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Closeness Centrality of Corona Product between Well-Known Graph and General Graph

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Abstract. Centrality measurement plays an important role to identify important/influential vertices and edges in a network or graph from different points of view. It also provide invaluable insights into the structure and functioning of interconnected systems, enabling researchers to identify critical nodes for targeted interventions, predict network behaviors, and optimize network performance. Though there are different centrality measurements in graphs theory, yet closeness centrality is widely used to analyze biological networks, social networks, fuzzy social network, transportation networks, etc. The closeness centrality of a node x in a network/graph is the unit fraction whose denominator is the sum of the distances from x to other nodes. This paper presents theoretical development to compute the closeness centrality of each node/vertex of different corona product graphs between well known graph (path graph, complete graph, cycle graph, wheel graph, star graph and complete bipartite graph) and general graph. Corona graph has lots of applications in signed networks, biotechnology, chemistry, small-world network, etc. We also present a suitable real application of our proposed results by which we can identify the influential nodes in small-world network.

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1 Introduction

Centrality measurements [3, 4, 8, 11, 12, 13, 33] are pivotal tools in network analysis, offering insights into the importance and influence [22, 29, 30, 31] of nodes within a network. By quantifying the relative significance of nodes based on various criteria such as degree, closeness, betweenness, and eigenvector centrality, researchers can uncover key players, pivotal connectors, and influential hubs within a network structure. These measurements are crucial for understanding the dynamics of complex systems, ranging from social networks to biological pathways and infrastructure networks. In this paper, we concentrate on vertex closeness centrality of graphs. The closeness centrality measurement is well-known to us for finding the important vertex in a complex network. The high centrality of a vertex gives it a positive influence on the network. It is used to analyse many networks, like the social networks [37, 38, 21], fuzzy social network [28], biological networks [45], transportation networks [35] etc. Else, it is used to choose potential leads in customer data and in bibliometrics [22]. The closeness centrality finds the suitable location in the real problem like facility location problem. It also finds the influence of a brain region in the brain network on other brain regions. Closeness centrality measures the connection between a street and all other neighbouring streets in the road network and also measures their accessibility. It also helps to identify influential nodes in small-world network [27].

Closeness centrality (C-centrality, in short) $C_C(v)$ of a node point v of a network is defined as the unit fraction whose denominator is the sum of the distances from x to other nodes. The mathematical expression of $C_C(v)$ is defined by $C_C(v) = \frac{1}{\sum_{x \in V} d(v,x)}$ where V is the vertex set of the network, $d(v,x)$ is the distance between the two vertices v and x. It is more receivable than degree centrality because it counts direct as well as indirect connections. Its aim is to recognize the suitable vertices in a network that can attain other vertices more quickly.

1.1 Review of the related works

Different centrality measurements were introduced and developed by lots of researchers to find the crucial nodes or edges in a network. Closeness centrality is one of them. In 1948, Bavelas [1] first introduced the concept of closeness centrality, and Sabidussi [42] first gave the definition of closeness centrality in 1966. Freeman [13] in 1978, delivered the mathematical expression of closeness centrality. After a few years, Newman [34] generalized the closeness centrality for weighted graphs by Dijkstra's shortest paths algorithm. In 2001, U. Brandes [3] presented a faster algorithm that takes $O(mn)$ time to calculate the C-centrality of any node in a network. In 2021, Eballe et al. [9] formulated the C-centrality of some graphs. Nandi [32] determined the C-centrality of the complete graph, wheel graph, and fan graph in 2022. Park [39] developed an algorithm for calculating the closeness centrality of a work-flow supported social network. Crescenzi et al. [7] proposed a greedy algorithm for calculating the increment of closeness centrality by adding new edges to it and applied it to real-world networks and synthetic graphs. In 2008, Okamoto et al. [36] designed an algorithm to find the top-k vertices in a network according to the highest C-centrality. Also, Kas er al. [20] worked on incremental closeness centrality for dynamically changing social networks. In the same year, Yen et al. [46] proposed an efficient approach to updating closeness centrality and average path length in dynamic networks. In 2014, Cohen et al. [23] Computed classic closeness centrality, at scale. Phuong-Hanh et al.[41] designed an efficient parallel algorithm for computing the closeness centrality in social networks in 2018. After that, in 2019, Mahapatra et al. [28] introduced a new concept of centrality measurement in fuzzy social networks. In the year 2019, Hu et al. [18] stydied closeness centrality measures in fuzzy enterprise technology innovation cooperation networks. Also, Hai et at. [17], in 2019, deeply studied parallel computation of hierarchical closeness centrality and applications. In [44], Shukla et al. developed an efficient parallel algorithms for betweenness and closeness-centrality in dynamic graphs. Besides these, Fushimi et al. [16] studied multiple perspective centrality measures (including closeness centrality) based on facility location problem under inter-group competitive environment. Regunta et al. [40] designed an efficient parallel algorithms for dynamic closeness-and betweenness centrality in 2021 Also, Elmezain et al. [10] presented Temporal Degree-Degree and Closeness-Closeness: A New Centrality Metrics for Social Network Analysis in 2021. Furthermore, O. Skibski [43] computed closeness centrality via the condorcet principle. In 2022, Evans et al. [11] showed by the shortest path tree approximation method, the inverse of C-centrality and the logarithm of degree centrality are linearly dependent. Freund et al. [14], in 2022, proposed an experimental study on the scalability of recent node centrality metrics in sparse complex networks. In the next year, Chen et al. [6] worked on normalized closeness centrality of urban networks: impact of the location of the catchment area and evaluation based on an idealized network. In the same year, Lopez et al. [25] presented efficient Data Transfer by Evaluating Closeness Centrality for Dynamic Social Complex Network-Inspired Routing. Also, closeness centrality on uncertain graphs was stydied by Liu et al. [24], in 2023.

1.2 Output

In our paper, we propose some new theoretical results to calculate vertex closeness centrality of different corona product graphs between well known graph and general graph. Here we consider well known graphs as path graph, complete graph, cycle graph, wheel graph, star graph and complete bipartite graph. We also apply our studied results to find the influential nodes of small-world network.

1.3 Structure of the paper

In our paper, we use some symbols that are presented in the Section 2. In the Section 3, we calculate the vertex closeness centrality different corona product between well known graph and general graph. Section 4 describes a suitable real application of our studied results. We write down our conclusion and future work in Section 5.

2 Some Notations

$C_C(v)$	:	C-centrality of the vertex v.
P_t	:	path graph having t nodes.
C_t	:	cycle graph having t nodes.
S_t	:	star graph having t nodes.
K_t	:	complete graph with t nodes.
W_t	:	wheel graph having $t+1$ nodes.
$K_{t,l}$	:	complete bipartite graph having $t+l$ nodes.
$deg(v)$	:	degree of the node v.

3 Closeness Centrality of Corona Product Graphs

Suppose G_{t_1} and T_{t_2} are two graphs with t_1 nodes, s_1 links/edges and t_2 nodes, s_2 edges, respectively. Now the corona product of two graphs G_{t_1} and T_{t_2} is symbolled by $G_{t_1} \odot T_{t_2}$, and it is made by sketching single copy of G_{t_1} and t_1 copies of T_{t_2} and connecting the j^{th} node point of G_{t_1} to each node point of the j^{th} copy of T_{t_2} by an edge. $G_{t_1} \odot T_{t_2}$ is also known as corona graph of G_{t_1} and T_{t_2}. Obviously, $|V(G_{t_1} \odot T_{t_2})| = t_1 + t_1 t_2$ and $|E(G_{t_1} \odot T_{t_2})| = s_1 + t_1 s_2 + t_1 t_2$. Frucht and Harary [15] first introduced the concept of corona of two graphs in 1970. Corona graph has lots of applications in signed networks [5], biotechnology [26], chemistry [19], etc.

3.1 Closeness centrality of corona product $P_t \odot G_s$

The graph $P_t \odot G_s$ is the corona product of P_t and G_s. The cardinality of the graph $P_t \odot G_s$ is $t + ts$. We assume that the degree of every vertices of G_s is known. Therefore, the degree of $v_{\alpha,\beta}$ in $P_t \odot G_s$ is 1 more than the degree of $v_{\alpha,\beta}$ in G_s. Let $V(P_t) = \{v_\alpha : \alpha = 1 : 1 : t\}$ and the node set of G_s corresponding to the node $v_\alpha, \alpha = 1 : 1 : t$ be $\{v_{\alpha,\beta} : \beta = 1 : 1 : s\}$. Figure 1 shows the corona graph $P_3 \odot G_5$.

Theorem 3.1. *The $C_C(v)$ of arbitrary vertex v of the graph $P_t \odot G_s$ is*

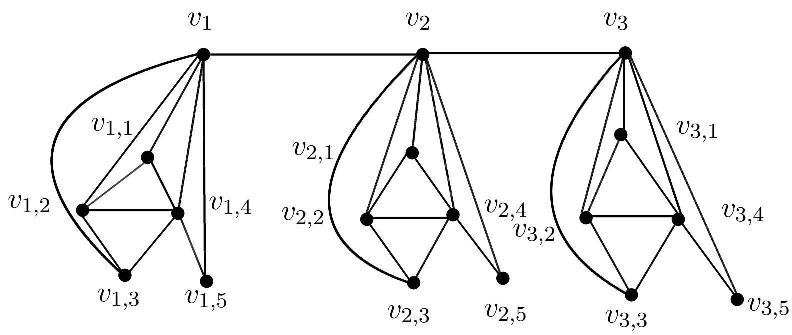

Figure 1: Corona graph $P_3 \odot G_5$

.

$$C_C(v) = \begin{cases} \frac{2}{(t-\alpha+1)[(t-\alpha)+s(t-\alpha+2)]+\alpha[(\alpha-1)+s(\alpha+1)]-2s}, & \text{if } v = v_\alpha \in P_t, \\ & \alpha = 1:1:t \\ \frac{2}{(t-\alpha+2)[(t-\alpha+1)+s(t-\alpha+3)]+(\alpha+1)[\alpha+s(\alpha+2)]-2[4s+deg(v_{\alpha,\beta}+2)]}, & \text{if} \\ & v = v_{\alpha,\beta} \in G_s, \alpha = 1:1:t,\ \beta = 1:1:s. \end{cases}$$

.

Proof. Let $\{v_\alpha : \alpha = 1:1:t\}$ and $\{v_{\alpha,\beta} : \alpha = 1:1:t; \beta = 1:1:s\}$ be the set of nodes of the path graph P_t and any graph G_s respectively. Also, let the degree of each vertex of G_s is known. If $v = v_\alpha$ be any vertex of P_t then

$\sum_{x\in V} d(v_\alpha, x) = \sum_{l=\alpha+1}^{t} d(v_\alpha, v_l) + \sum_{l=1}^{\alpha-1} d(v_\alpha, v_l) +$
$\sum_{l=\alpha}^{t} \sum_{\beta=1}^{s} d(v_\alpha, v_{l,\beta}) + \sum_{l=1}^{\alpha-1} \sum_{\beta=1}^{s} d(v_\alpha, v_{l,\beta})$
$= [1 + 2 + \cdots + (t-\alpha)] + [1 + 2 + \cdots + (\alpha-1)] + [s + 2s + \cdots + s(t - \alpha + 1)] + [2s + 3s + \cdots + s(\alpha - 1 + 1)]$
$= \frac{(t-\alpha)(t-\alpha+1)}{2} + \frac{\alpha(\alpha-1)}{2} + s\frac{(t-\alpha+1)(t-\alpha+2)}{2} + [s + 2s + 3s + \cdots + \alpha s] - s$
$= \frac{(t-\alpha)(t-\alpha+1)}{2} + \frac{\alpha(\alpha-1)}{2} + s\frac{(t-\alpha+1)(t-\alpha+2)}{2} + s\frac{\alpha(\alpha+1)}{2} - s$
$= \frac{(t-\alpha+1)[(t-\alpha)+s(t-\alpha+2)]+\alpha[(\alpha-1)+s(\alpha+1)]-2s}{2}$

Therefore, the C-centrality of v is

$$C_C(v) = \frac{1}{\sum_{x\in V} d(v,x)}$$

$$= \frac{2}{(t-\alpha+1)[(t-\alpha)+s(t-\alpha+2)]+\alpha[(\alpha-1)+s(\alpha+1)]-2s}.$$

If $v = v_{\alpha,\beta}$ is a node point of G_s where $\alpha = 1:1:t$; $\beta = 1:1:s$ then $\sum_{x\in V} d(v_{\alpha,\beta}, x) = \sum_{l=\alpha}^{t} d(v_{\alpha,\beta}, v_l) + \sum_{l=1}^{\alpha-1} d(v_{\alpha,\beta}, v_l) + \sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{\alpha,k})$ $+ \sum_{l=\alpha+1}^{t} \sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{l,k}) + \sum_{l=1}^{\alpha-1} \sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{l,k})$
$= [1+2+\cdots+(t-\alpha)+1] + [2+3+\cdots+(\alpha-1)+1] + \sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{\alpha,k}) +$ $[3s+4s+\cdots+s\{(t-\alpha)+2\}] + [3s+4s+\cdots+s\{(\alpha-1)+2\}]$.
The shortest distance between $v_{\alpha,\beta}$ of G_s and the adjacent vertices of $v_{\alpha,\beta}$ in G_s is 1 and the shortest distance between $v_{\alpha,\beta}$ and the remaining vertices of G_s is 2. i.e. $\{deg(v_{\alpha,\beta}) - 1\}$ vertices of G_s are at a distance 1 from $v_{\alpha,\beta}$ and $\{s - deg(v_{\alpha,\beta})\}$ vertices are at a distance 2.
Therefore,
$\sum_{x\in V} d(v_{\alpha,\beta}, x) = [deg(v_{\alpha,\beta}) - 1 + 2\{s - deg(v_{\alpha,\beta})\}] + \frac{(t-\alpha+1)(t-\alpha+2)}{2} +$ $[1+2+3+\cdots+\alpha] - 1 + [s+2s+3s+4s+\cdots+s(t-\alpha+2)] + [s+$ $2s+3s+4s+\cdots+s(\alpha+1)] - 3s - 3s$
$= [2s - deg(v_{\alpha,\beta}) - 1] + \frac{(t-\alpha+1)(t-\alpha+2)}{2} + \frac{\alpha(\alpha+1)}{2} + s\frac{(t-\alpha+2)(t-\alpha+3)}{2} +$ $s\frac{(\alpha+1)(\alpha+2)}{2} - 6s - 1$
$= \frac{(t-\alpha+2)}{2}[(t-\alpha+1)+s(t-\alpha+3)] + \frac{(\alpha+1)[\alpha+s(\alpha+2)]}{2} - [4s + deg(v_{\alpha,\beta}) + 2]$
$= \frac{(t-\alpha+2)[(t-\alpha+1)+s(t-\alpha+3)]+(\alpha+1)[\alpha+s(\alpha+2)]-2[4s+deg(v_{\alpha,\beta})+2]}{2}$.
Hence, $C_C(v)$, the C-centrality of v is
$\frac{1}{\sum_{x\in V} d(v,x)} = \frac{2}{(t-\alpha+2)[(t-\alpha+1)+s(t-\alpha+3)]+(\alpha+1)[\alpha+s(\alpha+2)]-2[4s+deg(v_{\alpha,\beta})+2]}$.
□

3.2 Closeness centrality of corona product $K_t \odot G_s$

The graph $K_t \odot G_s$ is the corona product of K_t and G_s. The cardinality of $K_t \odot G_s$ is $t + ts$. We assume that the degree of each vertex of G_s is known. Therefore, the degree of $v_{\alpha,\beta}$ in $K_t \odot G_s$ is 1 more than the degree of $v_{\alpha,\beta}$ in G_s. Let $\{v_\alpha : \alpha = 1:1:t\}$ and $\{v_{\alpha,\beta} : \beta = 1:1:s\}$ be the set of nodes of K_t and G_s corresponding to the node $v_\alpha, \alpha = 1:1:t$ respectively. $K_4 \odot G_4$ graph is displayed in Figure 2.

Theorem 3.2. *The $C_C(v)$ of any arbitrary node v of $K_t \odot G_s$ is*

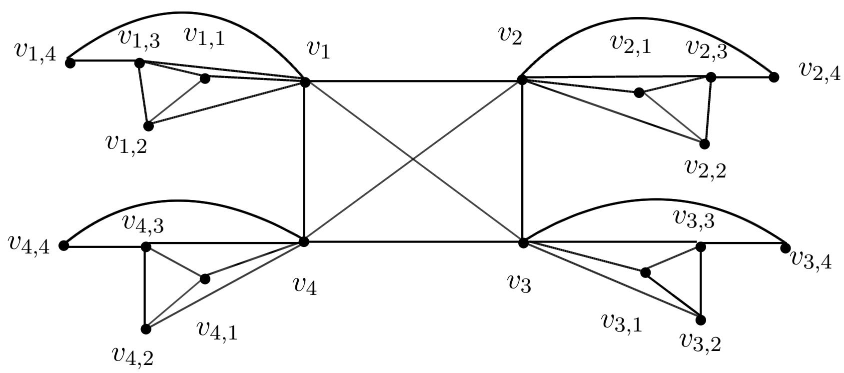

Figure 2: Corona graph $K_4 \odot G_4$

.

$$C_C(v) = \begin{cases} \frac{1}{3st+2t-s-deg(v_{\alpha,\beta})-2}, & \textit{if } v = v_{\alpha,\beta} \in G_s, \alpha = 1:1:t \textit{ and } \\ & \beta = 1:1:s \\ \frac{1}{2st+t-s-1}, & \textit{if } v = v_\alpha \in K_t, \alpha = 1:1:t. \end{cases}$$

.

Proof. Let $\{v_\alpha : \alpha = 1:1:t\}$ and $\{v_{\alpha,\beta} : \alpha = 1:1:t; \beta = 1:1:s\}$ be the set of nodes of the complete graph K_t and the graph G_s respectively. Also, let the degree of each vertex of G_s is known. If $v = v_\alpha$ be any vertex of K_t then
$\sum_{x\in V} d(v_\alpha, x)$
$= \sum_{l=1}^{t} d(v_\alpha, v_l) + [\sum_{l=\alpha}^{t} \sum_{\beta=1}^{s} d(v_\alpha, v_{l,\beta}) + \sum_{l=1}^{\alpha-1} \sum_{\beta=1}^{s} d(v_\alpha, v_{l,\beta})]$
$= (t-1) + [s + 2s(t-1)]$
$= 2st + t - s - 1$
Therefore, the closeness centrality of v is $C_C(v) = \frac{1}{\sum_{x\in V} d(v,x)} = \frac{1}{2st+t-s-1}$.

If $v = v_{\alpha,\beta}$ is a node point of G_s where $\alpha = 1:1:t$; $\beta = 1:1:s$

then
$\sum_{x\in V} d(v_{\alpha,\beta}, x)$
$= \sum_{l=1}^{t} d(v_{\alpha,\beta}, v_l) + \sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{\alpha,k}) + \sum_{l=1}^{t}\sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{l,k})$
$= [1 + 2(t-1)] + \sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{\alpha,k}) + 3s(t-1)$.
The shortest distance between $v_{\alpha,\beta}$ and the adjacent vertices of $v_{\alpha,\beta}$ in G_s is 1 and the shortest distance between $v_{\alpha,\beta}$ and the remaining vertices of G_s is 2. i.e. $\{deg(v_{\alpha,\beta})-1\}$ vertices of G_s have the shortest distance 1 from $v_{\alpha,\beta}$ and $\{s - deg(v_{\alpha,\beta})\}$ vertices have the shortest distance 2 from $v_{\alpha,\beta}$.
Therefore,
$\sum_{x\in V} d(v_{\alpha,\beta}, x)$
$= 2t - 1 + [deg(v_{\alpha,\beta}) - 1 + 2\{s - deg(v_{\alpha,\beta})\}] + 3st - 3s$
$= [2s - deg(v_{\alpha,\beta}) - 1] + 2t - 1 + 3st - 3s$
$= 3st + 2t - s - deg(v_{\alpha,\beta}) - 2$.
Hence, the closeness centrality of v is
$C_C(v) = \frac{1}{\sum_{x\in V} d(v,x)} = \frac{1}{3st+2t-s-deg(v_{\alpha,\beta})-2}$. □

Illustrative example 1: We consider the graph $K_4 \odot G_4$ displayed in the Figure 2. Now, if we apply the result of Theorem 3.2 for this corona product graph, then we have $s = 4; t = 4; \alpha = 1, 2, 3, 4; , \beta = 1, 2, 3, 4$. Also, $deg(v_{\alpha,1}) = 3; deg(v_{\alpha,2}) = 3; deg(v_{\alpha,3}) = 4; deg(v_{\alpha,4}) = 2$. So, using formula, we have $C_C(v_\alpha) = \frac{1}{2st+t-s-1} = \frac{1}{(2\times4\times4)+4-4-1} = \frac{1}{31}$.

Also,
$C_C(v_{\alpha,1}) = \frac{1}{3st+2t-s-deg(v_{\alpha,1})-2} = \frac{1}{(3\times4\times4)+(2\times4)-4-3-2} = \frac{1}{47}$.
$C_C(v_{\alpha,2}) = \frac{1}{3st+2t-s-deg(v_{\alpha,2})-2} = \frac{1}{(3\times4\times4)+(2\times4)-4-3-2} = \frac{1}{47}$.
$C_C(v_{\alpha,3}) = \frac{1}{3st+2t-s-deg(v_{\alpha,3})-2} = \frac{1}{(3\times4\times4)+(2\times4)-4-4-2} = \frac{1}{46}$.
$C_C(v_{\alpha,4}) = \frac{1}{3st+2t-s-deg(v_{\alpha,4})-2} = \frac{1}{(3\times4\times4)+(2\times4)-4-2-2} = \frac{1}{48}$.

These results are true for the graph we considered.

3.3 Closeness centrality of corona product $C_t \odot G_s$

The graph $C_t \odot G_s$ is the corona product of C_t and G_s. The cardinality of $C_t \odot G_s$ is $t + ts$. We assume that the degree of each vertex of G_s is known. Therefore, the degree of $v_{\alpha,\beta}$ in $C_t \odot G_s$ is 1 more than the degree of $v_{\alpha,\beta}$ in G_s. Let $V(C_t) = \{v_\alpha : \alpha = 1 : 1 : t\}$ and the node set of G_s corresponding to the node $v_\alpha, \alpha = 1 : 1 : t$ be $\{v_{\alpha,\beta} : \beta = 1 : 1 : s\}$.

$C_3 \odot G_4$ is displayed in Figure 3.

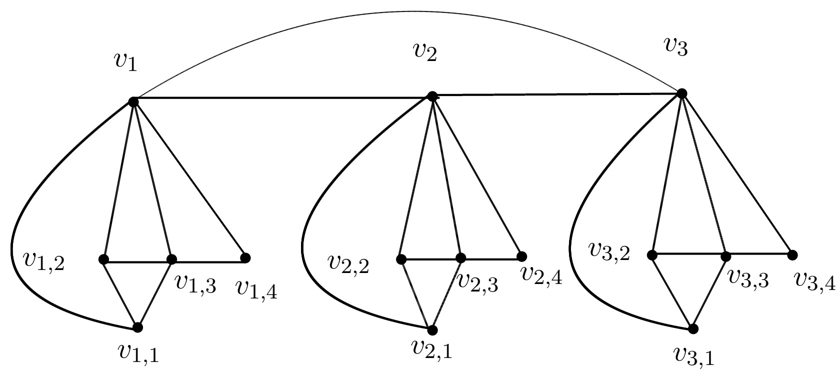

Figure 3: Corona graph $C_3 \odot G_4$

.

Theorem 3.3. *The $C_C(v)$ of any arbitrary node v of $C_t \odot G_s$ is*

$$C_C(v) = \begin{cases} \frac{4}{s(t+4)^2+(t+2)^2-4[4s+deg(v_{\alpha,\beta})+2]}, & \text{if } v = v_{\alpha,\beta} \in G_s, \alpha = 1:1:t; \\ & \beta = 1:1:s \text{ and } t \text{ is even} \\ \frac{4}{(t+3)[(t+1)+s(t+5)]-4[4s+deg(v_{\alpha,\beta})+2]}, & \text{if } v = v_{\alpha,\beta} \in G_s, \\ & \alpha = 1:1:t; \beta = 1:1:s \text{ and } t \text{ is odd} \\ \frac{4}{t^2+s(t+2)^2-4s}, & \text{if } v = v_\alpha \in C_t, \alpha = 1:1:t \text{ and } t \text{ is even} \\ \frac{4}{s(t^2+4t-1)+t^2-1}, & \text{if } v = v_\alpha \in C_t, \alpha = 1:1:t \text{ and } t \text{ is odd.} \end{cases}$$

.

Proof. Let $\{v_\alpha : \alpha = 1:1:t\}$ and $\{v_{\alpha,\beta} : \alpha = 1:1:t; \beta = 1:1:s\}$ be the set of nodes of the cycle graph C_t and the random graph G_s respectively. If $v = v_\alpha$ be any vertex of C_t and t is even then $\sum_{x \in V} d(v_\alpha, x) =$

$\sum_{\alpha=1}^{t} d(v_\alpha, v_k) + [\sum_{l=\alpha}^{t} \sum_{\beta=1}^{s} d(v_\alpha, v_{l,\beta}) + \sum_{l=1}^{\alpha-1} \sum_{\beta=1}^{s} d(v_\alpha, v_{l,\beta})]$.
We know for even cycle graph, $\sum_{\alpha=1}^{t} d(v_\alpha, v_k) = \frac{t^2}{4}$. As t is even, there are s vertices at distance 1, two sets of s vertices at distance $2, 3, \cdots, \frac{t}{2}$ and only s vertices at distance $\frac{t}{2}+1$ from v_α to $\{v_{\alpha,\beta} : \alpha = 1:1:t; \beta = 1:1:s\}$.
Therefore,
$\sum_{x\in V} d(v_\alpha, x)$
$= \frac{t^2}{4} + [s + 2s + 2s + 3s + 3s + \cdots + s \cdot \frac{t}{2} + s \cdot \frac{t}{2} + s(\frac{t}{2} + 1)]$
$= \frac{t^2}{4} + [s + 2s + 3s + \cdots + s \cdot \frac{t}{2} + s(\frac{t}{2} + 1)] + [2s + 3s + \cdots + s \cdot \frac{t}{2}]$
$= \frac{t^2}{4} + s[1 + 2 + \cdots + (\frac{t}{2} + 1)] + s[1 + 2 + 3 + \cdots + \frac{t}{2}] - s$
$= \frac{t^2}{4} + s\frac{(\frac{t}{2}+1)(\frac{t}{2}+2)}{2} + s\frac{\frac{t}{2}(\frac{t}{2}+1)}{2} - s$
$= \frac{t^2}{4} + \frac{s(t+2)(t+4)}{8} + \frac{st(t+2)}{8} - s$
$= \frac{t^2}{4} + \frac{s(t+2)(t+4+t)}{8} - s$
$= \frac{t^2}{4} + \frac{2s(t+2)^2}{8} - s$
$= \frac{t^2}{4} + \frac{s(t+2)^2}{4} - s$
$= \frac{t^2+s(t+2)^2-4s}{4}$.
Therefore, the closeness centrality of v is
$C_C(v) = \frac{1}{\sum_{x\in V} d(v,x)} = \frac{4}{t^2+s(t+2)^2-4s}$.

Let $v = v_\alpha$ be any vertex of C_t where t is odd. We know for the odd cycle graph C_t, the sum of the distances from any node to all other nodes is $\frac{(t-1)(t+1)}{4}$. As t is odd, there are two sets of s vertices of distance $2, 3, 4, \cdots, \frac{t+1}{2}$ and s vertices of distance 1 from v_α to $\{v_{\alpha,\beta} : \alpha = 1:1:t; \beta = 1:1:s\}$.
Therefore,
$\sum_{x\in V} d(v_\alpha, x)$
$= \frac{(t-1)(t+1)}{4} + [s + 2s + 2s + 3s + 3s + \cdots + s(\frac{t+1}{2}) + s(\frac{t+1}{2})]$
$= \frac{(t-1)(t+1)}{4} + [s + 2s + 3s + \cdots + s(\frac{t+1}{2})] + [2s + 3s + \cdots + s(\frac{t+1}{2)}]$
$= \frac{(t-1)(t+1)}{4} + s[1 + 2 + \cdots + \frac{t+1}{2}] + s[1 + 2 + 3 + \cdots + \frac{t+1}{2}] - s$
$= \frac{(t-1)(t+1)}{4} + 2s[1 + 2 + \cdots + \frac{t+1}{2}] - s$
$= \frac{(t-1)(t+1)}{4} + 2s\frac{\frac{t+1}{2}(\frac{t+1}{2}+1)}{2} - s$
$= \frac{(t-1)(t+1)}{4} + \frac{s(t+1)}{2}(\frac{t+1}{2} + 1) - s$

$= \frac{(t-1)(t+1)}{4} + s\{\frac{(t+1)^2}{4} + \frac{(t+1)}{2} - 1\}$
$= \frac{(t-1)(t+1)}{4} + \frac{s(t^2+4t-1)}{4}$
$= \frac{s(t^2+4t-1)+t^2-1}{4}$.
Therefore, the closeness centrality of v is
$C_C(v) = \frac{1}{\sum_{x\in V} d(v,x)} = \frac{4}{s(t^2+4t-1)+t^2-1}$.

If $v = v_{\alpha,\beta}$ is a node of G_s where $\alpha = 1:1:t$; $\beta = 1:1:s$ and t is even then
$\sum_{x\in V} d(v_{\alpha,\beta}, x)$
$= \sum_{l=1}^{t} d(v_{\alpha,\beta}, v_l) + \sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{\alpha,k}) + \sum_{l=1,\alpha\neq l}^{t} \sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{l,k})$.
The shortest distance between $v_{\alpha,\beta}$ and the adjacent vertices of $v_{\alpha,\beta}$ in G_s is 1 and the shortest distance between $v_{\alpha,\beta}$ and the remaining vertices of G_s is 2. i.e. $\{deg(v_{\alpha,\beta}) - 1\}$ vertices of G_s have the shortest distance 1 from $v_{\alpha,\beta}$ and $\{s - deg(v_{\alpha,\beta})\}$ vertices have the shortest distance 2 from $v_{\alpha,\beta}$. As t is even, so there are two sets of s vertices of distance $3, 4, \cdots, (\frac{t}{2}+1)$ and s vertices of distance $(\frac{t}{2}+2)$ from $v_{\alpha,\beta}$ to $v_{l,k}$.
Therefore,
$\sum_{x\in V} d(v_{\alpha,\beta}, x)$
$= [1 + 2 + 2 + 3 + 3 \cdots + \frac{t}{2} + \frac{t}{2} + (\frac{t}{2} + 1)] + [deg(v_{\alpha,\beta}) - 1 + 2\{s - deg(v_{\alpha,\beta})\}] + [3s + 3s + 4s + 4s + \cdots + s(\frac{t}{2}+1) + s(\frac{t}{2}+1) + s(\frac{t}{2}+2)]$
$= [1 + 2 + 3 + \cdots + \frac{t}{2} + (\frac{t}{2}+1)] + [2 + 3 + \cdots + \frac{t}{2}] + [2s - deg(v_{\alpha,\beta}) - 1] + [3s + 4s + \cdots + s(\frac{t}{2}+1) + s(\frac{t}{2}+2)] + [3s + 4s + \cdots + s(\frac{t}{2}+1)]$
$= [2s - deg(v_{\alpha,\beta}) - 1] + [1+2+\cdots+(\frac{t}{2}+1)] + [1+2+3+\cdots+\frac{t}{2}] - 1 + [s+2s+3s+4s+\cdots+s(\frac{t}{2}+1)+s(\frac{t}{2}+2)] - 3s + [s+2s+3s+4s+\cdots+s(\frac{t}{2}+1)] - 3s$
$= [2s - deg(v_{\alpha,\beta}) - 1] + \frac{(\frac{t}{2}+1)(\frac{t}{2}+2)}{2} + \frac{\frac{t}{2}(\frac{t}{2}+1)}{2} - 1 + s\frac{(\frac{t}{2}+2)(\frac{t}{2}+3)}{2} + s\frac{(\frac{t}{2}+1)(\frac{t}{2}+2)}{2} - 6s$
$= \frac{(t+2)(t+4)}{8} + \frac{t(t+2)}{8} + \frac{s(t+4)(t+6)}{8} + \frac{s(t+2)(t+4)}{8} - [4s + deg(v_{\alpha,\beta}) + 2]$
$= \frac{(t+2)(t+4+t)}{8} + \frac{s(t+4)(t+6+t+2)}{8} - [4s + deg(v_{\alpha,\beta}) + 2]$
$= \frac{s(t+4)^2+(t+2)^2}{4} - [4s + deg(v_{\alpha,\beta}) + 2]$
$= \frac{s(t+4)^2+(t+2)^2-4[4s+deg(v_{\alpha,\beta})+2]}{4}$.
Hence, the closeness centrality of v is
$C_C(v) = \frac{1}{\sum_{x\in V} d(v,x)} = \frac{4}{s(t+4)^2+(t+2)^2-4[4s+deg(v_{\alpha,\beta})+2]}$.

If $v = v_{\alpha,\beta}$ is a node of G_s where $\alpha = 1:1:t$; $\beta = 1:1:s$ and t is odd then

$\sum_{x\in V} d(v_{\alpha,\beta}, x)$
$= \sum_{l=1}^{t} d(v_{\alpha,\beta}, v_l) + \sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{\alpha,k}) + \sum_{l=1,\alpha\neq l}^{t} \sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{l,k})$.
The shortest distance between $v_{\alpha,\beta}$ and the adjacent vertices of $v_{\alpha,\beta}$ in G_s is 1 and the shortest distance between $v_{\alpha,\beta}$ and the remaining vertices of G_s is 2. i.e. $\{deg(v_{\alpha,\beta}) - 1\}$ vertices of G_s have the shortest distance 1 from $v_{\alpha,\beta}$ and $\{s - deg(v_{\alpha,\beta})\}$ vertices have the shortest distance 2 from $v_{\alpha,\beta}$. As t is odd, so there are two sets of s vertices of distance $3, 4, \cdots, (\frac{t+1}{2} + 1)$ from $v_{\alpha,\beta}$ to $v_{l,k}$.
Now,
$\sum_{x\in V} d(v_{\alpha,\beta}, x)$
$= [1 + 2 + 2 + 3 + 3 \cdots + \frac{t+1}{2} + \frac{t+1}{2}] + [2s - deg(v_{\alpha,\beta}) - 1] + [3s + 3s + 4s + 4s + \cdots + s(\frac{t+1}{2} + 1) + s(\frac{t+1}{2} + 1)]$
$= [2s - deg(v_{\alpha,\beta}) - 1] + 2\{1 + 2 + \cdots + \frac{t+1}{2}\} - 1 + 2\{s + 2s + 3s + 4s + \cdots + s(\frac{t+1}{2} + 1)\} - 6s$
$= 2 \cdot \frac{\frac{t+1}{2}(\frac{t+1}{2}+1)}{2} + 2s \cdot \frac{(\frac{t+1}{2}+1)(\frac{t+1}{2}+2)}{2} + [2s - deg(v_{\alpha,\beta}) - 1] - 6s - 1$
$= \frac{t+1}{2}(\frac{t+1}{2} + 1) + s(\frac{t+1}{2} + 1)(\frac{t+1}{2} + 2) - [deg(v_{\alpha,\beta}) + 4s + 2]$
$= \frac{(t+1)(t+3)}{4} + s\frac{(t+3)(t+5)}{4} - [deg(v_{\alpha,\beta}) + 4s + 2]$
$= \frac{(t+3)[(t+1)+s(t+5)]-4[deg(v_{\alpha,\beta})+4s+2]}{4}$.
Therefore, the C-centrality of v is
$C_C(v) = \frac{1}{\sum_{x\in V} d(v,x)} = \frac{4}{(t+3)[(t+1)+s(t+5)]-4[deg(v_{\alpha,\beta})+4s+2]}$. □

Illustrative example 2: We consider the graph $C_3 \odot G_4$ displayed in the Figure 3. Now, if we apply the result of Theorem 3.3 for this corona product graph, then we have $s = 4; t = 3(odd number); \alpha = 1, 2, 3; , \beta = 1, 2, 3, 4$. Also, $deg(v_{\alpha,1}) = 3; deg(v_{\alpha,2}) = 3; deg(v_{\alpha,3}) = 4; deg(v_{\alpha,4}) = 2$. So, using the formula, we have,
$C_C(v_\alpha) = \frac{4}{s(t^2+4t-1)+t^2-1} = \frac{4}{4(9+12-1)+9-1} = \frac{1}{22}$.
Also,

$$\begin{aligned} C_C(v_{\alpha,1}) &= \frac{4}{(t+3)[(t+1)+s(t+5)]-4[deg(v_{\alpha,1})+4s+2]} \\ &= \frac{4}{6[4+(4\times 8)]-4[16+3+2]} \\ &= \frac{1}{33}. \end{aligned}$$

$$\begin{aligned} C_C(v_{\alpha,2}) &= \frac{4}{(t+3)[(t+1)+s(t+5)]-4[deg(v_{\alpha,2})+4s+2]} \\ &= \frac{4}{6[4+(4\times 8)]-4[16+3+2]} \\ &= \frac{1}{33}. \end{aligned}$$

$$\begin{aligned}
C_C(v_{\alpha,3}) &= \tfrac{4}{(t+3)[(t+1)+s(t+5)]-4[deg(v_{\alpha,3})+4s+2]}\\
&= \tfrac{4}{(6[4+(4\times8)]-4[16+4+2]}\\
&= \tfrac{1}{32}.\\
C_C(v_{\alpha,4}) &= \tfrac{4}{(t+3)[(t+1)+s(t+5)]-4[deg(v_{\alpha,4})+4s+2]}\\
&= \tfrac{4}{6[4+(4\times8)]-4[16+2+2]}\\
&= \tfrac{1}{34}.
\end{aligned}$$

These results are true for the graph we considered.

3.4 Closeness centrality of corona product $W_t \odot G_s$

The graph $W_t \odot G_s$ is the corona product of the wheel graph W_t of $(t+1)$ vertices and the general graph G_s. The graph $W_t \odot G_s$ has $t+1+(t+1)s = (t+1)(s+1)$ node points. We assume that the degree of each vertex of G_s is known. Therefore, the degree of $v_{\alpha,\beta}$ in $W_t \odot G_s$ is 1 more than the degree of $v_{\alpha,\beta}$ in G_s. Let $\{v_\alpha : \alpha = 1:1:t+1\}$ and $\{v_{\alpha,\beta} : \beta = 1:1:s\}$ be the set of nodes of W_t and G_s respectively corresponding to the vertex $v_\alpha, \alpha = 1:1:t+1$ and v_1 is the central vertex of W_t. A corona graph $W_3 \odot G_4$ is displayed in Figure 4.

Theorem 3.4. *The $C_C(v)$ of arbitrary node v of $W_t \odot G_s$ is*

$$C_C(v) = \begin{cases}
\frac{1}{3st+2s+2t-deg(v_{\alpha,\beta})}, & \text{if } v = v_{\alpha,\beta} \in G_s, \alpha = 1:1:t+1;\\
& \beta = 1:1:s \text{ and } v_\alpha \text{ is central vertex of } W_t\\[2ex]
\frac{1}{4st+2t-s-deg(v_{\alpha,\beta})}, & \text{if } v = v_{\alpha,\beta} \in G_s, \alpha = 1:1:t+1;\\
& \beta = 1:1:s \text{ and } v_\alpha \text{ is non-central vertex of } W_t\\[2ex]
\frac{1}{2st+s+t}, & \text{if } v = v_\alpha \text{ is central vertex of } W_t,\ \alpha = 1:1:t+1\\[2ex]
\frac{1}{3st-2s+2t-3}, & \text{if } v = v_\alpha \text{ is non-central vertex of } W_t,\\
& \alpha = 1:1:t+1.
\end{cases}$$

.

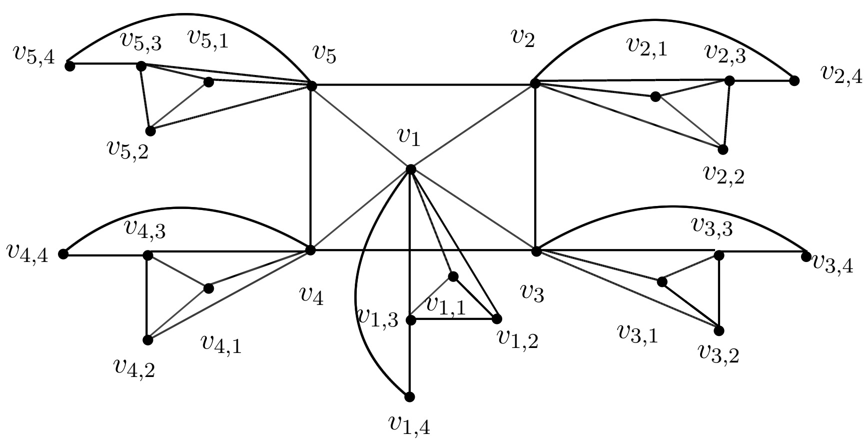

Figure 4: Corona graph $W_4 \odot G_4$

.

Proof. Let $\{v_\alpha : \alpha = 1 : 1 : t+1\}$ and $\{v_{\alpha,\beta} : \alpha = 1 : 1 : t+1; \beta = 1 : 1 : s\}$ be the set of node points of the wheel graph W_t and the graph G_s respectively. Also, let v_1 is the central vertex of W_t.
If $v = v_\alpha$ be any non-central vertex of W_t then
$\sum_{x\in V} d(v_\alpha, x) = \sum_{k=1}^{t+1} d(v_\alpha, v_k) + \sum_{k=1}^{t+1}\sum_{\beta=1}^{s} d(v_\alpha, v_{k,\beta})$
As three sets of s vertices of G_s are at 2 distances, s vertices of G_s are at 1 distance and remaining vertices of G_s are at 3 distances from v_α.
So, $\sum_{x\in V} d(v_\alpha, x) \quad = [3 + 2(t+1-4)] + 3\cdot 2s + [s + 3s(t+1-4)]$
$= 3 + 2t - 6 + 7s + 3st - 9s$
$= 3st - 2s + 2t - 3.$
Therefore, the C-centrality of v is $C_C(v) = \frac{1}{\sum_{x\in V} d(v,x)} = \frac{1}{3st-2s+2t-6}$.

If $v = v_1$ be the central node of W_t then
$\sum_{x\in V} d(v_1, x)$
$= \sum_{k=2}^{t+1} d(v_1, v_k) + \sum_{\beta=1}^{s} d(v_1, v_{1,\beta}) + \sum_{k=2}^{t+1}\sum_{\beta=1}^{s} d(v_1, v_{k,\beta})$
$= t + s + 2s\cdot t$
$= 2st + s + t.$

Hence, the C-centrality of v is $C_C(v) = \frac{1}{\sum_{x\in V} d(v,x)} = \frac{1}{2st+s+t}$.

If $v = v_{\alpha,\beta}$ is any node of G_s attached with the non-central vertex of W_t where $\alpha = 1:1:t+1$; $\beta = 1:1:s$ then
$\sum_{x\in V} d(v_{\alpha,\beta}, x)$
$= d(v_{\alpha,\beta}, v_\alpha) + d(v_{\alpha,\beta}, v_1) + \sum_{k=2,k\neq\alpha}^{t+1} d(v_{\alpha,\beta}, v_k) + d(v_{\alpha,\beta}, v_{\alpha,k})+$
$\sum_{l=1,l\neq\alpha}^{t+1}\sum_{k=1}^{s} d(v, v_{l,k})$.

The shortest distance between $v_{\alpha,\beta}$ and the adjacent vertices of $v_{\alpha,\beta}$ in G_s is 1 and the shortest distance between $v_{\alpha,\beta}$ and the remaining vertices of G_s is 2. i.e. $\{v_{\alpha,\beta} - 1\}$ vertices of G_s have the shortest distance 1 from $v_{\alpha,\beta}$ and $\{s - v_{\alpha,\beta}\}$ vertices have the shortest distance 2 from $v_{\alpha,\beta}$.
Now, $\sum_{x\in V} d(v_{\alpha,\beta}, x)$
$= 1 + 2 + [2 + 2 + 2(t + 1 - 4)] + [deg(v_{\alpha,\beta}) - 1 + 2(s - deg(v_{\alpha,\beta}))] + [3 \cdot 3s + 4s(t + 1 - 4)]$
$= 3 + (2t - 2) + [2s - deg(v_{\alpha,\beta}) - 1] + [9s + 4st - 12s]$
$= 4st + 2t - s - deg(v_{\alpha,\beta})$.
Hence, the C-centrality of v is $C_C(v) = \frac{1}{\sum_{x\in V} d(v,x)} = \frac{1}{4st+2t-s-deg(v_{\alpha,\beta})}$.

If $v = v_{1,\beta}$ is any node of G_s attached with the central node point v_1 of W_t where $\beta = 1:1:s$ then $\sum_{x\in V} d(v_{1,\beta}, x) = d(v_{1,\beta}, v_1) + \sum_{\alpha=2}^{t+1} d(v_{1,\beta}, v_\alpha) + \sum_{k=1}^{s} d(v_{1,\beta}, v_{1,k}) + \sum_{\alpha=2}^{t+1}\sum_{k=1}^{s} d(v_{1,\beta}, v_{\alpha,k})$.
The shortest distance between $v_{1,\beta}$ and the adjacent vertices of $v_{1,\beta}$ in G_s is 1 and the shortest distance between $v_{1,\beta}$ and the remaining vertices of G_s is 2. i.e. $deg(v_{1,\beta}) - 1$ vertices of G_s have the shortest distance 1 from $v_{1,\beta}$ and $\{s - deg(v_{1,\beta})\}$ vertices have the shortest distance 2 from $v_{1,\beta}$.
Now,
$\sum_{x\in V} d(v_{1,\beta}, x)$
$= 1 + 2t + [deg(v_{1,\beta}) - 1 + 2(s - deg(v_{1,\beta}))] + 3st$
$= 3st + 2s + 2t - deg(v_{1,\beta})$.
Hence, the C-centrality of v is
$C_C(v) = \frac{1}{\sum_{x\in V} d(v,x)} = \frac{1}{3st+2s+2t-deg(v_{1,\beta})}$. □

3.5 Closeness centrality of corona product $S_t \odot G_s$

The graph $S_t \odot G_s$ is the product of the star graph S_t of t vertices and the random graph G_s. The cardinality of the corona graph $S_t \odot G_s$ is $t + ts$. We assume that the degree of each vertex of G_s is known. Therefore, the degree of $v_{\alpha,\beta}$ in $S_t \odot G_s$ is 1 more than the degree of $v_{\alpha,\beta}$ in G_s. Let the node set of S_t and G_s(corresponding to the vertex $v_\alpha, \alpha = 1 : 1 : t$) be $\{v_\alpha : \alpha = 1 : 1 : t\}$ and $\{v_{\alpha,\beta} : \beta = 1 : 1 : s\}$, respectively. Also, let v_1 be the cental vertex of S_t. Figure 5 shows the corona graph $S_3 \odot G_4$.

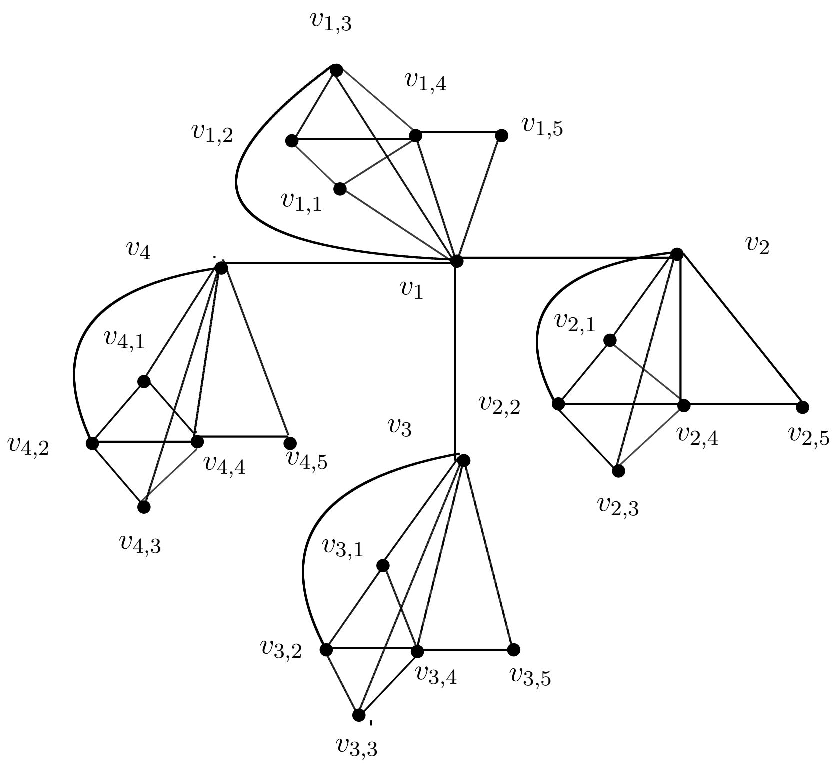

Figure 5: Corona graph $S_4 \odot G_5$

.

Theorem 3.5. *The $C_C(v)$ of arbitrary vertex v of $S_t \odot G_s$ is given by*

$$C_C(v) = \begin{cases} \frac{1}{2st+t-s-1}, \textit{ if } \ v = v_\alpha \textit{ is central node of } S_t,\ \alpha = 1:1:t \\ \frac{1}{3st-3s+2t-3}, \textit{ if } v = v_\alpha \textit{ is non-central node of } S_t,\ \alpha = 1:1:t \\ \frac{1}{4st+3t-3s-deg(v_{\alpha,\beta})-4}, \quad \textit{if } \ v = v_{\alpha,\beta} \in G_s,\ \alpha = 1:1:t; \\ \qquad \beta = 1:1:s \textit{ and } v_\alpha \textit{ is non-central node of } S_t \\ \frac{1}{3st-s+2t-deg(v_{1,\beta})-2}, \quad \textit{if } \ v = v_{1,\beta} \in G_s, \\ \qquad \beta = 1:1:s \textit{ and } v_1 \textit{ is central node of } S_t. \end{cases}$$

.

Proof. Let $\{v_\alpha : \alpha = 1:1:t\}$ and $\{v_{\alpha,\beta} : \alpha = 1:1:t; \beta = 1:1:s\}$ be the set of nodes of the star graph S_t and the random graph G_s respectively. Also let, v_1 is the central vertex of S_t. If $v = v_\alpha$ is the central vertex of S_t i.e., if $v = v_1$ then
$\sum_{x\in V} d(v_1, x)$
$= \sum_{\alpha=2}^{t} d(v_1, v_\alpha) + \sum_{\beta=1}^{s} d(v_1, v_{1,\beta}) + \sum_{l=2}^{t}\sum_{\beta=1}^{s} d(v_1, v_{l,\beta})$
$= (t-1) + s + 2s(t-1)$
$= t - 1 + s(1 + 2t - 2)$
$= 2st - s + t - 1.$
Therefore, the C-centrality of v is $C_C(v) = \frac{1}{\sum_{x\in V} d(v_\alpha,x)} = \frac{1}{2st-s+t-1}$.

If $v = v_\alpha$ is a non-central node of S_n then
$\sum_{x\in V} d(v_\alpha, x)$
$= \sum_{k=1}^{t} d(v_\alpha, v_k) + \sum_{\beta=1}^{s} d(v_\alpha, v_{\alpha,\beta}) + [\sum_{l=\alpha+1}^{t}\sum_{\beta=1}^{s} d(v_\alpha, v_{l,\beta}) +$
$\sum_{l=1}^{\alpha-1}\sum_{\beta=1}^{s} d(v_\alpha, v_{l,\beta})]$.
The shortest distance from v_α to each vertex of S_t except central vertex (v_1) is 2 and from v_α to central vertex (v_1) is 1. So, $\sum d(v_\alpha, v_k) = 1 + 2(t-2)$. Also, the shortest distance from v_α to each vertex of $\{v_{\alpha,\beta} : \beta = 1:1:s\}$ is 1 and the shortest distance from v_α to each vertex of $\{v_{\alpha,\beta} : \alpha = 1, 2, \cdots, \alpha-1, \alpha+1, \cdots, t; \beta = 1:1:s\} - \{v_{1,\beta} : \beta = 1:1:s\}$ is 3. The shortest distance from v_α to each vertex of $\{v_{1,\beta} : \beta = 1:1:s\}$ is 2.
Hence,

$\sum_{x\in V} d(v_\alpha, x)$
$= [1 + 2(t-2)] + s + [2s + 3s(t-2)]$
$= 2t - 3 + 3s(t-1)$
$= 3st - 3s + 2t - 3.$
Therefore, the C-centrality of v is $C_C(v) = \frac{1}{\sum_{x\in V} d(v_\alpha,x)} = \frac{1}{3st-3s+2t-3}$.

If $v = v_{\alpha,\beta}$ attached with any non-central node of S_t where $\alpha = 1 : 1 : t$; $\beta = 1 : 1 : s$ then
$\sum_{x\in V} d(v_{\alpha,\beta}, x)$
$= \sum_{p=1}^{t} d(v_{\alpha,\beta}, v_p) + \sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{1,k}) + \sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{\alpha,k}) +$
$[\sum_{l=\alpha+1}^{t} \sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{l,k}) + \sum_{l=2}^{\alpha-1} \sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{l,k})]$.
The shortest distance between $v_{\alpha,\beta}$ and the adjacent vertices of $v_{\alpha,\beta}$ in G_s is 1 and the shortest distance between $v_{\alpha,\beta}$ and the remaining vertices of G_s is 2. i.e. $\{deg(v_{\alpha,\beta})-1\}$ vertices of G_s have the shortest distance 1 from $v_{\alpha,\beta}$ and $\{s - deg(v_{\alpha,\beta})\}$ vertices have the shortest distance 2 from $v_{\alpha,\beta}$.
So,
$\sum_{x\in V} d(v_{\alpha,\beta}, x)$
$= 1 + [2 + 3(t-2)] + 3s + [deg(v_{\alpha,\beta}) - 1 + 2(s - deg(v_{\alpha,\beta}))] + 4s(t-2)$
$= 3t - 3 + 3s + 2s - deg(v_{\alpha,\beta}) - 1 + 4st - 8s$
$= 4st - 3s + 3t - deg(v_{\alpha,\beta}) - 4.$
Hence, the C-centrality of v is $C_C(v) = \frac{1}{\sum_{x\in V} d(v,x)} = \frac{1}{4st-3s+3t-deg(v_{\alpha,\beta})-4}$.

If $v = v_{\alpha,\beta}$ is attached with the central node v_1 of S_t where $\beta = 1 : 1 : s$ then
$\sum_{x\in V} d(v_{1,\beta}, x) = d(v_{1,\beta}, v_1) + \sum_{p=2}^{t} d(v_{1,\beta}, v_p) + \sum_{k=1}^{s} d(v_{1,\beta}, v_{1,k}) + \sum_{l=2}^{t} \sum_{k=1}^{s} d(v_{1,\beta}, v_{l,k})$.
The shortest distance between $v_{1,\beta}$ and the adjacent vertices of $v_{1,\beta}$ is in G_s 1 and the shortest distance between $v_{1,\beta}$ and the remaining vertices of G_s is 2. i.e. $deg(v_{1,\beta}) - 1$ vertices of G_s have the shortest distance 1 from $v_{1,\beta}$ and $\{s - deg(v_{1,\beta})\}$ vertices have the shortest distance 2 from $v_{1,\beta}$.
Now,
$\sum_{x\in V} d(v_{1,\beta}, x)$
$= 1 + 2(t-1) + [deg(v_{1,\beta}) - 1 + 2\{s - deg(v_{1,\beta})\}] + [3s(t-1)]$
$= 2t - 1 + 2s - deg(v_{1,\beta}) - 1 + 3st - 3s$

$= 3st - s + 2t - deg(v_{1,\beta}) - 2.$
Hence, the C-centrality of v is $C_C(v) = \frac{1}{\sum_{x\in V} d(v,x)} = \frac{1}{3st-s+2t-deg(v_{1,\beta})-2}$.
□

3.6 Closeness centrality of corona product $K_{t,l} \odot G_s$

The graph $K_{t,l} \odot G_s$ is the corona product of $K_{t,l}$ and G_s. The cardinality of the corona graph $K_{t,l} \odot G_s$ is $t + l + (t + l)s = (t + l)(s + 1)$. We assume that the degree of each vertex of the G_s is known. Therefore, the degree of $v_{\alpha,\beta}$ in $K_{t,l} \odot G_s$ is 1 more than the degree of $v_{\alpha,\beta}$ in G_s. Let the node set of $K_{t,l}$ and G_s(corresponding to the node $v_\alpha, \alpha = 1, 2, \cdots, t, t+1, t+2, \cdots, t+l$) be $\{v_\alpha : \alpha = 1, 2, \cdots, t, t+1, t+2, \cdots, t+l\}$ and $\{v_{\alpha,\beta} : \beta = 1 : 1 : s\}$, respectively. Figure 6 shows the graph $K_{3,2} \odot G_4$.

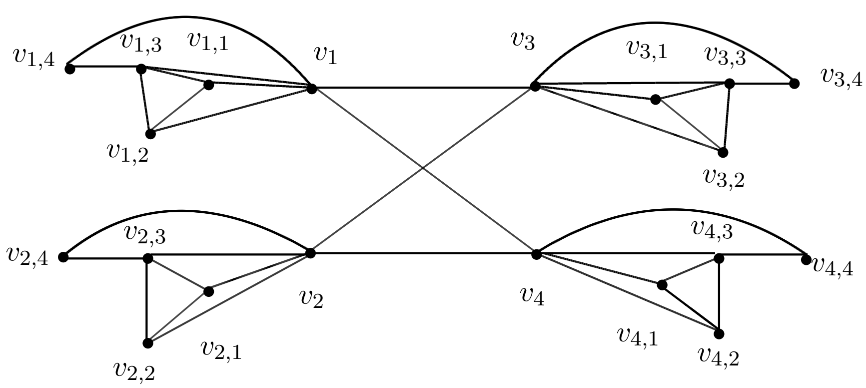

Figure 6: Corona graph $K_{2,2} \odot G_4$

.

Theorem 3.6. *The $C_C(v)$ of arbitrary vertex v of $K_{t,l} \odot G_s$ is*

$$C_C(v) = \begin{cases} \frac{1}{3st+2sl+2t-2s+l-2}, \textit{if} \ \ v = v_\alpha \in K_{t,l} \ \textit{and} \ \alpha = 1:1:s \\ \\ \frac{1}{2st+3sl+2l-2s+t-2}, \textit{if} \ \ v = v_\alpha \in K_{t,l} \ \textit{and} \ \alpha = t+1:1:t+l \\ \\ \frac{1}{4st+3sl-2s+3t+2l-deg(v_{\alpha,\beta})-3}, \textit{if} \ \ v = v_{\alpha,\beta} \in G_s, \ \alpha = 1:1:t \\ \qquad\qquad \textit{and} \ \beta = 1:1:s \\ \\ \frac{1}{4sl+3st+3l+2t-2s-deg(v_{\alpha,\beta})-3}, \qquad \textit{if} \ \ v = v_{\alpha,\beta} \in G_s, \\ \qquad \alpha = t+1:1:t+l \ \textit{and} \ \beta = 1:1:s. \end{cases}$$

.

Proof. Let $\{v_1, v_2, \cdots, v_t, v_{t+1}, \cdots, v_{t+l}\}$ and $\{v_{\alpha,\beta} : \alpha = 1, 2, \cdots, t, t+1, \cdots, t+l; \beta = 1 : 1 : s\}$ be the set of nodes of the complete bipartite graph $K_{t,l}$ and the random graph G_s respectively. If $v = v_\alpha$ where $\alpha = 1 : 1 : t$ then
$\sum_{x \in V} d(v_\alpha, x) = \sum_{k=1}^{t} d(v_\alpha, v_k) + \sum_{k=t+1}^{t+l} d(v_\alpha, v_k) + [\sum_{j=1}^{s} d(v_\alpha, v_{\alpha,\beta}) + \sum_{k=1,k\neq\alpha}^{t} \sum_{\beta=1}^{s} d(v_\alpha, v_{k,\beta})] + \sum_{k=t+1}^{t+l} \sum_{\beta=1}^{s} d(v_\alpha, v_{k,\beta})$
$= 2(t-1) + l + [s + 3s(t-1)] + 2sl$
$= 2t - 2 + l + s + 3st - 3s + 2sl$
$= 3st + 2sl - 2s + 2t + l - 2.$
So, the C-centrality of v is $C_C(v) = \frac{1}{\sum_{x \in V} d(v_\alpha, x)} = \frac{1}{3st+2sl-2s+2t+l-2}$.

If $v = v_\alpha$, $\alpha = t+1, t+2, \cdots, t+l$ then
$\sum_{x \in V} d(v_\alpha, x)$
$= \sum_{k=1}^{t} d(v_\alpha, v_k) + \sum_{k=t+1}^{t+l} d(v_\alpha, v_k) +$
$[\sum_{\beta=1}^{s} d(v_\alpha, v_{\alpha,\beta}) + \sum_{k=t+1}^{t+l} \sum_{\beta=1}^{s} d(v_\alpha, v_{k,\beta})] + \sum_{k=1}^{t} \sum_{\beta=1}^{s} d(v_\alpha, v_{k,\beta})$
$= t + 2(l-1) + [s + 3s(l-1)] + 2st$
$= t + 2l - 2 + s + 3sl - 3s + 2st$
$= 3sl + 2st - 2s + t + 2l - 2.$
So, the C-centrality of v is $C_C(v) = \frac{1}{\sum_{x \in V} d(v_\alpha, x)} = \frac{1}{3sl+2st-2s+t+2l-2}$.

If $v = v_{\alpha,\beta}$ where $\alpha = 1 : 1 : t$; $\beta = 1 : 1 : s$ then
$\sum_{x \in V} d(v_{\alpha,\beta}, x) = [d(v_{\alpha,\beta}, v_\alpha) + \sum d(v_{\alpha,\beta}, v_{\alpha+1}) + \cdots + d(v_{\alpha,\beta}, v_t)$

$+d(v_{\alpha,\beta}, v_{\alpha-1}) + d(v_{\alpha,\beta}, v_{\alpha-2}) + \cdots + d(v_{\alpha,\beta}, v_1)] + [d(v_{\alpha,\beta}, v_{t+1})+$
$d(v_{\alpha,\beta}, v_{t+2})+d(v_{\alpha,\beta}, v_{t+l})]+\sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{\alpha,k})+[\sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{\alpha+1,k})+$
$\cdots + \sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{t,k}) + \sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{\alpha-1,k}) + \sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{\alpha-2,k}) +$
$\cdots + \sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{1,k})] + [\sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{t+1,\beta}) + \sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{t+2,\beta}) +$
$\cdots + \sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{t+l,\beta})]$.
The shortest distance between $v_{\alpha,\beta}$ and the adjacent vertices of $v_{\alpha,\beta}$ in G_s is 1 and the shortest distance between $v_{\alpha,\beta}$ and the remaining vertices of G_s is 2. i.e. $(deg(v_{\alpha,\beta})-1)$ vertices of G_s have the shortest distance 1 from $v_{\alpha,\beta}$ and $\{s-deg(v_{\alpha,\beta})\}$ vertices have the shortest distance 2 from $v_{\alpha,\beta}$.
Therefore,
$\sum_{x\in V} d(v_{\alpha,\beta}, x)$
$= [1+3(t-1)]+2l+[deg(v_{\alpha,\beta})-1+2\{s-deg(v_{\alpha,\beta})\}]+4s(t-1)+3sl$
$= 3t-2+2l+2s-deg(v_{\alpha,\beta})-1+4st-4s+3sl$
$= 4st+3sl-2s+3t+2l-deg(v_{\alpha,\beta})-3$.
Hence, the C-centrality of v is
$C_C(v) = \frac{1}{\sum_{x\in V} d(u,x)} = \frac{1}{4st+3sl-2s+3t+2l-deg(v_{\alpha,\beta})-3}$.

If $v = v_\alpha$, $\alpha = t+1, t+2, \cdots, t+l$; $\beta = 1:1:s$ then
$\sum_{x\in V} d(v_{\alpha,\beta}, x)$
$= [d(v_{\alpha,\beta}, v_\alpha) + \sum d(v_{\alpha,\beta}, v_{\alpha+1}) + \cdots + d(v_{\alpha,\beta}, v_{t+l}) + d(v_{\alpha,\beta}, v_{\alpha-1}) +$
$d(v_{\alpha,\beta}, v_{\alpha-2})+\cdots+d(v_{\alpha,\beta}, v_{t+1})]+[d(v_{\alpha,\beta}, v_1)+d(v_{\alpha,\beta}, v_2)+\cdots+d(v_{\alpha,\beta}, v_t)]$
$+\sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{\alpha,k})+[\sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{\alpha+1,k})+\cdots+\sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{t+l,k})+$
$\sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{\alpha-1,k})+\sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{\alpha-2,k})+\cdots+\sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{t+1,k})]+$
$[\sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{1,\beta}) + \sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{2,\beta}) + \cdots + \sum_{k=1}^{s} d(v_{\alpha,\beta}, v_{t,\beta})]$.
The shortest distance between $v_{\alpha,\beta}$ and the adjacent vertices of $v_{\alpha,\beta}$ in G_s is 1 and the shortest distance between $v_{\alpha,\beta}$ and the remaining vertices of G_s is 2. i.e. $(deg(v_{\alpha,\beta})-1)$ vertices of G_s have the shortest distance 1 from $v_{\alpha,\beta}$ and $\{s-deg(v_{\alpha,\beta})\}$ vertices have the shortest distance 2 from $v_{\alpha,\beta}$.
Therefore,
$\sum_{x\in V} d(v_{\alpha,\beta}, x)$
$= [1+3(l-1)]+2t+[deg(v_{\alpha,\beta})-1+2\{s-deg(v_{\alpha,\beta})\}]+4s(l-1)+3st$
$= 3l-2+2t+2s-deg(v_{\alpha,\beta})-1+4sl-4s+3st$
$= 4sl+3st-2s+3l+2t-deg(v_{\alpha,\beta})-3$.
Hence, the C-centrality of v is
$C_C(v) = \frac{1}{\sum_{x\in V} d(u,x)} = \frac{1}{4sl+3st-2s+3l+2t-deg(v_{\alpha,\beta})-3}$. □

4 Real Application

We consider a small-world network. It is a special type of network where most nodes are not directly connected to one another but can be reached from any other node through a relatively short chain of intermediate nodes. These networks exhibit a high degree of clustering, where nodes tend to cluster together into tightly-knit groups, yet also possess short average path lengths between nodes. Small-world networks are prevalent in various fields, including social networks, the internet, and neural networks, playing a crucial role in understanding connectivity patterns and information flow in complex systems. In 2015, Lv et al [27] introduced recursive corona product graphs as a new model of small-world networks. They denoted a g^{th} generation of recursive corona graph by $C_q(g+1)$, where $C_q(g+1) = C_q(g) \odot K_q, g \geq 0, k \geq 2$ with the initial condition $C_q(0) = K_q$. They also studied different characteristics like order and size, degree distribution, average path length, clustering coefficient, etc. of their proposed graph model based on corona product graphs. They also established some results for all the quantities of the recursive corona graphs, which are similar to those observed in real-life networks. Now, if we define a g^{th} generation of recursive corona graph $C_q(g+1)$, where $C_q(g+1) = K_q \odot C_q(g)$, remaining all other conditions same, then it also represents a graph model of small-world network. Obviouly, in that recurssive corona graph, first graph is known (a complete graph) and second graph is an arbitrary graph. So, we can easily identify the influential nodes of this type of small-world network based on our proposed results presented in this paper.

5 Conclusion and Future Work

Closeness centrality is a popular variant of centrality measurement used to recognize the characteristic of a vertex in a network. It helps to find the significant node point in biological networks, social networks, transportation networks, etc. In this present paper, we propose some new theorems related to the C-centrality of different types of corona graphs between well-known graph and random graph. In future, we have a plan to formulate the C-centrality of the corona product of two general graphs with the help of these results. We also try to find influential nodes of the small-world network based on corona graphs of two fuzzy graphs.

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