarily a linear operator and a module map in the sense that $\Phi(ax) = \phi(a)\Phi(x)$ for all $x \in E, a \in A$. Applying norm continuity of ϕ, the calculation $\|\Phi(x)\|^2 = tr([\Phi(x)|\Phi(x)]) = tr(\phi([x|x])) = \|\phi(a)\|^2 \leqslant \|\phi\|^2\|a\|^2 = \|\phi\|^2\|x\|^2$, where $[x|x] = a^*a$ for some $a \in A$, shows that Φ is continuous too.
If E, F and G are Hilbert modules over H^*-algebras A, B and C respectively, $\phi_1 : A \to B$ and $\phi_2 : B \to C$ are morphisms of H^*-algebras and $\Phi_1 : E \to F$ and $\Phi_2 : F \to G$ are ϕ_1-morphism and ϕ_2-morphism respectively, then it is straightforward to show that $\Phi_2\Phi_1 : E \to G$ is a $\phi_2\phi_1$-morphism.
In what follows we give an analogue of [7, Proposition 2.2] in the framework of Hilbert H^*-modules.

Proposition 2.5. *Let A and B be proper H^*-algebras, E and F be full Hilbert module and Hilbert module over A and B respectively. Also let $\Phi : E \to F$ be a continuous bijective linear map and $\phi : A \to B$ be a map in which $\overline{\phi(\tau(A))}^{\tau_B} = \phi(\tau(A))$, $\Phi(ax) = \phi(a)\Phi(x)$ and $[\Phi(x)|\Phi(y)] = \phi([x|y])$, for each $a \in A$ and $x, y \in E$. Then F is full if and only if $\phi|_{\tau(A)}$ is a (τ_A, τ_B)-continuous isomorphism.*

Proof. Suppose that F is full. Let $a_1, a_2 \in A$ and $\alpha \in \mathbb{C}$, then $(\phi(\alpha a_1 + a_2) - \alpha\phi(a_1) - \phi(a_2))\Phi(x) = 0$ and $(\phi(a_1a_2) - \phi(a_1)\phi(a_2))\Phi(x) = 0$, for each $x \in E$. Since Φ is surjective and F is full, we deduce from Lemma 2.2 that ϕ is linear and preserves multiplication. We are going to show that ϕ is injective.
Let $\phi(a) = 0$ ($a \in A$), then for each $x \in E$, $\Phi(ax) = 0$. Injectivity of Φ implies that $ax = 0$ for each $x \in E$. Applying again Lemma 2.2, we obtain that $a = 0$. It is clear that $\phi|_{\tau(A)}$ denotes a linear map such as $\phi_1 : \tau(A) \to \tau(B)$ such that $\phi_1(a) = \phi(a)$ for all $a \in \tau(A)$. Now let $b \in \tau(B)$, then fullness of F implies that $b = \lim_{n\to\infty}{}^{\tau_B}[\Phi(x_n)|\Phi(y_n)] = \lim_{n\to\infty}{}^{\tau_B}\phi_1([x_n|y_n])$ for some $x_n, y_n \in E$. From this fact and taking into account that $\phi(\tau(A))$ is τ_B-closed, we conclude that ϕ_1 is surjective. Next we will show that ϕ_1 is (τ_A, τ_B)-continuous. Assume that $\{a_n\}$ is a sequence in $\tau(A)$ such that $\lim_{n\to\infty}{}^{\tau_A}a_n = 0$ and $\lim_{n\to\infty}{}^{\tau_B}\phi_1(a_n) = b$ for some $b \in \tau(B)$. Then by the comment after Definition 1.2, $\lim_{n\to\infty} a_nx = 0$ and continuity of Φ forces that $0 = \lim_{n\to\infty}\Phi(a_nx) = \lim_{n\to\infty}\phi_1(a_n)\Phi(x) = b\Phi(x)$

for all $x \in E$. Since Φ is surjective and F is full, $b = 0$ and it follows from closed graph theorem that ϕ_1 is (τ_A, τ_B)-continuous. A similar argument shows that ϕ is continuous too. By above discussion it is enough to show that ϕ_1 preserves adjoint. Before proving this we remind that the equalities $\|a\| = \|a^*\|$ and $\tau_A(b) = \tau_A(b^*)$ ($a \in A$, $b \in \tau(A)$), imply that the map which takes a to a^* ($a \in A$) and its restriction to $\tau(A)$ are norm continuous and (τ_A, τ_B)-continuous respectively. Let $a \in \tau(A)$, then we may assume that $a = \lim_{n\to\infty}{}^{\tau_A} u_n$, each u_n is of the form $u_n = \sum_{i=1}^{k_n}[x_{i,n}|y_{i,n}]$ for some $x_{i,n}, y_{i,n} \in E$. Hence

$$\phi_1(a^*) = \lim_{n\to\infty}{}^{\tau_B}\ \phi_1(u_n^*) = \lim_{n\to\infty}{}^{\tau_B} \sum_{i=1}^{k_n} \phi_1([y_{i,n}|x_{i,n}])$$

$$\begin{aligned}
&= \lim_{n\to\infty}{}^{\tau_B} \sum_{i=1}^{k_n}[\Phi(y_{i,n})|\Phi(x_{i,n})] = (\lim_{n\to\infty}{}^{\tau_B} \sum_{i=1}^{k_n}[\Phi(x_{i,n})|\Phi(y_{i,n})])^* \\
&= (\lim_{n\to\infty}{}^{\tau_B} \sum_{i=1}^{k_n} \phi_1([x_{i,n}|y_{i,n}]))^* = (\phi_1(\lim_{n\to\infty}{}^{\tau_A} \sum_{i=1}^{k_n}[x_{i,n}|y_{i,n}]))^* \\
&= \phi_1(a)^*.
\end{aligned}$$

The second equality in the last line holds since by the inverse mapping theorem $(\phi_1)^{-1}$ is a (τ_B, τ_A)-continuous operator. Therefore ϕ_1 preserves adjoint and so it is a (τ_A, τ_B)-continuous isomorphism. Since A is proper, so $\overline{\tau(A)} = A$ ([1, Lemma 2.7]), hence if $a \in A$, then there exists a sequence $\{a_n\} \subseteq \tau(A)$ such that $a = \lim_{n\to\infty} a_n$. By morphism of ϕ_1 and continuity of ϕ we obtain the equality $\phi(a^*) = \phi(\lim_{n\to\infty} a_n^*) = \lim_{n\to\infty} \phi(a_n)^* = (\lim_{n\to\infty} \phi(a_n))^* = (\phi(a))^*$, which proves that ϕ is a morphism too.
Conversely, if ϕ_1 is (τ_A, τ_B)-continuous isomorphism, then $(\phi_1)^{-1}$ is a (τ_B, τ_A)-continuous isomorphism. Thus we have

$$\overline{[F|F]}^{\tau_B} = \overline{[\Phi(E)|\Phi(E)]}^{\tau_B} = \overline{\phi_1([E|E])}^{\tau_B} = \phi_1(\overline{[E|E]}^{\tau_A}) = \phi_1(\tau(A)) = \tau(B),$$

it means that F is full and our goal is achieved. □

In the following theorem we investigate some conditions under which a ϕ-morphism takes an orthonormal basis to an orthonormal basis. For this purpose, we need to recall some assertions. Firstly, if A is a simple H^*-algebra and $\{e_i\}_{i\in I}$ is a maximal family of doubly orthogonal minimal projections for A, then it is the orthogonal sum of minimal ideals Ae_i s' ([4, Theorem

5.34.16]). Secondly, if E is a Hilbert A-module, then for each minimal projection $e_i \in A$ $(i \in I)$ there exists an orthonormal basis $\{u_{\lambda,i}\}_{\lambda\in\Lambda}$ in E such that $[u_{\lambda,i}|u_{\lambda,i}] = e_i$ for each $\lambda \in \Lambda$ ([3, Proposition 1.5]).

Theorem 2.6. *Let $\phi : A \to B$ be a continuous morphism of simple H^*-algebras, $\Phi : E \to F$ be a ϕ-morphism of Hilbert H^*-modules and $\{e_i\}_{i\in I}$ and $\{u_{\lambda,i}\}_{\lambda\in\Lambda}$ be as above. If ϕ is injective in which $\phi|_{\tau(A)}$ has τ_B-closed range, Φ is surjective and F is full, then $\{\phi(e_i)\}_{i\in I}$ and $\{\Phi(u_{\lambda,i})\}_{\lambda\in\Lambda}$ are maximal family of doubly orthogonal minimal projections for B and orthonormal basis for F respectively.*

Proof. At first we show that $\{\phi(e_i)\}_{i\in I}$ is a maximal family of doubly orthogonal minimal projections.
Step 1. $\phi(e_i)$ $(i \in I)$ is a minimal projection. Obviously $\phi(e_i)$ is a projection. We will show that it is minimal. If $b \in B$, then fullness of F implies that $b\phi(e_i) = \lim_{n\to\infty}{}^{\tau_B} \sum_{j=1}^{k_n} [y_{j,n}|y'_{j,n}]$ for some $y_{j,n}, y'_{j,n}$ in F. It follows by surjectivity of Φ that

$$b\phi(e_i) = \lim_{n\to\infty}{}^{\tau_B} \sum_{j=1}^{k_n} [\Phi(x_{j,n})|\Phi(x'_{j,n})] = \lim_{n\to\infty}{}^{\tau_B} \sum_{j=1}^{k_n} \phi([x_{j,n}|x'_{j,n}]) \in \overline{\phi(\tau(A))}^{\tau_B} = \phi(\tau(A))$$

for some $x_{j,n}, x'_{j,n}$ in E. Thus $b\phi(e_i) = \phi(a)$ for some $a \in \tau(A)$. Then

$$\phi(e_i)b\phi(e_i) = \phi(e_i)b\phi(e_i^2) = \phi(e_i)(b\phi(e_i))\phi(e_i) = \phi(e_i a e_i) = \lambda\phi(e_i),$$

for some $\lambda \in \mathbb{C}$. It gives that $\phi(e_i)B\phi(e_i) = \mathbb{C}\phi(e_i)$.
Step 2. $\phi(e_i)$ s' are doubly orthogonal, since for $i \neq j$, $\phi(e_i)\phi(e_j) = \phi(e_i e_j) = 0$ and also we have

$$\begin{aligned} \langle\phi(e_i), \phi(e_j)\rangle &= \langle\phi(e_i)\phi(e_i), \phi(e_j)\rangle \\ &= \langle\phi(e_i), \phi(e_i^*)\phi(e_j)\rangle \\ &= \langle\phi(e_i), \phi(e_i e_j)\rangle = 0. \end{aligned}$$

Step 3. $\{\phi(e_i)\}_{i\in I}$ is a maximal family of doubly orthogonal minimal projections. If on the contrary there is a minimal projection e_0 in F that is doubly orthogonal to each element of $\{\phi(e_i)\}_{i\in I}$, then by fullness of F and surjectivity of Φ, we have $e_0 = \lim_{n\to\infty}{}^{\tau_B} \sum_{j=1}^{k_n} [\Phi(t_{j,n})|\Phi(t'_{j,n})] = \lim_{n\to\infty}{}^{\tau_B} \sum_{j=1}^{k_n} \phi([t_{j,n}|t'_{j,n}])$ for some $t_{j,n}, t'_{j,n}$ in E. By the argument applied in Proposition 2.5 $\phi|_{\tau(A)}$ is (τ_A, τ_B)-continuous and using inverse mapping theorem we obtain that $(\phi|_{\tau(A)})^{-1}$ is

(τ_B, τ_A)-continuous. So $e_0 = \phi(\lim_{n\to\infty}{}^{\tau_A}\sum_{j=1}^{k_n}[t_{j,n}|t'_{j,n}])$. Put $a = \lim_{n\to\infty}{}^{\tau_A}\sum_{j=1}^{k_n}[t_{j,n}|t'_{j,n}]$. By the preceding assertions, $a = \sum_{i\in I} a_i e_i$ for some $a_i \in A$ and by continuity of ϕ, $e_0 = \phi(a) = \phi(\sum_{i\in I} a_i e_i) = \sum_{i\in I}\phi(a_i)\phi(e_i)$. It yields that

$$\|e_0\|^2 = \langle e_0, e_0\rangle = \langle \sum_{i\in I}\phi(a_i)\phi(e_i), e_0\rangle = \sum_{i\in I}\langle \phi(a_i), e_0\phi(e_i)\rangle = 0.$$

Then $e_0 = 0$ which is a contradiction.
Finally we are going to show that $\{\Phi(u_{\lambda,i})\}_{\lambda\in\Lambda}$ is an orthonormal basis for F. To see this, we have

(i) $[\Phi(u_{\lambda,i})|\Phi(u_{\lambda',i})] = \phi([u_{\lambda,i}|u_{\lambda',i}]) = 0$, for $\lambda \neq \lambda'$.

(ii) $[\Phi(u_{\lambda,i})|\Phi(u_{\lambda,i})] = \phi([u_{\lambda,i}|u_{\lambda,i}]) = \phi(e_i)$.

(iii) Let $y \in F$ be arbitrary. By surjectivity of Φ, $y = \Phi(x)$, for some $x \in X$. If $x = \lim_{n\to\infty}\sum_{\lambda_n=1}^{k_n} a_{\lambda_n} u_{\lambda_n,i}$ for some $a_{\lambda_n} \in A$, then $y = \lim_{n\to\infty}\sum_{\lambda=1}^{k_n}\phi(a_{\lambda_n})\Phi(u_{\lambda_n,i})$ by continuity of Φ. It means that $\{\Phi(u_{\lambda,i})\}_{\lambda\in\Lambda}$ generates a dense submodule of F. $\square$

Theorem 2.7. *Let E and F be Hilbert modules over simple H^*-algebras A and B respectively, $\phi : A \to B$ be a continuous morphism, $\{e_i\}_{i\in I}$ and $\{u_{\lambda,i}\}_{\lambda\in\Lambda}$ be as before and $\Phi : E \to F$ be a ϕ-morphism. If $\{\phi(e_i)\}_{i\in I}$ and $\{\Phi(u_{\lambda,i})\}_{\lambda\in\Lambda}$ are maximal family of doubly orthogonal minimal projections for B and orthogonal basis for F respectively, then ϕ is injective and $\overline{[F|F]} = B$.*

Proof. Suppose that $a \in A$ and $\phi(a) = 0$. We know that $a = \sum_{i\in I} a_i e_i$ (see step 3 in the proof of Theorem 2.6). Since $\{\phi(e_i)\}_{i\in I}$ is doubly orthogonal, we have

$$\begin{aligned} \|\phi(a)\|^2 &= \langle \phi(a), \phi(a)\rangle \\ &= \langle \phi(\sum_{i\in I} a_i e_i), \phi(\sum_{j\in I} a_j e_j)\rangle \\ &= \sum_{i\in I}\langle \phi(a_i)\phi(e_i), \phi(a_i)\phi(e_i)\rangle = 0. \end{aligned}$$

Thus $\|\phi(a_i)\phi(e_i)\| = 0$ for each $i \in I$, and $\phi(a_i e_i^2 a_i^*) = 0$. Let i be an arbitrary fixed element of I. By [9, Lemma 1], $a_i e_i^2 a_i^* = a_i e_i^* e_i a_i^* = \sum_{j\in J} \lambda_j e'_j$ for some maximal family $\{e'_j\}_{j\in J}$ of doubly orthogonal projections and some positive scalars λ_j. It follows from continuity of ϕ that $\phi(a_i e_i^* e_i a_i^*) = \sum_{j\in J} \lambda_j \phi(e'_j) = 0$ for each $i \in I$. Multiplying this relation by $\phi(e'_m)$ ($m \in J$ is arbitrary) we get $\lambda_m = 0$. Note that $\phi(e'_m) \neq 0$, because by assumption, A and B have the same cardinal of maximal family of doubly orthogonal projections (see also the comment before [9, Lemma 1]). Consequently $a_i e_i^* e_i a_i^*$ and so $a_i e_i$ are equal to zero by [1, Lemma 2.2]. Since $i \in I$ is arbitrary, $a_i e_i = 0$ for each $i \in I$. It follows that $a = 0$.

For the second part, let $b \in B$. Then $b = \sum_{i\in I} b_i \phi(e_i) = \sum_{i\in I} b_i \phi([u_{\lambda,i}|u_{\lambda,i}]) = \sum_{i\in I} [b_i \Phi(u_{\lambda,i})|\Phi(u_{\lambda,i})]$ for an arbitrary fixed element $\lambda \in \Lambda$. It means that $b \in \overline{[F|F]}$ and the proof is completed. □

Proposition 2.8. *Let E and F be Hilbert modules over H^*-algebras A and B respectively, $\phi : A \to B$ be a morphism and $\Phi : E \to F$ be a ϕ-morphism. If Φ is surjective, F is full and $\phi(\tau(A))$ is τ_B-closed in $\tau(B)$, then $\phi|_{\tau(A)} : \tau(A) \to \tau(B)$ is surjective.*

Proof. By the assumptions one obtains that
$\tau(B) = \overline{[F|F]}^{\tau_B} = \overline{[\Phi(E)|\Phi(E)]}^{\tau_B} = \overline{\phi([E|E])}^{\tau_B} \subseteq \overline{\phi(\tau(A))}^{\tau_B} = \phi(\tau(A)) \subseteq \tau(B)$. It implies that $\phi(\tau(A)) = \tau(B)$. □

We specialize a result of [6] to Hilbert H^*-modules.

Lemma 2.9. (see[6, lemma 2.10.]) *Let E and F be Hilbert module and full Hilbert module over H^*-algebras A and B respectively, ϕ_i s' $(i = 1, 2)$ be maps from A to B and $\Phi : E \to F$ be a surjective map satisfies $\Phi(ax) = \phi_i(a)\Phi(x)$ $(i = 1, 2)$ for all $x \in E$ and $a \in A$. Then $\phi_1 = \phi_2$.*

Definition 2.10. *Let E and F be Hilbert modules over H^*-algebras A and B respectively. A linear operator $\Phi : E \to F$ is said to be a unitary operator if there exists an injective morphism $\phi : A \to B$ such that Φ is a surjective ϕ-morphism and $\phi|_{\tau(A)}$ is (τ_A, τ_B)-continuous.*

From (τ_A, τ_B)-continuity of $\phi|_{\tau(A)}$ one conclude that ϕ is continuous. Indeed, there exists $M \geqslant 0$ in which $\tau_B(\phi(b)) \leqslant M\tau_A(b)$ for each $b \in \tau(A)$. Thus for each $a \in A$, we have $\|\phi(a)\|^2 = \tau_B(\phi(a^*a)) \leqslant M\tau_A(a^*a) = M\|a\|^2$.

For example let $B = \mathbb{C}^{n+1}$ $(n \geqslant 2)$, $A = E = \{\{a_i\}_{i=1}^n \in \mathbb{C}^n : a_1 = 0\}$ and

$F = \{\{d_i\}_{i=1}^{n+1} \in B : d_1 = d_2 = 0\}$. Then $\tau(A) = A$, $\tau(B) = B$, E is a full Hilbert A-module when $[\{a_i\}_{i=1}^n|\{c_i\}_{i=1}^n] = \{a_i\overline{c_i}\}_{i=1}^n$ and F is a Hilbert B-module (but not full) when $[\{d_i\}_{i=1}^{n+1}|\{b_i\}_{i=1}^{n+1}] = \{d_i\overline{b_i}\}_{i=1}^{n+1}$. Let $\Phi : E \to F$ defined by $\Phi(\{a_i\}_{i=1}^n) = \{b_i\}_{i=1}^{n+1}$ where $b_1 = 0$ and $b_i = a_{i-1}$ for $i = 2, ..., n+1$ and let $\phi : A \to B$ in which $\phi = \Phi$. Clearly ϕ (and so Φ) is a continuous isomorphism and moreover $[\Phi(\{a_i\}_{i=1}^n)|\Phi(\{c_i\}_{i=1}^n)] = \phi([\{a_i\}_{i=1}^n|\{c_i\}_{i=1}^n])$ for all $\{a_i\}_{i=1}^n$ and $\{c_i\}_{i=1}^n$ in E. Using Example 2. and continuity of ϕ, it is easy to verify that ϕ is (τ_A, τ_B)-continuous, therefore Φ is a unitary operator.
In what follows we state [12, Theorem 3.5] for Hilbert H^*-modules.

Theorem 2.11. *Suppose that E and F are Hilbert modules over H^*-algebras A and B respectively and $\phi : A \to B$ is a surjective morphism in which $\phi|_{\tau(A)}$ is isometry. Then linear operator $\Phi : E \to F$ which satisfies $\Phi(ax) = \phi(a)\Phi(x)$ for all $x \in E$ and $a \in A$, is a ϕ-morphism if and only if it is isometry.*

Proof. If Φ is a ϕ-morphism, then by Remark 2.3 for each $x \in E$, $\|\Phi(x)\|_F = tr([\Phi(x)|\Phi(x)])^{\frac{1}{2}} = tr(\phi([x|x]))^{\frac{1}{2}} = tr([x|x])^{\frac{1}{2}} = \|x\|_E$, so Φ is an isometry. Conversely, let Φ be an isometry, we will show that $[\Phi(x)|\Phi(x)] = \phi([x|x])$. Let $x \in E$ and $b \in B$, then by surjectivity of ϕ, there exists $a \in A$ such that $\phi(a) = b$. Applying again Remark 2.3 we have

$$\begin{aligned} tr(b[\Phi(x)|\Phi(x)]b^*) &= tr([b\Phi(x)|b\Phi(x)]) = tr([\phi(a)\Phi(x)|\phi(a)\Phi(x)]) \\ &= tr([\Phi(ax)|\Phi(ax)]) = \|\Phi(ax)\|^2 \\ &= \|ax\|^2 = tr([ax|ax]) = tr(\phi([ax|ax])) \\ &= tr(\phi(a[x|x]a^*)) = tr(b\phi([x|x])b^*). \end{aligned} \tag{3}$$

By the Definition 1.2, $[\Phi(x)|\Phi(x)] = c^*c$ and $\phi([x|x]) = d^*d$ for some c and d in B. Then from (3) we deduce that $tr(bc^*cb^*) = tr(bd^*db^*)$ and so $\langle bc^*, bc^*\rangle = \langle bd^*, bd^*\rangle$ for each $b \in B$. From this we conclude that

$$\langle b^*b, c^*c - d^*d\rangle = 0. \tag{4}$$

Replace b by c and then by d in (4) and subtract the obtained relations we get $\langle c^*c - dd^*, c^*c - d^*d\rangle = 0$ and so $c^*c = d^*d$. Consequently $[\Phi(x)|\Phi(x)] = \phi([x|x])$. □

Corollary 2.12. *Suppose that E and F are full Hilbert modules over H^*-algebras A and B respectively, $\phi : A \to B$ is a map and $\phi|_{\tau(A)}$ is isometry and further $\Phi : E \to F$ is a linear operator in which $\Phi(ax) = \phi(a)\Phi(x)$ for all $x \in E$ and $a \in A$. If Φ is unitary, then it is surjective and isometry. Conversely, if in addition ϕ is surjective, then Φ is unitary.*

Proof. If Φ is a unitary operator, then there exists an injective morphism $\varphi : A \to B$ such that Φ is a surjective φ-morphism and $\varphi|_{\tau(A)}$ is (τ_A, τ_B)-continuous. By Lemma 2.9, $\varphi = \phi$. On the other hand the argument applied in Theorem 2.11 shows that Φ is isometry. Conversely, let Φ be surjective and isometry and also ϕ be surjective, according to the proof of Proposition 2.5, ϕ is a continuous injective morphism and also $\phi|_{\tau(A)}$ is (τ_A, τ_B)-continuous. By Theorem 2.11, Φ is a ϕ-morphism, hence Φ is a unitary operator. $\square$

The following three propositions are the versions of some results appeared in [7, 12] in the framework of Hilbert H^*-modules. The proofs are omitted.

Proposition 2.13. *Let E, F be full Hilbert modules over H^*-algebras A and B respectively and $\Phi : E \to F$ be a continuous linear operator. Then the following assertions are equivalent:*
(i) Φ is a unitary operator.
(ii) Φ is bijective and there is a map $\phi : A \to B$ such that $\Phi(ax) = \phi(a)\Phi(x)$ and $[\Phi(x)|\Phi(y)] = \phi([x|y])$ for all $a \in A$ and $x, y \in E$.

Proposition 2.14. *Let E with $[.|.]_A : E \times E \to \tau(A)$ be a full Hilbert A-module and with $[.|.]_B : E \times E \to \tau(B)$ be a full Hilbert B-module. Then id_E (identity operator on E) is a unitary operator if and only if there is a map $\phi : A \to B$ such that $\phi|_{\tau(A)}$ is (τ_A, τ_B)-continuous, $ax = \phi(a)x$ and $\phi([x|y]_A) = [x|y]_B$ for all $a \in A$ and $x, y \in E$.*

Proposition 2.15. *Suppose that E and F are full Hilbert modules over H^*-algebra A and $\Phi : E \to F$ is a surjective and isometry A-linear map. Then Φ is a unitary operator and identity map is only morphism which makes Φ to ϕ-morphism.*

We terminate this discussion with a result concerned with faithful Hilbert H^*-modules [3]. For this purpose, we need to state some comments.

Let A and B be simple proper H^*-algebras and ϕ be a surjective morphism from A into B. If e is a minimal projection in A, then it is easy to check that $\phi(e)$ is a minimal projection in B. If A is a commutative simple proper H^*-algebra, then by [1, Theorem 4.1], $A = Ae$ for some minimal projection e in A and further $A = Ae = Ae^2 = eAe = \mathbb{C}e$.
Suppose that A and B are commutative simple proper H^*-algebras, $\phi : A \to B$ is a nonzero morphism and e,e' are minimal projections in A and B respectively. Then for some complex number λ, $\phi(\lambda e) = e'$. It implies that every nonzero morphism ϕ is a surjection. One can easily conclude that ϕ is an injection, too.
Recall that a Hilbert A-module X is faithful if $\{a \in A : \ aX = \{0\}\} = \{0\}$. By [3, Remark 1.6] (see also [5]), for each faithful Hilbert H^*-module X over

a proper H^*-algebra A, there exists a family $\{X_i\}_{i\in I}$ of Hilbert H^*-modules where each X_i is a Hilbert H^*-module over a simple H^*-algebra A_i, such that X is equal to the mixed product of the family $\{X_i\}_{i\in I}$,

$$X = \bigotimes_{i\in I} X_i = \{\{x_i\} \in \prod_{i\in I} X_i : \sum_{i\in I} \|x_i\|^2 < \infty\}.$$

Theorem 2.16. *Suppose that A and B are commutative proper H^*-algebras in which they have the same cardinal of doubly orthogonal minimal projections, E and F are faithful Hilbert modules over A and B respectively and $\phi : A \to B$ is a continuous morphism. Assume that $\Phi : E \to F$ is a surjective ϕ-morphism. Then Φ is a unitary operator.*

Proof. Suppose that $\{e_i\}_{i\in I}$ and $\{e'_i\}_{i\in I}$ $(= \{\phi(e_i)\}_{i\in I})$ are the maximal family of doubly orthogonal minimal projections for A and B respectively. Also suppose that e_i $(i \in I)$ is an arbitrary minimal projection in A, $\phi(e_i) = e'_i$, $E_{e_i} = \{x \in E : [x|x] = \lambda e_i, \lambda \geqslant 0\}$ and $F_{e'_i} = \{y \in F : [y|y] = \lambda e'_i, \lambda \geqslant 0\}$, then $Ae_i = \mathbb{C}e_i$, $Be'_i = \mathbb{C}e'_i$ (by the previous comment), E_{e_i} $(F_{e'_i})$ is a full Hilbert module over Ae_i (Be'_i) and $\Phi_{e_i} = \Phi|_{E_{e_i}} : E_{e_i} \to F_{e'_i}$ is well defined. Indeed for each $x \in E_{e_i}$, $[\Phi(x)|\Phi(x)] = \phi([x|x]) = \phi_{e_i}(\lambda e_i) = \lambda e'_i$ for some positive number λ, where $\phi_{e_i} = \phi|_{Ae_i}$. It forces that $\Phi(x) \in F_{e'_i}$. Obviously Φ_{e_i} is a ϕ_{e_i}-morphism. By the above comment ϕ_{e_i} is an isomorphism. Also it is (τ_A, τ_B)-continuous. Since $\tau(\mathbb{C}e_i) = \mathbb{C}e_i$, then $\tau(Ae_i) = Ae_i$ and so for each $a \in Ae_i$, $a^*a = \overline{\lambda} e_i \lambda e_i = |\lambda|^2 e_i$, $[a] = |\lambda| e_i$ and $\tau_A(a) = tr([a]) = tr(|\lambda| e_i) = |\lambda| tr(e_i) = |\lambda| \|e_i\|^2$. Now let $\epsilon > 0$ be given. Put $\delta \leqslant \frac{\epsilon \|e_i\|^2}{\|e'_i\|^2}$. In this case inequality $\tau_A(\lambda e_i) < \delta$ implies that $\tau_B(\phi_{e_i}(\lambda e_i)) = \tau_B(\lambda e'_i) = |\lambda| \|e'_i\|^2 < \epsilon$. As we mentioned, the faithful Hilbert H^*-module $E(F)$ is equal to the mixed product of the family $\{E_{e_i}\}_{i\in I}(\{F_{e'_i}\}_{i\in I})$, where each $E_{e_i}(F_{e'_i})$ is a faithful Hilbert H^*-module over a simple H^*-algebra $A_{e_i}(B_{e'_i})$. Also $A = \sum_{i\in I} Ae_i$ and $B = \sum_{i\in I} Be'_i$ ([1, Theorem 4.1]).

Injectivity of ϕ_{e_i} s' $(i \in I)$ implies that ϕ is injective too. We will show that for each $i \in I$, Φ_{e_i} is surjective. Since for any arbitrary element $y \in F_{e'_i}$, by surjectivity of Φ there exists $x \in E$ such that $y = \Phi(x)$. We have $[y|y] = [\Phi(x)|\Phi(x)] = \phi([x|x])$. Furthermore for some positive number λ, we have $[y|y] = \lambda e'_i = \lambda \phi(e_i) = \phi(\lambda e_i)$. Then $[x|x] = \lambda e_i$, so $x \in E_{e_i}$ and Φ_{e_i} is surjective. From the above discussion we conclude that for each minimal projection e_i in A, Φ_{e_i} is a unitary operator. Now since $\Phi(\{x_i\}_{i\in I}) = \{\Phi_{e_i}(x_i)\}_{i\in I}$, so Φ is a unitary operator. $\square$

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