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The Category Rel(Nom)

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Abstract. The category Rel($\mathcal{C}$) may be formed for any category $\mathcal{C}$ with finite limits using the same objects as $\mathcal{C}$ but whose morphisms from X to Y are binary relations in $\mathcal{C}$, that is, subobjects of $X \times Y$. In this paper, concerning the topos **Nom**, we study the category **Rel**(**Nom**). In this category, we define and investigate certain morphisms, such as deterministic morphisms. Then, stochastic mappings between nominal sets are defined by exploiting the underlying relation of functions between nominal sets. This allows one to reinterpret concepts and earlier results in terms of morphisms.

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1 Introduction

Finitely supported mathematics (or theory of nominal sets, when dealing with computer science applications) provides a framework for working with infinitely structured hierarchically constructed by involving some basic elements (called atoms) by dealing only with a finite number of entities that form their supports, see [2]. This theory is related to the recent development of Fraenkel-Mostowski's set theory, which works with "nominal sets" and deals with binding and new names in computer science, and developing by studying the category of nominal sets and equivariant functions between them, see [16]. But some very common mathematical structures are not functions. Therefore, in this paper, we introduce the category **Rel**(**Nom**) consisting of nominal sets and equivariant relations between them which can have several advantages and is more expressive than the category of nominal sets alone, as it allows one to reason about relations between elements, not just the elements themselves. When working in **Rel**(**Nom**), it is possible to reason about how permutations work on the elements of the sets and the relations between them, which can be useful in fields such as physics and computer science. In type theory, the category **Rel**(**Nom**) can be used to model dependent types, which are types that depend on values, not just other types. This allows one to reason about the properties of programs that depend on input data. Also, the category **Rel**(**Nom**) can be used to represent mathematical structures such as algebraic data types and reasoning about them in an equivalent way.

Although the category **Rel**(**Nom**) is not a topos, see Remark 3.5, but presheaf representation of nominal sets in **Rel**(**Nom**) allows one to understand the mathematical structure of these objects in a more general and abstract way. Additionally, studying the equivariant relations between different presheaves can provide insight into the permutations and invariances of the sets being studied. Furthermore, in the field of computer science, in particular domain-specific languages, nominal sets and their presheaf representation can be used to reason about the syntax and semantics of programming languages in a more formal and rigorous way. Therefore, we devote Section 2 to introduce the category of **Rel**(**Nom**) and explore some properties of its morphisms. We then, in Section 3, discuss the presheaf representation of the objects of **Rel**(**Nom**). Finally, in Section 4, we introduce another presheaf representation of nominal sets in **Rel**(**Nom**) and introduce the natural deterministic and stochastic morphisms in this category.

1.1 The category G-Set

This subsection is devoted to the needed facts about G-sets. We refer interested readers to [7] and [13] for more information.

A non-empty set X equipped with a map $G \times X \longrightarrow X$ (action of a group G on X) mapping (g, x) to gx is called a *G-set* if for every $g_1, g_2 \in G$ and every $x \in X$, we have $g_1(g_2x) = (g_1g_2)x$ and $ex = x$, in which "e" is the identity of the group G. For G-sets X and Y, a map $f : X \to Y$ is called an *equivariant map* if $f(gx) = gf(x)$, for all $x \in X$ and $g \in G$. The category of all G-sets with equivariant maps between them is denoted by G-**Set**.

An element x of a G-set X is called a *zero* (or a *fixed*) element if $gx = x$, for all $g \in G$. We denote the set of all zero elements of a G-set X by $\mathcal{Z}(X)$.

The G-set X all of whose elements are zero is called *discrete*, or a G-set with the *identity action*.

A subset Y of a G-set X is an *equivariant subset* (or a *G-subset*) of X if for all $g \in G$ and $y \in Y$ we have $gy \in Y$. The subset $\mathcal{Z}(X)$ of X is a G-subset.

Given a G-set X and $x \in X$, the set $Gx = \{gx : g \in G\}$ is called the *orbit* of x. Note that the class $\{Gx\}_{x \in X}$ is the corresponding partition of the equivalence relation $\sim$ over X defined by $x \sim x'$ if and only if there exists $g \in G$ with $gx = x'$, for which the class $x \backslash \sim = Gx$ is denoted by $\mathsf{orb}x$.

Given a G-set X and $x \in X$, the set $G_x = \{g \in G : gx = x\}$ is a subgroup of G fixing x . Also, for every $Y \subseteq X$, we define:

$$G_Y = \{g \in G : gy = y,\ \forall y \in Y\} = \bigcap_{y \in Y} G_y.$$

1.2 The category of nominal sets

In this subsection, we briefly recall relevant definitions concerning nominal sets. For the most part, we follow [11, 15, 16].

From now on $\mathbb{D}$ denotes a fixed, countably infinite set whose elements $a, b, c, \ldots$ are called *atomic names*. A *permutation* π of $\mathbb{D}$ is a bijective map from $\mathbb{D}$ to itself. All permutations of $\mathbb{D}$ with the composition of maps as the binary operation form a group called the *symmetric group* on $\mathbb{D}$ and denoted by $\mathrm{Sym}(\mathbb{D})$.

A permutation $\pi \in \mathrm{Sym}(\mathbb{D})$ is *finitary* if the set $\{d \in \mathbb{D} : \pi d \neq d\}$ is a finite subset of $\mathbb{D}$. It is clear that the set $\mathrm{Perm}(\mathbb{D})$ consisting of all finitary permutations is a subgroup of $\mathrm{Sym}(\mathbb{D})$.

Let X be a set equipped with an action of the group $\mathrm{Perm}(\mathbb{D})$, $\mathrm{Perm}(\mathbb{D}) \times X \longrightarrow X$ mapping $(\pi, x) \rightsquigarrow \pi x$. By definition of action of the group $\mathrm{Perm}(\mathbb{D})$ over the set X, we have:

(i) $\pi_1(\pi_2 x) = (\pi_1 \circ \pi_2)x$

(ii) $\mathrm{id}x = x$,

for every $\pi_1, \pi_2 \in \mathrm{Perm}(\mathbb{D})$ and every $x \in X$.

The set $\mathbb{D}$ together with the specified action given in Example 1.6(i) provide the most natural example of a $\mathrm{Perm}(\mathbb{D})$-set. In this case, for a given $C \subseteq \mathbb{D}$, using the notation G_x given in Subsection 1.1, we have:

$$(\mathrm{Perm}(\mathbb{D}))_C = \{\pi \in \mathrm{Perm}(\mathbb{D}) : \pi(d) = d,\ \forall d \in C\}.$$

Given a $\mathrm{Perm}(\mathbb{D})$-set X, a set of atomic names $C \subseteq \mathbb{D}$ is a *support* for an element $x \in X$ if for all $\pi \in \mathrm{Perm}(\mathbb{D})$, we have:

$$[\forall d \in C\ : \pi(d) = d)] \Longrightarrow \pi x = x.$$

In other words,

$$\pi \in (\mathrm{Perm}(\mathbb{D}))_C \Longrightarrow \pi x = x.$$

Given a $\mathrm{Perm}(\mathbb{D})$-set X, we say an element $x \in X$ is *finitely supported* if x has a finite support.

Definition 1.1. [16] *A nominal set is a* $\mathrm{Perm}(\mathbb{D})$*-set, in which each element is finitely supported.*

Nominal sets are the objects of a category, denoted by **Nom**, whose morphisms are equivariant maps and whose composition and identities are as in the category of $\mathrm{Perm}(\mathbb{D})$-**Set**. The category **Nom** is a full subcategory of the category $\mathrm{Perm}(\mathbb{D})$-**Set**.

Definition 1.2. [16] *The transposition (also known as swapping) of a pair of elements $d_1, d_2 \in \mathbb{D}$ is the finitary permutation $(d_1\ d_2) \in \mathrm{Perm}(\mathbb{D})$ given for all $d \in \mathbb{D}$ by*

$$(d_1\ d_2)d = \begin{cases} d_2 & \text{if } d = d_1 \\ d_1 & \text{if } d = d_2 \\ d & \text{otherwise.} \end{cases}$$

Remark 1.3. [16, Propositions 2.1, 2.3] *Suppose X is a nominal set and $x \in X$.*

(i) *A finite subset $C \subseteq \mathbb{D}$ supports x if and only if $(d_1\ d_2)x = x$, for all $d_1, d_2 \notin C$.*

(ii) *The intersection of any number of finite supports of x is a support of x.*

(iii) *By (ii), x has the least finite support and is denoted by* $\operatorname{supp} x$*. In fact,* $\operatorname{supp}_X x = \bigcap\{C : C \text{ is a finite support of} x\}$*.*

Lemma 1.4. *[16] If X is a* $\mathrm{Perm}(\mathbb{D})$*-set, then the subset*

$$X_{\mathrm{fs}} = \{x \in X : x \text{ is finitely supported in } X\}$$

of X, consisting of all finitely supported elements of X, is a nominal set.

Remark 1.5. [16](i) *Given a* $\mathrm{Perm}(\mathbb{D})$*-set X, the set $\mathcal{P}(X) = \{Y : Y \subseteq X\}$ with the following action*

$$\mathrm{Prem}(\mathbb{D}) \times \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$$
$$(\pi, Y) \rightsquigarrow \pi \cdot Y = \{\pi y : y \in Y\}$$

is a $\mathrm{Perm}(\mathbb{D})$*-set. A set of atomic names C supports $Y \in \mathcal{P}(X)$ if and only if*

$$(\forall \pi \in \mathrm{Perm}(\mathbb{D}))((\forall d \in C)\ \pi(d) = d) \Longrightarrow (\forall y \in Y)\ \pi y \in Y.$$

(ii) *The equivariant subsets of X are exactly the zero elements of $\mathcal{P}(X)$. Hence, we have* $\operatorname{supp} Y = \emptyset$ *if and only if Y is an equivariant subset of X, for every $Y \in \mathcal{P}(X)$. Particularly, X is supported by the empty set in $\mathcal{P}(X)$.*

(iii) *The finitely supported elements of* $\mathcal{P}(\mathbb{D})$ *are finite and cofinite subsets of* X*; more explicitly,* $C \in \mathcal{P}(\mathbb{D})$ *is finitely supported if either* C *or* $\mathbb{D} \setminus C$ *is finite.*

In the following, we give some examples of nominal sets.

Example 1.6. (i) The set $\mathbb{D}$ is a nominal set, with the action

$$\mathrm{Perm}(\mathbb{D}) \times \mathbb{D} \longrightarrow \mathbb{D}$$

$$(\pi, d) \rightsquigarrow \pi(d).$$

Indeed, the set $\{d\}$ is a finite support of d, for every $d \in \mathbb{D}$.

(ii) Every discrete $\mathrm{Perm}(\mathbb{D})$-set X is a nominal set. Indeed, the empty set is a finite support for each element $x \in X$.

(iii) Each finite element of $\mathcal{P}(\mathbb{D})$ is supported by itself. So we get the nominal set $\mathcal{P}_{\mathrm{f}}(\mathbb{D})$ of all finite subsets of $\mathbb{D}$ with $\pi \cdot C = \{\pi d : d \in C\}$ and $\mathrm{supp}\, C = C$.

Remark 1.7. [16, Proposition 2.11] *If* X *is a nominal set and* $x \in X$*, then* $\pi \mathrm{supp}\, x = \mathrm{supp}\, \pi x$*, for every* $\pi \in \mathrm{Perm}(\mathbb{D})$*.*

We recall the following definition from [13].

Definition 1.8. *A nominal set* X *is called*

(i) *decomposable if there exist non-empty equivariant subsets* X_1, X_2 *of* X*, such that* $X = X_1 \cup X_2$ *and* $X_1 \cap X_2 = \emptyset$*. Otherwise,* X *is indecomposable.*

(ii) *cyclic if it is generated by only one element, i.e.* $X = \mathrm{Perm}(\mathbb{D})x$*, for some* $x \in X$*.*

Lemma 1.9. *A non-empty nominal set* X *is indecomposable if and only if* X *is cyclic and has no pure equivariant subset.*

Proof. ($\Rightarrow$) Let $x \in X$ and X be indecomposable. Then, $\mathrm{Perm}(\mathbb{D})x \subseteq X$. If $X \neq \mathrm{Perm}(\mathbb{D})x$, then $X = \mathrm{Perm}(\mathbb{D})x \cup (X \setminus \mathrm{Perm}(\mathbb{D})x)$ which is a contradiction. Note that since $\mathrm{Perm}(\mathbb{D})$ is a group, $X \setminus \mathrm{Perm}(\mathbb{D})x$ is an equivariant subset of X. Now, suppose A is an equivariant subset of $X = \mathrm{Perm}(\mathbb{D})x$. Let $a \in A$. Then, $a \in \mathrm{Perm}(\mathbb{D})x$ and so there exists $\pi \in \mathrm{Perm}(\mathbb{D})$ with $a = \pi x$. Thus, $x = \pi^{-1} a \in A$ and so $X = A$.

($\Leftarrow$) The converse is obviously true. □

Lemma 1.10. *Suppose X and Y are two nominal sets. Also, suppose $X' \in \mathcal{P}_{\mathrm{fs}}(X)$ and $Y' \in \mathcal{P}_{\mathrm{fs}}(Y)$. If $f : X' \longrightarrow Y'$ is a finitely supported map as a subset of the nominal set $X \times Y$, then* $\operatorname{supp} f(X') \subseteq \operatorname{supp} f \cup \operatorname{supp} X'$. *Furthermore,* $\operatorname{supp} f(x) \subseteq \operatorname{supp} f \cup \operatorname{supp} x$ *when $X' = \{x\}$.*

Proof. Let $d_1, d_2 \notin \operatorname{supp} f \cup \operatorname{supp} X'$. Then, $(d_1\ d_2)f = f$ and $(d_1\ d_2)X' = X'$. Since $(d_1\ d_2)f = f$, we have $f((d_1\ d_2)x) = (d_1\ d_2)f(x)$, for all $x \in X'$. Let $f(x) \in f(X')$ with $x \in X'$. Then, $(d_1\ d_2)X' = X'$ implies that $(d_1\ d_2)x \in X'$ and so $(d_1\ d_2)f(x) = f((d_1\ d_2)x) \in f(X')$. Thus, $(d_1\ d_2)f(X') = f((d_1\ d_2)X')$. □

1.3 The category Rel

In this subsection, we review some elementary facts concerning the category of sets and relations, denoted by **Rel**, from [1, 4, 6, 8]. The category **Rel** is a category whose objects are sets and morphisms are relations, $R \subseteq X \times Y$. Here, the set of relations from X to Y is denoted by $\mathcal{R}(X,Y)$. By $R : X \to Y$, we mean $R \in \mathcal{R}(X,Y)$. The composition of morphisms $R \in \mathcal{R}(X,Y)$ and $S \in \mathcal{R}(Y,Z)$ is the relational composition $S \circ R \in \mathcal{R}(X,Z)$, defined by

$$(x,z) \in S \circ R \Longleftrightarrow \exists y \in Y; (x,y) \in R \text{ and } (y,z) \in S.$$

The identity morphism $id_X : X \to X$ is the identity relation $\Delta_X = \{(x,x) : x \in X\}$. A relation $R : X \to Y$ is called a *partial map* whenever it is welldefined, i.e $(x,y_1),(x,y_2) \in R$ implies $y_1 = y_2$. A partial map $R : X \to Y$ is said to be *total*, and referred to as a (*total*) *map*, whenever its domain of definition coincides with X. The category **Set** is a wide subcategory of **Rel**.

Definition 1.11. [4] *Suppose $R \in \mathcal{R}(X,Y)$.*

(i) *For given $S \subseteq X$, the set $\overrightarrow{R}(S) = \{y \in Y : \exists x \in S; (x,y) \in R\}$ is called the direct image of S under R. Particularly, $\overrightarrow{R}(X)$ is called the image of R and is denoted by* $\mathrm{Im}R$. *For the singleton subset $\{x\} \subseteq X$, the set $\overrightarrow{R}(\{x\})$ is denoted by $\overrightarrow{R}(x)$.*

(ii) *For given $T \subseteq Y$, the set $\overleftarrow{R}(T) = \{x \in X : \exists y \in T; (x,y) \in R\}$ is called the inverse image of T under R. Particularly, $\overleftarrow{R}(Y)$ is called the domain of R and is denoted by* $\mathrm{Dom}R$.

Definition 1.12. [4] *A relation $R \in \mathcal{R}(X,Y)$ is said to be*

(i) *injective if $(x,y) \in R$ and $(x',y) \in R$ implies that $x = x'$.*

(ii) *surjective if for every $y \in Y$ there is some $x \in X$ so that $(x,y) \in R$.*

(iii) *total injective if for every $x \in X$ there exists $y \in Y$ such that x is the only element related to y. That is, if $(x,y) \in R$ and $(x',y) \in R$, then $x = x'$.*

(iv) *partial surjective map if for every $y \in Y$ there exists $x \in X$ such that y is the only element related to x. That is, if $(x,y) \in R$ and $(x,y') \in R$, then $y = y'$.*

(v) *monic if it is left cancelable; that is $R \circ S = R \circ T$ implies $S = T$.*

(vi) *epic if it is right cancelable; that is $S \circ R = T \circ R$ implies $S = T$.*

(vii) *well-defined if $(x,y) \in R$ and $(x,y') \in R$ implies that $y = y'$.*

Remark 1.13. *If $R \in \mathcal{R}(X,Y)$ is a partial surjective map, then $\overleftarrow{R}(A) \cap \overleftarrow{R}(B) = \overleftarrow{R}(A \cap B)$. To prove the non-trivial part, if $x \in \overleftarrow{R}(A) \cap \overleftarrow{R}(B)$, then there exist $a \in A$ and $b \in B$ with $(x,a),(x,b) \in R$. Now, since R is a partial surjective map, $a = b \in A \cap B$ and so $x \in \overleftarrow{R}(A \cap B)$.*

Lemma 1.14. *Suppose X and Y are two sets and $R \in \mathcal{R}(X,Y)$ is a relation.*

(i) *Let R be injective. Then, $S = S'$, if $\overrightarrow{R}(S) = \overrightarrow{R}(S')$, for every $S, S' \subseteq \mathrm{Dom}R$.*

(ii) *Let R be a partial surjective map. Then, $T = T'$, if $\overleftarrow{R}(T) = \overleftarrow{R}(T')$, for every $T, T' \subseteq \mathrm{Im}R$.*

Proof. (i) Let $x \in S$. Then, there exists $y \in Y$ with $(x,y) \in R$ and so $y \in \overrightarrow{R}(S) = \overrightarrow{R}(S')$. Thus, $y \in \overrightarrow{R}(S')$ and so there exists $x' \in S'$ with $(x',y) \in R$. Now, since R is injective, we get that $x = x'$. So, $x \in S'$. Similarly, we get that $S' \subseteq S$.

(ii) Let $y \in T$. Then, there exists $x \in X$ with $(x,y) \in R$ and so $x \in \overleftarrow{R}(T) = \overleftarrow{R}(T')$. Thus, $x \in \overleftarrow{R}(T')$ and so there exists $y' \in T'$ with $(x,y') \in R$. Now, since R is a partial surjective map, $y = y'$. So, $y \in T'$. Similarly, $T' \subseteq T$. □

Definition 1.15. [4] *Let X, Y be two sets and $R \in \mathcal{R}(X,Y)$. Then,*

(i) *R has a right inverse if there exists $S \in \mathcal{R}(Y,X)$ with $R \circ S = id_{\mathrm{Dom}S}$.*

(ii) *R has a left inverse if there exists $S \in \mathcal{R}(Y, X)$ with $S \circ R = id_{\mathrm{Dom}R}$.*

Remark 1.16. *Let X, Y be two sets and $R \in \mathcal{R}(X, Y)$. Then,*
(i) *if R has a right inverse, then R is surjective.*
(ii) *if R has a left inverse, then R is injective.*

We also recall from [12] that the faithful functor $F : \mathbf{Set} \longrightarrow \mathbf{Rel}$, which is the identity on objects and takes each map $f : X \longrightarrow Y$ to its underlying relation $\{(x, y) \in X \times Y : f(x) = y\}$, is a left adjoint to the powerset (or image) functor $P : \mathbf{Rel} \longrightarrow \mathbf{Set}$ which maps every set X to its power set $\mathcal{P}(X)$ and every relation $R : X \to Y$ to the direct image mapping $\overrightarrow{R} : \mathcal{P}(X) \to \mathcal{P}(Y)$. This adjunction induces covariant powerset monad on **Set**. Moreover, **Rel** is isomorphic to the Kleisli category for this monad.

2 The Category Rel(Nom)

In this section, we focus on the equivariant relations between the nominal sets rather than the equivariant functions between them and take into considering the category **Rel**(**Nom**).

Definition 2.1. *For given G-sets X and Y, the set $\mathcal{R}(X, Y)$ equipped with the action*

$$\cdot : G \times \mathcal{R}(X, Y) \to \mathcal{R}(X, Y), \quad g \cdot R = \{(gx, gy) : (x, y) \in R\},$$

is a G-set.

Remark 2.2. *Given the nominal sets X and Y,*
(i) *a finite set $A \subseteq \mathbb{D}$ is a finite support for $R \in \mathcal{R}(X, Y)$ whenever,*

$$\begin{aligned} \pi \in (\mathrm{Perm}(\mathbb{D}))_A \quad &\Longrightarrow \quad \pi \cdot R = R \\ &\Longrightarrow \quad R(x) = \pi(\overrightarrow{R}(\pi^{-1}x)), \end{aligned}$$

for every $x \in \mathrm{Dom}R$.

So, a relation $R : X \longrightarrow Y$ is equivariant if $\pi \cdot R = R$, for every $\pi \in \mathrm{Perm}(\mathbb{D})$.

(ii) *if* A *is a finite support for the relation* $R : X \longrightarrow Y$*, then* πA *is a finite support of* $\pi \cdot R$.

(iii) *Since, by Remark 1.5(i),* $\mathcal{P}(X \times Y)$ *is a* $\mathrm{Perm}(\mathbb{D})$*-set, using Lemma 1.4, the set of all finitely supported relations from* X *to* Y*, denoted by* $\mathcal{R}_{\mathrm{fs}}(X, Y)$*, is a nominal set.*

Definition 2.3. *Suppose* X *and* Y *are two nominal sets (or two* G*-sets) and* $R \in \mathcal{R}(X, Y)$*. The relation* R *is equivariant if it is an equivariant subset of* $X \times Y$.

Now, we give some examples of equivariant relations.

Example 2.4. Given nominal sets X and Y,

(i) the relation $\{(x, A) \in X \times \mathcal{P}_{\mathrm{f}}(\mathbb{D}) : \mathrm{supp}_X x \subseteq A\}$, denoted by $\mathrm{inc} : X \longrightarrow \mathcal{P}_{\mathrm{f}}(\mathbb{D})$, is an equivariant element of $X \longrightarrow \mathcal{P}_{\mathrm{f}}(\mathbb{D})$.

(ii) the relation $\{(x, x') \in X \times X : \mathrm{supp}\, x \subseteq \mathrm{supp}\, x'\}$, denoted by $\leqslant : X \longrightarrow X$, defined in [11], is an equivariant relation on X.

(iii) *the support relation* $\{(x, d) \in X \times \mathbb{D} : d \in \mathrm{supp}_X x\}$, denoted by *supp* $: X \longrightarrow \mathbb{D}$, is an equivariant element of $X \longrightarrow \mathbb{D}$.

(iv) the *freshness relation* $\{(x, y) \in X \times Y : \mathrm{supp}\, x \cap \mathrm{supp}\, y = \emptyset\}$, denoted by $\sharp_{X,Y} : X \longrightarrow Y$, defined in [16], is an equivariant relation on $X \times Y$ and it is said that x is *fresh* for y. To simplify, we denote the equivariant relation $\sharp_{X,X} : X \longrightarrow X$ by $\sharp_X$.

Among the various examples in the preceding example, the freshness relation is a significant and useful one [15]. We discuss further conditions for freshness relation in certain circumstances in the next theorem.

Theorem 2.5. *Given non-empty nominal sets* X *and* Y,

(i) *the relation* $\sharp_X \neq \emptyset$ *and* $\mathrm{Dom}\sharp_X = X$.

(ii) *the relation* $\sharp_X$ *is symmetric, that is,* $\sharp_X^{-1} = \sharp_X$.

(iii) *if* $\mathcal{Z}(X) \neq \emptyset$*, then* $\sharp_{X,Y}$ *is a surjective relation.*

(iv) *the relation* $\sharp_{\mathbb{D},X}$ *is surjective.*

(v) *for the nominal set* $\mathrm{Perm}(\mathbb{D})$*, if* $(d, \pi_1), (d, \pi_2) \in \sharp_{\mathbb{D},\mathrm{Perm}(\mathbb{D})}$*, then* $(d, \pi_1 \circ \pi_2) \in \sharp_{\mathbb{D},\mathrm{Perm}(\mathbb{D})}$.

(vi) *the nominal set* X *is discrete if and only if* $\sharp_X$ *is a reflexive relation.*

Proof. (i) If $\mathcal{Z}(X) \neq \emptyset$, then $(\mathcal{Z}(X)\times X)\cup(X\times\mathcal{Z}(X)) \subseteq \sharp_X$. If $\mathcal{Z}(X) = \emptyset$, then for each $x \in X$ with $\operatorname{supp} x \neq \emptyset$ there exists $\pi \in \mathrm{Perm}(\mathbb{D})$ such that $\operatorname{supp} \pi x \cap \operatorname{supp} x = \emptyset$ and hence, $(x, \pi x) \in \sharp_X$.

(ii) This is trivial.

(iii) The zero elements are fresh for every $y \in Y$, so $\sharp_{X,Y}$ is surjective. Furthermore, if $X = \{\theta\}$ is a singleton nominal set, then $\sharp_{\{\theta\},Y}$ is also injective.

(iv) Using the Choose-a-Fresh-Name Principle, there exists $d \in \mathbb{D}$ with $d \notin \operatorname{supp}_X x$, for every $x \in X$, which means $(d, x) \in \sharp_{\mathbb{D},X}$, the result is obtained.

(v) Since $d \notin \operatorname{supp} \pi_1 \cup \operatorname{supp} \pi_2$ and $\operatorname{supp} \pi_1 \circ \pi_2 \subseteq \operatorname{supp} \pi_1 \cup \operatorname{supp} \pi_2$, we get the desired result.

(vi) The relation $\sharp_X$ is reflexive if and only if $(x, x) \in \sharp_X$, for every $x \in X$, if and only if $\operatorname{supp} x = \emptyset$, for every $x \in X$. □

Remark 2.6. (i) *The composition of two binary equivariant relations is an equivariant relation.*

(ii) *Nominal sets (G-sets) and the equivariant relations between them, together with the relational composition and diagonal relations as identities, form a category* **Rel**(**Nom**) (**Rel**-(G-**Set**)).

Here, we are going to study some categorical properties in **Rel**(**Nom**).

Theorem 2.7. *Let X and Y be two G-sets. Then, the set $\mathcal{R}(X, Y)$ with the action $* : G \times \mathcal{R}(X, Y) \longrightarrow \mathcal{R}(X, Y)$ defined by*

$$(g, R) \rightsquigarrow g * R = \{(x, gy) : (g^{-1}x, y) \in R\},$$

is a G-set.

Proof. For every $x \in X$ and $g_1, g_2 \in G$, we have:

$$\begin{aligned}
(x, y) \in (g_1g_2) * R &\iff \exists y' \in Y,\ y = (g_1g_2)y';\ \ ((g_1g_2)^{-1}x, y') \in R\\
&\iff \exists y' \in Y,\ y = (g_1g_2)y';\ ((g_2{}^{-1}g_1{}^{-1})x, y') \in R\\
&\iff \exists y' \in Y,\ y = (g_1g_2)y';\ (g_2{}^{-1}(g_1{}^{-1}x), y') \in R\\
&\iff \exists y' \in Y,\ y = (g_1g_2)y';\ (g_1{}^{-1}x, g_2y') \in g_2 * R\\
&\iff \exists y' \in Y, y = (g_1g_2)y'; (x, g_1(g_2y')) \in g_1 * (g_2 * R)\\
&\iff (x, y) \in g_1 * (g_2 * R).
\end{aligned}$$

So, $(g_1g_2) * R = g_1 * (g_2 * R)$. Also,

$$e * R = \{(x, e\,y) : (e\,x, y) \in R\} = \{(x, y) : (x, y) \in R\} = R.$$

Therefore, $\mathcal{R}(X, Y)$ is a G-set. □

Theorem 2.8. *Let X, Y be two G-set and $R \in \mathcal{R}(X, Y)$. Then, the following statements are equivalent*

(i) *The relation R is equivariant;*

(ii) *The relation R^{-1} is equivariant;*

(iii) *For every $g \in G$ and $y \in \mathrm{Im}R$, $g(\overleftarrow{R}(y)) = \overleftarrow{R}(gy)$;*

(iv) *For every $g \in G$ and $x \in \mathrm{Dom}R$, $g(\overrightarrow{R}(x)) = \overrightarrow{R}(gx)$;*

(v) *For every $g \in G$, $g * R = R$, in which the action "$*$" is defined in* Theorem 2.7*;*

(vi) *The relation R is equivariant; i.e. R is a zero element of $\mathcal{R}(X, Y)$, for the action "$*$" defined in* Theorem 2.7.

Proof. (i)⇒(ii) Given each $(x, y) \in R^{-1}$ and $g \in G$, we have $(y, x) \in R$. Since R is equivariant, $(gy, gx) \in R$ and so $(gx, gy) \in R^{-1}$.

(ii)⇒(iii) Suppose R^{-1} is an equivariant relation and $g \in G$, $y \in \mathrm{Im}R$. Then we have:

$$\begin{aligned} x \in g(\overleftarrow{R}(y)) &\Longleftrightarrow g^{-1}x \in \overleftarrow{R}(y) \\ &\Longleftrightarrow (g^{-1}x, y) \in R \\ &\Longleftrightarrow (y, g^{-1}x) \in R^{-1} \\ &\Longleftrightarrow (gy, x) \in R^{-1} \\ &\Longleftrightarrow (x, gy) \in R \\ &\Longleftrightarrow x \in \overleftarrow{R}(gy). \end{aligned}$$

(iii)⇒(iv) Let $g \in G$ and $x \in \mathrm{Dom}R$. Then

$$\begin{aligned} y \in g(\overrightarrow{R}(x)) &\iff g^{-1}y \in \overrightarrow{R}(x) \\ &\iff (x, g^{-1}y) \in R \\ &\iff x \in \overleftarrow{R}(g^{-1}y) \\ &\iff x \in g^{-1}\overleftarrow{R}(y) \\ &\iff gx \in \overleftarrow{R}(y) \\ &\iff y \in \overrightarrow{R}(gx). \end{aligned}$$

(iv)⇒(v) Let $g \in G$ and $x \in \mathrm{Dom}R$. Then we have:

$$\begin{aligned} (x, y) \in g * R &\iff \exists\, y' \in Y,\ \ y = gy' \ \text{ and } \ (g^{-1}x, y') \in R \\ &\iff \exists\, y' \in Y,\ y = gy' \ \text{ and } \ y' \in \overrightarrow{R}(g^{-1}x) \\ &\iff \exists\, y' \in Y,\ y = gy' \ \text{ and } \ y' \in g^{-1}(\overrightarrow{R}(x)) \\ &\iff \exists\, y' \in Y,\ y = gy' \text{ and } \ gy' \in \overrightarrow{R}(x) \\ &\iff y \in \overrightarrow{R}(x) \\ &\iff \ (x, y) \in R. \end{aligned}$$

(v)⇒(vi) This is trivial.

(vi)⇒(i) Let $g \in G$. Then, using the assumption, we have:

$$\begin{aligned} (x, y) \in R &\iff \ (x, y) \in g * R, \\ &\iff \ \exists y' \in Y,\ y = gy' \text{ and } \ (g^{-1}x, y') \in R \\ &\iff \ \exists y' \in Y,\ y' = g^{-1}y \text{ and } \ y' \in \overrightarrow{R}(g^{-1}x) \\ &\iff \ g^{-1}y \in \overrightarrow{R}(g^{-1}x) \\ &\iff \ (g^{-1}x, g^{-1}y) \in R. \end{aligned}$$

Thus $g * R = R$ implies that the relation R is equivariant. $\square$

Corollary 2.9. (i) *According to* Theorem 2.8(v), *for any nominal sets (or in general G-sets) X and Y, the zero elements of $\mathcal{R}(X,Y)$ (or equivalently, the relations with empty support) are exactly the equivariant relations from X to Y.*

(ii) *For any nominal sets X and Y, $\mathcal{Z}(\mathcal{R}_{\mathrm{fs}}(X,Y)) = \mathcal{Z}(\mathcal{R}(X,Y))$.*

Theorem 2.10. *Let X and Y be G-sets and $R \in \mathcal{R}(X,Y)$. Then, $g{*}R = g \cdot R$, for every $g \in G$.*

Proof. Suppose X and Y are G-sets and $R \in \mathcal{R}(X,Y)$, for every $x \in X$, we have:

$$\begin{aligned} y \in (\overrightarrow{g * R})(x) &\Longleftrightarrow (x,y) \in g * R \\ &\Longleftrightarrow \exists y' \in Y,\ y = gy',\ (g^{-1}x, y') \in R \\ &\Longleftrightarrow \exists y' \in Y,\ y = gy',\ y' \in \overrightarrow{R}(g^{-1}x) \\ &\Longleftrightarrow \exists y' \in Y,\ y' = g^{-1}y,\ g^{-1}y \in \overrightarrow{R}(g^{-1}x) \\ &\Longleftrightarrow y \in g\overrightarrow{R}(g^{-1}x) \\ &\Longleftrightarrow y \in (\overrightarrow{g \cdot R})(x). \end{aligned}$$

So for every $g \in G$, $g * R = g \cdot R$. □

Proposition 2.11. *Suppose X, Y are nominal sets and $R \in \mathcal{R}_{\mathrm{fs}}(X,Y)$. Then*

(i) *if $S \in \mathcal{P}_{\mathrm{fs}}(X)$, then* $\operatorname{supp} \overrightarrow{R}(S) \subseteq \operatorname{supp} R \cup \operatorname{supp} S$.

(ii) *if $S' \in \mathcal{P}_{\mathrm{fs}}(Y)$, then* $\operatorname{supp} \overleftarrow{R}(S') \subseteq \operatorname{supp} R \cup \operatorname{supp} S'$.

Proof. (i) Let $\pi \in \mathrm{Perm}(\mathbb{D})$ with $\pi(d) = d$, for all $d \in \operatorname{supp} R \cup \operatorname{supp} S$. Thus, $\pi * R = R$ and $\pi S = S$. Since $\pi * R = R$, by the definition of "$*$" in Theorem 2.7, $(\pi x, y) \in \pi * R = R$ if and only if $(x, \pi^{-1} y) \in R$. Now, we have:

$$\begin{aligned} y \in \pi(\overrightarrow{R}(S)) &\Longleftrightarrow \pi^{-1}y \in \overrightarrow{R}(S) \\ &\Longleftrightarrow \exists\, x \in S,\ (x, \pi^{-1}y) \in R \\ &\Longleftrightarrow \exists\, x \in S,\ (\pi x, y) \in R \\ &\Longleftrightarrow (x', y) \in R,\ x' = \pi x \in S \\ &\Longleftrightarrow y \in \overrightarrow{R}(S). \end{aligned}$$

(ii) The proof is similar to part (i). □

Corollary 2.12. *Suppose X, Y are nominal sets. If $R \in \mathcal{R}_{\mathrm{fs}}(X,Y)$ and $(x,y) \in R$, then*

(i) $\operatorname{supp} \overrightarrow{R}(x) \subseteq \operatorname{supp} R \cup \operatorname{supp} x$.

(ii) $\operatorname{supp} \overleftarrow{R}(y) \subseteq \operatorname{supp} R \cup \operatorname{supp} y$.

Proof. Let $S = \{x\}$ and $S' = \{y\}$. Then, applying Proposition 2.11, we get the result. □

Corollary 2.13. *Suppose X and Y are nominal sets and $R \in \mathcal{R}(X,Y)$ is equivariant.*

(i) *If $X' \in \mathcal{Z}(\mathcal{P}(X))$ and $Y' \in \mathcal{Z}(\mathcal{P}(Y))$, then $\overrightarrow{R}(X')$ and $\overleftarrow{R}(Y')$ are, respectively, equivariant subsets of Y and X.*

(ii) $\operatorname{supp} \overrightarrow{R}(x) \subseteq \operatorname{supp} x$.

(iii) $\operatorname{supp} \overleftarrow{R}(y) \subseteq \operatorname{supp} y$.

Proof. (i) Since $X' \in \mathcal{Z}(\mathcal{P}(X))$ and $R \in \mathcal{R}(X,Y)$ is an equivariant relation, $\operatorname{supp} X' = \operatorname{supp} R = \emptyset$. Now, applying Proposition 2.11(i), we get $\operatorname{supp} \overrightarrow{R}(X') = \emptyset$. Analogously, one can prove that $\overleftarrow{R}(Y')$ is an equivariant subset of Y.

(ii) Since R is equivariant, $\operatorname{supp} R = \emptyset$. Now, applying Corollary 2.12(i), $\operatorname{supp} \overrightarrow{R}(x) \subseteq \operatorname{supp} x$.

(iii) The proof is similar to (ii). □

Lemma 2.14. *Given two nominal sets X, Y and an injective relation $R \in \mathcal{R}_{\mathrm{fs}}(X,Y)$ and $(x,y) \in R$, the following assertions hold:*

(i) $\operatorname{supp} x \subseteq \operatorname{supp} \overrightarrow{R}(x) \cup \operatorname{supp} R$.

(ii) $\operatorname{supp} x \subseteq \operatorname{supp} y \cup \operatorname{supp} R$.

(iii) $\operatorname{supp} x = \operatorname{supp} \overleftarrow{R}(y)$.

Proof. (i) Let $d, d' \notin \operatorname{supp} \overrightarrow{R}(x) \cup \operatorname{supp} R$. Then $(d\ d')\overrightarrow{R}(x) = \overrightarrow{R}(x)$ and $R = (d\ d') \cdot R$. By Remark 2.2(i), we have $\overrightarrow{R}(x) = \overrightarrow{((d\ d') \cdot R)}(x) = (d\,d')(\overrightarrow{R}(d\,d')x)$. Thus, $(d\,d')\overrightarrow{R}(x) = \overrightarrow{R}(d\,d')x$ and so $\overrightarrow{R}(x) = \overrightarrow{R}(d\,d')x$. Now, since R is injective, $(d\,d')x = x$. So, by Remark 1.3 (i), $\operatorname{supp} \overrightarrow{R}(x) \cup \operatorname{supp} R$ supports x.

(ii) Let $d, d' \notin \operatorname{supp} y \cup \operatorname{supp} R$. Then, $(d\ d')y = y$ and $R = (d\ d') \cdot R$. By Remark 2.2(i), we have $(d\ d')\overrightarrow{R}(x) = \overrightarrow{R}(d\ d')x$. Thus, $y \in \overrightarrow{R}(x)$ implies that $y = (d\ d')y \in (d\ d')\overrightarrow{R}(x) = \overrightarrow{R}(d\ d')x$. So, $(x, y), ((d\ d')x, y) \in R$. Now, since R is injective, $(d\ d')x = x$. So, by Remark 1.3(i), $\operatorname{supp} y \cup \operatorname{supp} R$ supports x.

(iii) Since R is injective and $x \in \overleftarrow{R}(y)$, $\overleftarrow{R}(y) = \{x\}$. Thus, $\operatorname{supp} x = \operatorname{supp} \overleftarrow{R}(y)$. □

Corollary 2.15. *Let X, Y be two nominal sets, $R \in \mathcal{R}(X, Y)$ be an equivariant injective relation, and $(x, y) \in R$. Then,*

(i) $\operatorname{supp} x = \operatorname{supp} \overrightarrow{R}(x)$.

(ii) $\operatorname{supp} x \subseteq \operatorname{supp} y$.

Proof. Notice that, since R is an equivariant relation, $\operatorname{supp} R = \emptyset$. So,

(i) applying Lemma 2.14(i), we have $\operatorname{supp} x \subseteq \operatorname{supp} \overrightarrow{R}(x)$ and Corollary 2.13(ii) implies that $\operatorname{supp} \overrightarrow{R}(x) \subseteq \operatorname{supp} x$. Thus $\operatorname{supp} \overrightarrow{R}(x) = \operatorname{supp} x$.

(ii) the result follows from Lemma 2.14(ii). □

Corollary 2.16. *Let $R \in \mathcal{R}(X, \mathbb{D})$ be an equivariant injective relation and $(x, d) \in R$. Then, $\operatorname{supp} x \subseteq \{d\}$ and so $x \in \mathcal{Z}(X)$ or $\operatorname{supp} x = \{d\}$.*

The following example shows that the converse of Corollary 2.15(ii) is not true in general.

Example 2.17. Suppose $R \in \mathcal{R}(\mathbb{D}, \mathbb{D}^{(2)})$ is a relation defined by

$$R = \{(d, (x, y)) : d = x \vee d = y,\ x \neq y\}.$$

For all $(d, (x, y)) \in R$, we have $\{d\} = \operatorname{supp} d \subseteq \{x, y\} = \operatorname{supp}(x, y)$, but R is not injective, since $(d, (d, d')), (d', (d, d')) \in R$.

Proposition 2.18. *Let X be a nominal set. Then,*

(i) *the relation $\leqslant$, given in Example 2.4(ii), is a reflexive and transitive relation on X.*

(ii) *if R is a non-empty injective equivariant relation on X, then $R \subseteq\ \leqslant$.*

(iii) *the set $\mathcal{R}^e_{inj}$ of all injective equivariant relations on X is a nominal set and $\leqslant$ is an upper bound for $\mathcal{R}^e_{inj}$.*

Proof. (i) This is Clear.

(ii) Let $(a,b) \in R$. Since R is injective, applying Corollary 2.15(ii), $\operatorname{supp} a \subseteq \operatorname{supp} b$. So, $R \subseteq \leqslant$.

(iii) Let $\pi \in \mathrm{Perm}(\mathbb{D})$ and $R \in \mathcal{R}^{e}_{inj}$ with $(x,y),(x',y) \in \pi * R$. Since R is equivariant, $(\pi^{-1}x, \pi^{-1}y), (\pi^{-1}x', \pi^{-1}y) \in R$. Now, since R is injective, $\pi^{-1}x = \pi^{-1}x'$ and so $x = x'$. Also, $\leqslant$ is an upper bound for $\mathcal{R}^{e}_{inj}$ by (ii). □

Remark 2.19. (i) *The only equivariant injective relation on* $\mathbb{D}$ *is* $\Delta_{\mathbb{D}}$*. To examine, let* R *be an injective equivariant relation on* $\mathbb{D}$ *and* $(d,d') \in R$*. Then,* $\operatorname{supp} d \subseteq \operatorname{supp} d'$ *and so* $d = d'$*.*

(ii) *Let a relation* $R \in \mathcal{R}(\mathbb{D}, \mathbb{D}^{(2)})$ *be equivariant and injective, and* $(x,y) \in R$*. Then, there exist* $a, b \in \mathbb{D}$ *where* $a \neq b$ *such that either* $x = a$ *or* $x = b$ *and* $y = (a,b)$ *with* $(a,(a,b))$*,* $(b,(a,b)) \notin R$*.*

Proposition 2.20. *Suppose* X *is a nominal set and* $R \in \mathcal{R}(X,X)$ *is an equivariant injective relation and* $(x,y) \in R$*. If* R *is a symmetric relation, then*

(i) $\operatorname{supp} x = \operatorname{supp} y$*.*

(ii) $\operatorname{supp} \overrightarrow{R}(x) = \operatorname{supp} y$*.*

Proof. (i) Suppose $(x,y) \in R$. Since R is symmetric, $(y,x) \in R$. By Corollary 2.15(ii) and injectivity of R, we get $\operatorname{supp} x = \operatorname{supp} y$.

(ii) The proof follows from part (i) and Corollary 2.15(i). □

According to [16, Lemma 2.12 (iii)], if $f : X \to Y$ is a surjective equivariant map, in which X is a nominal set and Y is a $\mathrm{Perm}(\mathbb{D})$-set, then Y is a nominal set. The next example demonstrates that this is not the case when f an equivariant relation.

Example 2.21. Let $R : \mathbb{D} \longrightarrow [\mathcal{P}(\mathbb{D}) \setminus \emptyset]$ be a relation defined by $R = \{(d,A) : d \in A\}$. Then, R is surjective and equivariant. Notice that, by Remark 1.5(iii), $\mathcal{P}(\mathbb{D})$ is not a nominal set.

Proposition 2.22. *Let* X *and* Y *be two nominal sets and* $R \in \mathcal{R}(X,Y)$ *be equivariant. Then,*

(i) *the map* $\overrightarrow{R} : \mathcal{P}_{\mathrm{fs}}(X) \to \mathcal{P}_{\mathrm{fs}}(Y)$ *mapping each* $A \in \mathcal{P}_{\mathrm{fs}}(X)$ *to* $\overrightarrow{R}(A)$ *is equivariant.*

(ii) *the map* $\overleftarrow{R} : \mathcal{P}_{\rm fs}(Y) \to \mathcal{P}_{\rm fs}(X)$ *defined by* $\overleftarrow{R}(B) = \{x \in X : \exists y \in B; (x, y) \in R\}$*, for each* $B \in \mathcal{P}_{\rm fs}(Y)$*, is equivariant.*

Proof. (i) Let $\pi \in \mathrm{Perm}(\mathbb{D})$. We show that $\pi\overrightarrow{R}(A) = \overrightarrow{R}(\pi A)$. To this end, let $\pi y \in \pi\overrightarrow{R}(A)$. Then, there exists $x' \in A$ with $(x', y) \in R$. Since R is equivariant, $(\pi x', \pi y) \in R$. Thus, $\pi x' \in \pi A$, and so, $\pi y \in \overrightarrow{R}(\pi A)$. Similarly, one can show that $\overrightarrow{R}(\pi A) \subseteq \pi\overrightarrow{R}(A)$.

(ii) The proof is similar to part (i). □

Lemma 2.23. *Let X and Y be two G-sets and $R \in \mathcal{R}(X, Y)$ be equivariant. Then,*

(i) *if R is a partial surjective map and X is indecomposable, then $\overrightarrow{R}(X)$ is indecomposable.*

(ii) *if R is non-empty and Y is indecomposable, then R is surjective.*

Proof. (i) On the contrary, suppose there exist disjoint equivariant subsets A and B of Y with $\overrightarrow{R}(X) = A \cup B$. Since R is a partial surjective map, $\overleftarrow{R}(A)$ and $\overleftarrow{R}(B)$ are non-empty equivariant subsets of X. Since X is indecomposable, by Lemma 1.9, $X = \overleftarrow{R}(A) = \overleftarrow{R}(B)$ which is a contradiction. This is because, $\overleftarrow{R}(A) \cap \overleftarrow{R}(B) = \overleftarrow{R}(A \cap B) = \emptyset$, by Remark 1.13.

(ii) By Lemma 1.9, there exists $y \in Y$ with $Y = Gy$. Let $t \in Y$ and $(x, z) \in R$. Then, there exist $g_1, g_2 \in G$ with $t = g_1 y$ and $z = g_2 y$. Now, since R is equivariant, $(g_1 g_2^{-1} x, t) = (g_1 g_2^{-1} x, g_1 y) = (g_1 g_2^{-1} x, g_1 g_2^{-1} z) \in R$; meaning that R is surjective. □

Proposition 2.24. *Let X, Y be nominal sets and $R \in \mathcal{R}(X, Y)$ be equivariant. If R is epic, then R is surjective.*

Proof. Notice that, if Y is indecomposable, then R is surjective, by Lemma 2.23(ii). So, suppose Y is decomposable and R is epic. If R is not surjective, then there exists $y \in Y$, but $y \notin \mathrm{Im}R$. Take $Z = \{\theta_1, \theta_2\}$ be a discrete nominal set and $R_1 : \mathrm{Im}R \longrightarrow \{\theta_1\}$ where $R_1 = \mathrm{Im}R \times \{\theta_1\}$ and $R_2 : \mathrm{Im}R \longrightarrow \{\theta_1, \theta_2\}$ where $R_2 = (\mathrm{Im}R \times \{\theta_1\}) \cup (\mathrm{Perm}(\mathbb{D})y \times \{\theta_2\})$. Notice that, since $\mathrm{Im}R$ is an equivariant subset of Y, R_1 and R_2 are equivariant relations. For given $x \in \mathrm{Dom}R$, we have $\overrightarrow{R}(x) \subseteq \mathrm{ImR}$ and so $R_1 \circ R = R_2 \circ R$. Since R is epic, we obtain $R_1 = R_2$, which is a contradiction. $\square$

Now, we show that the converse of Proposition 2.24 is not true in general.

Example 2.25. Suppose $\mathbb{D}$ is a nominal set. Consider the equivariant relations $\sharp_{\mathbb{D}} = \{(d, d') : d \neq d'\} = \mathbb{D}^{(2)}$ and $R = \mathbb{D}^2 = \mathbb{D} \times \mathbb{D}$. By Theorem 2.5(iv), $\sharp_{\mathbb{D}}$ is surjective. We have $R \circ \sharp_{\mathbb{D}} = R = \sharp_{\mathbb{D}} \circ \sharp_{\mathbb{D}}$, but $R \neq \sharp_{\mathbb{D}}$ meaning that $\sharp_{\mathbb{D}}$ is not epic.

Note 2.26. (i) If a relation R has a right inverse, then R is surjective. This is because morphisms having a right inverse are epimorphisms and so by Proposition 2.24, we get the result.

(ii) Example 2.25 also shows that the converse of (ii) does not hold. Indeed, if S is an equivariant relation on $\mathbb{D}$ with $\sharp_{\mathbb{D}} \circ S = id_{\mathbb{D}}$, since $(d, d) \in \sharp_{\mathbb{D}} \circ S$ and $\sharp_{\mathbb{D}}$ is surjective, there exists $d' \neq d$ with $(d', d) \in \sharp_{\mathbb{D}}$ and $(d, d') \in S$. For given $d'' \neq d', d$, we have $(d', d'') \in \sharp_{\mathbb{D}}$. Thus $(d, d'') \in \sharp_{\mathbb{D}} \circ S = id_{\mathbb{D}}$ meaning that $d = d''$, which is a contradiction.

By the same scheme of [9] but different in details we have the following theorem.

Theorem 2.27. *Let X, Y be nominal sets and $R \in \mathcal{R}(X, Y)$ be equivariant. Then, the following statements are equivalent.*

(i) *R is monic and* $\mathrm{Dom}R = X$.

(ii) *The map $\overrightarrow{R} : \mathcal{P}_{\mathrm{fs}}(X) \to \mathcal{P}_{\mathrm{fs}}(Y)$ is injective and* $\mathrm{Dom}R = X$.

(iii) *R is total injective.*

Proof. (i⇒ii) Suppose $\overrightarrow{R}(U) = \overrightarrow{R}(V)$ for some $U, V \in \mathcal{P}_{\mathrm{fs}}(X)$. We show $U = V$. For this, we consider finitely supported relations $S = \{(*, u) : u \in U\}$ and $T = \{(*, v) : v \in V\}$. Then, clearly, $R \circ S = R \circ T$. Hence $S = T$, since R is monic. Thus $U = V$.

(ii⇒iii) Notice that, $X - \{x\} \in \mathcal{P}_{\mathrm{fs}}(X)$, for every $x \in X$. To do so, let C be a finite support of x and $d_1, d_2 \notin C$. Then, $(d_1\ d_2)x = x$. Now, for all $y \in X - \{x\}$, we have $x = (d_1\ d_2)x \neq (d_1\ d_2)y \in X - \{x\}$ meaning that C is a finite support for $X - \{x\}$. Since $\overrightarrow{R}$ is injective and $X \neq X - \{x\}$, we have $\overrightarrow{R}(X) \neq \overrightarrow{R}(X - \{x\})$. For given $x \in X$, since $\mathrm{Dom}R = X$, there exists $y \in Y$ with $(x, y) \in R$. Now, if there exists $x'' \in X - \{x\}$ with $(x'', y) \in R$, then $y \in \overrightarrow{R}(X - \{x\})$, which is a contradiction.

(iii⇒i) First, notice that since R is total, $\mathrm{Dom}R = X$. Now, suppose $R_1, R_2 \in \mathcal{R}(Z, X)$ are equivariant with $R \circ R_1 = R \circ R_2$. Let $(z, x) \in R_1$. Then, by the assumption, there exists $y \in Y$ with $(x, y) \in R$. Thus $(z, y) \in R \circ R_1 = R \circ R_2$ and so there exists $x' \in X$ with $(z, x') \in R_2$ and $(x', y) \in R$. Since R is injective and $(x, y), (x', y) \in R$, $x = x'$. So, $(z, x) = (z, x') \in R_2$ which implies that $R_1 \subseteq R_2$. Analogously, $R_2 \subseteq R_1$, and we are done. □

Lemma 2.28. *Let X, Y be nominal sets and $R \in \mathcal{R}(X, Y)$ be equivariant. Then, the following statements are equivalent.*

(i) *The map $\overleftarrow{R} : \mathcal{P}_{\mathrm{fs}}(Y) \to \mathcal{P}_{\mathrm{fs}}(X)$ is injective and* $\mathrm{Im}R = Y$.
(ii) *R is partial surjective.*

Proof. The proof is similar to that of Theorem 2.27. □

Remark 2.29. *If R is a partial surjective map, then similarly to the proof of (iii⇒i) in Theorem 2.27, it is proved that R is epic, but the converse is not true (see Example 2.31*(ii)*).*

Corollary 2.30. *Suppose $R \in \mathcal{R}(X, Y)$ is a partial surjective equivariant map between nominal sets X and Y. Let $(x, y) \in R$. Then,*

(i) $\mathrm{supp}\, \overleftarrow{R}(y) = \mathrm{supp}\, y$.
(ii) $\mathrm{supp}\, y \subseteq \mathrm{supp}\, x$.

Proof. (i) By Lemma 2.28, $\overleftarrow{R}$ is an injective equivariant map. By Corollary 2.15(i), we get the result.

(ii) Similar to the proof of Corollary 2.15(ii). □

Example 2.31. (i) The relation $\sharp_{\mathbb{D}} = \{(d, d') : d \neq d'\}$ is a surjective equivariant relation on $\mathbb{D}$, but the map $\overleftarrow{\sharp_{\mathbb{D}}} : \mathcal{P}_{\mathrm{fs}}(\mathbb{D}) \to \mathcal{P}_{\mathrm{fs}}(\mathbb{D})$ is not injective. To do so, let $A, B \in \mathcal{P}_{\mathrm{fs}}(\mathbb{D})$. Then A and B are finite or cofinite, by Remark 1.5(iii). Take $A = \{d_1, d_2\}$ and $B = \{d_3, d_4\}$. Now, $\overleftarrow{\sharp_{\mathbb{D}}}(A) = \{d \in \mathbb{D} : (\exists a \in A)\, a \sharp_{\mathbb{D}} d\} = \mathbb{D}$ and similarly $\overleftarrow{\sharp_{\mathbb{D}}}(B) = \mathbb{D}$. So $\overleftarrow{\sharp_{\mathbb{D}}}(A) = \overleftarrow{\sharp_{\mathbb{D}}}(B)$, but $A \neq B$.

(ii) Let $R \in \mathcal{R}(\mathbb{D}, \mathbb{D}^{(2)})$ be equivariant defined by $R = \{(d, (d, d')) : d \neq d'\}$. Then, R is surjective but not a partial map. This is because, $(d, (d, d')), (d, (d, d'')) \in R$ for $d'' \neq d'$. On the other hand, $S = \{((d, d'), d) : d \neq d'\}$ is an equivariant relation and $R \circ S = id_{\mathbb{D}^{(2)}}$, meaning that R is epic. So, epic and partial surjective maps are not equivalent.

3 Sheaf Representation Of Nominal Sets In Rel(Nom)

The sheaf representation of nominal sets provides a more general and abstract setting, which can be useful in various areas of mathematics and computer science, such as the study of programming languages with binding constructs. Recall from [16, Theorem 6.8] that the category **Nom** can be considered as a sheaf-subcategory of $\mathbf{Set}^{\mathcal{P}_{\mathrm{f}}(\mathbb{D})}$, by the adjunction $I_* : \mathbf{Nom} \longrightarrow \mathbf{Set}^{\mathcal{P}_{\mathrm{f}}(\mathbb{D})}$ and $I^* : \mathbf{Set}^{\mathcal{P}_{\mathrm{f}}(\mathbb{D})} \longrightarrow \mathbf{Nom}$ in which $I^* \dashv I_*$ and $\mathbb{D}$ is the set of atomic names. Hence, **Nom** is a topos. In this section, although the category **Rel**(**Nom**) is not a topos (see Remark 3.5) we are going to examine the counterpart of functors I^* and I_*, denoted by $\mathcal{P}^*_{\mathrm{fs}}$ and $\mathcal{P}_{\mathrm{fs}*}$ respectively, for their advantages.

Lemma 3.1. (i) *There is an obvious inclusion functor* $I : \mathbf{Nom} \hookrightarrow \mathbf{Rel}(\mathbf{Nom})$ *that is identity on objects and takes an equivariant map* $f : X \to Y$ *to its underling relation.*
(ii) *The inclusion functor* I *is a left adjoint to the functor* $\mathcal{P}_{\mathrm{fs}} : \mathbf{Rel}(\mathbf{Nom}) \longrightarrow \mathbf{Nom}$ *that takes every object* $X \in \mathbf{Rel}(\mathbf{Nom})$ *to* $\mathcal{P}_{\mathrm{fs}}(X)$ *and every equivqrint relation* $R : X \to Y$ *maps to* $\overrightarrow{R} : \mathcal{P}_{\mathrm{fs}}(X) \to \mathcal{P}_{\mathrm{fs}}(Y)$. *That is, we have:*

$$\mathcal{P}_{\mathrm{fs}} : \mathbf{Rel}(\mathbf{Nom}) \longrightarrow \mathbf{Nom}$$

$$\begin{array}{ccc} X & \rightsquigarrow & \mathcal{P}_{\mathrm{fs}}(X) \\ {\scriptstyle R}\downarrow & \rightsquigarrow & \downarrow{\scriptstyle \overrightarrow{R}} \\ Y & \rightsquigarrow & \mathcal{P}_{\mathrm{fs}}(Y) \end{array}$$

Proof. (i) This is Clear.

(ii) First, it should be noted that $\mathcal{P}_{\mathrm{fs}}(X)$, for any nominal set X, is a nominal set according to Remark 1.5(i) and Lemma 1.4, and that $\overrightarrow{R}$ is an equivariant map according to Proposition 2.22(i). As a result, by a simple verification, $\mathcal{P}_{\mathrm{fs}}$ is a functor. Now, to prove $I \dashv \mathcal{P}_{\mathrm{fs}}$, we show that $\eta_X : X \to \mathcal{P}_{\mathrm{fs}}(I(X))$ defined by $\eta_X(x) = \{x\}$ is a universal $\mathcal{P}_{\mathrm{fs}}$-arrow and $\eta = (\eta_X)_{X \in \mathbf{Nom}}$ is a natural transformation. Indeed, for every equivariant map $f : X \to \mathcal{P}_{\mathrm{fs}}(Y)$, in which $Y \in \mathbf{Rel}(\mathbf{Nom})$, we define the relation $R_f = \{(x, y) \; : \; y \in f(x)\} \in \mathcal{R}(I(X), Y)$. Since f is equivariant, so is R_f. We also have $\overrightarrow{R_f} \circ \eta_X(x) = \overrightarrow{R_f}(\{x\}) = f(x)$. That is, the following triangle is commutative:

$$
\begin{array}{ccc}
X & \xrightarrow{\eta_X} & \mathcal{P}_{\mathrm{fs}}(I(X)) \\
& {\scriptstyle f}\searrow & \Big\downarrow {\scriptstyle \overrightarrow{R_f} = \mathcal{P}_{\mathrm{fs}}(R_f)} \\
& & \mathcal{P}_{\mathrm{fs}}(Y)
\end{array}
\qquad\qquad
\begin{array}{c}
I(X) \\
\Big\downarrow {\scriptstyle \exists ! R_f} \\
Y
\end{array}
$$

The uniqueness of R_f with $\overrightarrow{R_f} \circ \eta_X = f$ follows from its definition and the naturality of η can be easily checked. $\square$

Remark 3.2. *By Theorem 6.8 of [16], the composition functor* $\mathcal{P}_{\mathrm{fs}*} : \mathbf{Rel}(\mathbf{Nom}) \xrightarrow{\mathcal{P}_{\mathrm{fs}}} \mathbf{Nom} \xrightarrow{I_*} \mathbf{Set}^{\mathcal{P}_{\mathrm{f}}(\mathbb{D})}$*, defined by*

$$
\mathcal{P}_{\mathrm{fs}*} : \mathbf{Rel}(\mathbf{Nom}) \longrightarrow \mathbf{Set}^{\mathcal{P}_{\mathrm{f}}(\mathbb{D})}
$$

$$
\begin{array}{ccc}
X & \rightsquigarrow & \mathcal{P}_{\mathrm{fs}*}X \\
{\scriptstyle R}\Big\downarrow & \rightsquigarrow & \Big\downarrow{\scriptstyle R_*} \\
Y & \rightsquigarrow & \mathcal{P}_{\mathrm{fs}*}Y
\end{array}
$$

in which $\mathcal{P}_{\mathrm{fs}*}X : \mathcal{P}_{\mathrm{f}}(\mathbb{D}) \to \mathbf{Set}$ *mapping each* $A \in \mathcal{P}_{\mathrm{f}}(\mathbb{D})$ *to the set* $\{k \in \mathcal{P}_{\mathrm{fs}}X \; : \; \mathrm{supp} k \subseteq A\}$ *and each equivariant relation* $R : X \to Y$ *to the the natural transformation* $R_* = \{R_{*A}\}_{A \in \mathcal{P}_{\mathrm{f}}(\mathbb{D})}$ *defined by*

$$
R_{*A} = \overrightarrow{R} : \mathcal{P}_{\mathrm{fs}*}X(A) \to \mathcal{P}_{\mathrm{fs}*}Y(A), \; \textit{for every } A \in \mathcal{P}_{\mathrm{f}}(\mathbb{D}),
$$

assigns every object in $\mathbf{Rel}(\mathbf{Nom})$ *to a sheaf.*

Theorem 3.3. *The composion functor* $\mathcal{P}_{\mathrm{fs}*} : \mathbf{Rel}(\mathbf{Nom}) \to \mathbf{Set}^{\mathcal{P}_{\mathrm{f}}(\mathbb{D})}$ *has a left adjoint, which is denoted by* ${\mathcal{P}_{\mathrm{fs}}}^*$*.*

Proof. First we note that since $\mathcal{P}_{\mathrm{f}}(\mathbb{D})$ is an up-directed set, the image of every functor F is up-directed, for every presheaf $F \in \mathbf{Set}^{\mathcal{P}_{\mathrm{f}}(\mathbb{D})}$. Now we consider the assignment

$$\mathcal{P}_{\mathrm{fs}}{}^{*} : \mathbf{Set}^{\mathcal{P}_{\mathrm{f}}(\mathbb{D})} \longrightarrow \mathbf{Rel}(\mathbf{Nom})$$

$$\begin{array}{ccc} F & \rightsquigarrow & \overrightarrow{\lim}_{A \in \mathcal{P}_{\mathrm{f}}(\mathbb{D})} FA \\ \tau \big\downarrow & \rightsquigarrow & \big\downarrow \tau^{*} \\ G & \rightsquigarrow & \overrightarrow{\lim}_{A \in \mathcal{P}_{\mathrm{f}}(\mathbb{D})} GA \end{array}$$

in which $\overrightarrow{\lim}_{A \in \mathcal{P}_{\mathrm{f}}(\mathbb{D})} FA$ is direct limit (or directed colimit) of the diagram $\{FA\}_{A \in \mathcal{P}_{\mathrm{f}}(\mathbb{D})}$ which is the quotient $\bigcup_{A \in \mathcal{P}_{\mathrm{f}}(\mathbb{D})} FA/\!\sim$ (see [18]) and the relation τ^{*} is defined by

$$(x/\!\sim, y/\!\sim) \in \tau^{*} \Leftrightarrow \ \tau_{A}(x) = y$$

in which x maps to $x/\!\sim$ by the colimit injection. It is worth noting that $\overrightarrow{\lim}_{A \in \mathcal{P}_{\mathrm{f}}(\mathbb{D})} FA$ together with the action $\cdot : \mathrm{Perm}(\mathbb{D}) \times \overrightarrow{\lim}_{A \in \mathcal{P}_{\mathrm{f}}(\mathbb{D})} FA \to \overrightarrow{\lim}_{A \in \mathcal{P}_{\mathrm{f}}(\mathbb{D})} FA$ mapping each $(\pi, x/\!\sim)$ to $F(\pi|_{A})(x)/\!\sim$ is a nominal set. See [16, lemma 6.7]. Also naturality of τ indicates that τ^{*} is well-defined and functoriality of F implies that τ^{*} is equivariant. To prove that $\mathcal{P}_{\mathrm{fs}}{}^{*}$ is a left adjoint to $\mathcal{P}_{\mathrm{fs}*}$, we give the natural transformation $\eta_{F} : F \to \mathcal{P}_{\mathrm{fs}*}\mathcal{P}_{\mathrm{fs}}{}^{*}(F)$ to be $\eta_{F\,B} : FB \to \mathcal{P}_{\mathrm{fs}*}\overrightarrow{\lim}_{A \in \mathcal{P}_{\mathrm{f}}(\mathbb{D})} FA(B)$, mapping each $x \in FB$ to $\{x/\!\sim\}$, in each level $B \in \mathcal{P}_{\mathrm{f}}(\mathbb{D})$, for every functor $F \in \mathbf{Set}^{\mathcal{P}_{\mathrm{f}}(\mathbb{D})}$. Notice that, by the definition of action of $\overrightarrow{\lim}_{A \in \mathcal{P}_{\mathrm{f}}(\mathbb{D})} FA$, $\mathrm{supp}\{x/\!\sim\} \subseteq B$, for every $x \in FB$. Indeed, if $\pi|_{B} = id_{B}$, then $F(\pi|_{B}) = id_{FB}$ and hence $F(\pi|_{B})(x) = x$. Also, for every $x \in FB$ and for the inclusion function $i : B \hookrightarrow C$ we have:

$$\begin{aligned} \mathcal{P}_{\mathrm{fs}*}(i)\eta_{F\,B}(x) &= \overrightarrow{Fi}(\{x/\!\sim\}) \\ &= \{Fi(x)/\!\sim\} \\ &= \eta_{F\,C}(Fi(x)). \end{aligned}$$

This indicates the naturality of η_{F}. Now we show that η_{F} is a universal arrow, for every functor $F \in \mathbf{Set}^{\mathcal{P}_{\mathrm{f}}(\mathbb{D})}$. To do so, let $\iota : F \to \mathcal{P}_{\mathrm{fs}*}Y$ be a natural transformation, for some $y \in \mathbf{Rel}(\mathbf{Nom})$. Then there exists

$$\bar{\iota} := \{(x/\sim, y) \ : \ \iota_A(x) = y\}$$

in which $x \in FA$ maps to $x/\sim$ by the colimit injection. Naturality of ι implies that $\bar{\iota}$ is well-defined and functoriality of F implies that $\bar{\iota}$ is equivariant. Also we have

$$\begin{aligned} \mathcal{P}_{\mathrm{fs}*}(\bar{\iota})_A \circ \eta_{FB}(x) &= \mathcal{P}_{\mathrm{fs}*}(\bar{\iota})_A(\{x/\sim\}) \\ &= \overrightarrow{\bar{\iota}}(\{x/\sim\}) \\ &= y \\ &= \iota_A(x). \end{aligned}$$

One can easily check the uniqueness of $\bar{\iota}$ with $\mathcal{P}_{\mathrm{fs}*}(\bar{\iota})_A \circ \eta_{FB}(x) = \iota_A(x)$. □

The following example shows that the functor $\mathcal{P}_{\mathrm{fs}*} : \mathbf{Rel}(\mathbf{Nom}) \to \mathbf{Set}^{\mathcal{P}_{\mathrm{f}}(\mathbb{D})}$ is not faithful.

Example 3.4. Suppose $R, R' \in \mathcal{R}(\mathbb{D}, \mathbb{D}^{(2)})$ with $R = \{(d, (d, d')) : d \neq d'\}$ and $R' = \{(d', (d, d')) : d \neq d'\}$. It is clear that $R, R' \in \mathbf{Rel}(\mathbf{Nom})$ and $R_{*A} = R'_{*A} = \mathbb{D}^{(2)}$, for every $A \in \mathcal{P}_{\mathrm{f}}(\mathbb{D})$, but $R \neq R'$. So the functor $\mathcal{P}_{\mathrm{fs}*}$ is not faithful.

Remark 3.5. *It is worth noting that $\emptyset$ is a zero object in* $\mathbf{Rel}(\mathbf{Nom})$*, that is, both initial and terminal. Now, since the only toposes with a zero object are ones equivalent to the trivial, that is one-object-one-morphism, and the category* $\mathbf{Rel}(\mathbf{Nom})$ *is patently not equivalent to that,* $\mathbf{Rel}(\mathbf{Nom})$ *is not a topos.*

4 Natural Deterministic Morphisms In Rel(Nom)

In the category $\mathbf{Rel}(\mathbf{Nom})$ there are several types of morphisms, each with their advantages. Each of these types can be used in combination to gain a deeper understanding of the underlying objects. This section is devoted to an important kind of these morphisms which is called natural deterministic morphism.

Notation 4.1. Let X be a nominal set, and $A \in \mathcal{P}_{\text{fs}}(X)$. The notation $\mathcal{R}_{\text{fs}}(A, B)$, in this section, refers to the set of all finitely supported relations from X to A.

Definition 4.2. *Let X and Y be two nominal sets. A pair (R, σ) is called a natural deterministic morphism if $R \in \mathcal{R}_{\text{fs}}(X, Y)$ and $\sigma : \epsilon_Y \circ \overrightarrow{R} \to \epsilon_X$ is a natural transformation, in which, for every nominal set X, ϵ_X is a functor from a subcategory $\mathcal{T}(X)$ of $(\mathcal{P}_{\text{fs}}(X), \subseteq)$ to the category* **Nom** *defined by the following diagram, for every $A, B \in \mathcal{T}(X)$.*

$$\begin{array}{ccc} A & \rightsquigarrow & \epsilon_X(A) = \mathcal{R}_{\text{fs}}(X, A) \\ \big\downarrow & & \big\downarrow \\ B & \rightsquigarrow & \epsilon_X(B) = \mathcal{R}_{\text{fs}}(X, B) \end{array}$$

Note 4.3. In this paper, we either assume $\mathcal{T}(X) = (\mathcal{P}_{\text{fs}}(X), \subseteq)$ and consider the functor $\epsilon_X : \mathcal{P}_{\text{fs}}(X) \to \textbf{Nom}$, or suppose $\mathcal{T}(X)$ to be the set of all equivariant subsets of X, denoted by $\text{Eqsub}(X)$. In the latter case we denote ϵ_X by ϵ_X^{eq} for emphasis.

Remark 4.4. (i) *By Proposition 2.22, every finitely supported relation $R : X \longrightarrow Y$ induces two functors*

$$\overrightarrow{R} : (\mathcal{P}_{\text{fs}}(X), \subseteq) \longrightarrow (\mathcal{P}_{\text{fs}}(Y), \subseteq), \quad \overleftarrow{R} : (\mathcal{P}_{\text{fs}}(Y), \subseteq) \longrightarrow (\mathcal{P}_{\text{fs}}(X), \subseteq).$$

(ii) *For every $S \in \text{Eqsub}(X)$ and $\rho \in \mathcal{R}_{\text{fs}}(X, S)$, since* $\text{supp}\,(X, S) = \text{supp}\, X \cup \text{supp}\, S$ *and* $\text{supp}\, X = \text{supp}\, S = \emptyset$, *empty set supports ρ and we have $\pi\rho = \rho$, for every $\pi \in \text{Perm}(\mathbb{D})$.*

Proposition 4.5. *Each equivariant $R \in \mathcal{R}(X, Y)$ determines a natural deterministic morphism.*

Proof. Define the natural transformation $((\sigma_R)_S)_{S \in \mathcal{P}_{\text{fs}}(X)}$, in which $(\sigma_R)_S : \epsilon_Y(\overrightarrow{R}(S)) \to \epsilon_X(S)$ assigns every $\rho \in \mathcal{R}_{\text{fs}}(Y, \overrightarrow{R}(S))$ to $(\sigma_R)_S(\rho)$, defined by

$$(x, s) \in (\sigma_R)_S(\rho) \Leftrightarrow \text{ there exist } y_1 \in Y, y_2 \in \overrightarrow{R}(S) \text{ such that } \begin{array}{ccc} & s & \\ & & \searrow^{R} \\ x & & y_2. \\ {}_{R}\searrow & & \nearrow_{\rho} \\ & y_1 & \end{array}$$

It is clear that $(\sigma_R)_S$'s are maps. The naturality of σ_R is also easily obtained. □

Proposition 4.6. *Let X be a nominal set and $R \in \mathcal{R}(X,X)$ be an equivariant injective relation. Then, $(\sigma_R)_X(\sharp_X) \subseteq \sharp_X$.*

Proof. Applying Proposition 4.5 we have:

$$(x,y) \in (\sigma_R)_X(\sharp_X) \Leftrightarrow \text{ there exist } y_1 \in X, y_2 \in \overrightarrow{R}(X) \text{ such that } \begin{array}{ccc} & y & \\ & & \searrow^{R} \\ x & & y_2. \\ {}_{R}\searrow & & \nearrow_{\sharp_X} \\ & y_1 & \end{array}$$

Since R is injective, by Corollary 2.15(ii), $\operatorname{supp} x \subseteq \operatorname{supp} y_1$ and $\operatorname{supp} y \subseteq \operatorname{supp} y_2$. Now, since $\operatorname{supp} x \cap \operatorname{supp} y \subseteq \operatorname{supp} y_1 \cap \operatorname{supp} y_2$ and $(y_1, y_2) \in \sharp_X$, we get $(x,y) \in \sharp_X$. □

Theorem 4.7. *Suppose X, Y are nominal sets and $R \in \mathcal{R}(X,Y)$ is equivariant. Then*

(i) *$(\sigma_R)_S$ is order-preserving, for every $S \in \mathcal{P}_{\mathrm{fs}}(X)$.*

(ii) *If $R, T \in \mathcal{R}(X,Y)$ are equivariant relations and $R \subseteq T$, then $(\sigma_R)_S(\rho) \subseteq (\sigma_T)_S(\rho)$, for every $S \in \mathcal{P}_{\mathrm{fs}}(X)$.*

Proof. (i) The proof is straightforward.

(ii) For every $S \in \mathcal{P}_{\mathrm{fs}}(X)$ and $\rho \in \epsilon_Y(\overrightarrow{R}(S))$ we have:

$$(x,s) \in (\sigma_R)_S(\rho) \Leftrightarrow \text{ there exist } y_1 \in Y, y_2 \in \overrightarrow{R}(S) \text{ such that } \begin{array}{ccc} & s & \\ & & \searrow^{R} \\ x & & y_2 \\ {}_{R}\searrow & & \nearrow_{\rho} \\ & y_1 & \end{array}$$

$$\Rightarrow \text{ there exist } y_1 \in Y, y_2 \in \overrightarrow{R}(T) \text{ such that } \begin{array}{ccc} & s & \\ & & \searrow^{T} \\ x & & y_2 \\ {}_{T}\searrow & & \nearrow_{\rho} \\ & y_1 & \end{array}$$

$$\Leftrightarrow (x,s) \in (\sigma_T)_S(\rho).$$

So $(\sigma_R)_S(\rho) \subseteq (\sigma_T)_S(\rho)$. □

4.1 The properties of natural deterministic morphisms

Proposition 4.8. *Suppose X and Y are nominal sets and $\rho \in \mathcal{R}_{\mathrm{fs}}(X,Y)$. If $S \in \mathcal{P}_{\mathrm{fs}}(X)$ and $\overrightarrow{R}(S) \neq \emptyset$, then*

(i) $\operatorname{supp}(\sigma_R)_S(\rho) \subseteq \operatorname{supp}(\sigma_R)_S \cup \operatorname{supp}\rho$.

(ii) $\operatorname{supp}(\sigma_R)_S \subseteq \operatorname{supp} R \cup \operatorname{supp} S$.

Proof. (i) Since $(\sigma_R)_S$'s are maps, by Lemma 1.10, we have $\operatorname{supp}(\sigma_R)_S(\rho) \subseteq \operatorname{supp}(\sigma_R)_S \cup \operatorname{supp}\rho$.

(ii) Let $d_1, d_2 \notin \operatorname{supp} R \cup \operatorname{supp} S$. Then, $(d_1\ d_2)S = S$ and $\overrightarrow{R}((d_1\ d_2)x) = (d_1\ d_2)\overrightarrow{R}(x)$. We show that $(d_1\ d_2)(\sigma_R)_S(\rho) = (\sigma_R)_S((d_1\ d_2)\rho)$. To do so, let $(x,s) \in (\sigma_R)_S((d_1\ d_2)\rho)$. Then, there exist $y_1 \in Y$, $y_2 \in \overrightarrow{R}(S)$ with $(x,y_1),(s,y_2) \in R$ and $(y_1,y_2) \in (d_1\ d_2)\rho$. So, $((d_1\ d_2)y_1,(d_1\ d_2)y_2) \in \rho$. Since $(d_1\ d_2)S = S$ and $\overrightarrow{R}((d_1\ d_2)x) = (d_1\ d_2)\overrightarrow{R}(x)$, we have $(d_1\ d_2)s \in S$ and $((d_1\ d_2)x,(d_1\ d_2)y_1) \in R$. Thus, $((d_1\ d_2)x,(d_1\ d_2)s) \in (\sigma_R)_S(\rho)$ and so $(x,s) \in (d_1\ d_2)(\sigma_R)_S(\rho)$. The other side is proved similarly. □

Note 4.9. Suppose X, Y are nominal sets. Then

(i) if $R \in \mathcal{R}_{\mathrm{fs}}(X,Y)$, then $(\sigma_R)_S$'s are finitely supported.

(ii) if $R \in \mathcal{R}(X,X)$ is equivariant, then $(\sigma_R)_X$ and $(\sigma_R)_X(\Delta_X)$ are equivariant, too.

Remark 4.10. *Suppose X is a nominal set and $R \in \mathcal{R}(X,X)$ is equivariant. Then*

(i) $\mathrm{Dom}(\sigma_R)_X(R) \subseteq \mathrm{Dom}R$.

(ii) $\mathrm{Im}(\sigma_R)_X(R) \subseteq \mathrm{Dom}R$.

Proof. (i) Suppose $x \in \mathrm{Dom}(\sigma_R)_X(R)$, so there exists $y \in X$, such that $(x,y) \in (\sigma_R)_X(R)$. Then we have:

$$(x,y) \in (\sigma_R)_X(R) \Leftrightarrow \text{ there exist } y_1 \in Y, y_2 \in \overrightarrow{R}(X) \text{ such that } \begin{array}{ccc} & y & \\ & & \searrow^{R} \\ x & & y_2 \\ {}_{R}\searrow & & \nearrow_{R} \\ & y_1 & \end{array}.$$

Therefore $x \in \mathrm{Dom}R$ and $\mathrm{Dom}(\sigma_R)_X(R) \subseteq \mathrm{Dom}R$.

(ii) Suppose $y \in \mathrm{Im}(\sigma_R)_X(R)$, so there exists $x \in X$, such that $(x, y) \in (\sigma_R)_X(R)$. Then we have:

$$(x, y) \in (\sigma_R)_X(R) \Leftrightarrow \text{ there exist } y_1 \in Y, y_2 \in \overrightarrow{R}(X) \text{ such that } \begin{array}{ccc} & y & \\ & {}_{R}\searrow & \\ x & & y_2 \\ {}_{R}\searrow & & \nearrow_{R} \\ & y_1 & \end{array}.$$

Therefore $y \in \mathrm{Dom}R$ and $\mathrm{Im}(\sigma_R)_X(R) \subseteq \mathrm{Dom}R$. $\square$

Proposition 4.11. *Suppose X is a nominal set and $R \in \mathcal{R}(X, X)$ is equivariant. If R is injective, then*
(i) $(\sigma_R)_X(R) \subseteq R$.
(ii) *the relation* $(\sigma_R)_X(R)$ *is injective.*

Proof. (i) Suppose R is injective and $(x, y) \in (\sigma_R)_X(R)$. So we have:

$$(x, y) \in (\sigma_R)_X(R) \Leftrightarrow \text{ there exist } x_1 \in X, x_2 \in \overrightarrow{R}(X) \text{ such that } \begin{array}{ccc} & y & \\ & {}_{R}\searrow & \\ x & & x_2 \\ {}_{R}\searrow & & \nearrow_{R} \\ & x_1 & \end{array}.$$

Since R is injective, so $y = x_1$. Then $(x, y) \in R$ and $(\sigma_R)_X(R) \subseteq R$.
(ii) This follows from part (i). $\square$

Corollary 4.12. *Suppose X is a nominal set and $R \in \mathcal{R}(X, X)$ is an equivariant injective relation. If $(x, y) \in (\sigma_R)_X(R)$, then* $\mathrm{supp}\, x \subseteq \mathrm{supp}\, y$.

Proof. Since R is an equivariant injective relation, by Proposition 4.11(ii), $(\sigma_R)_X(R)$ is injective. Now, using Corollay 2.15(ii), we get that $\mathrm{supp}\, x \subseteq \mathrm{supp}\, y$. $\square$

Proposition 4.13. *Let X be a nominal set and $R \in \mathcal{R}(X, X)$ be equivariant. Then R is injective if and only if $(\sigma_R)_X(\Delta_X) \subseteq \Delta_X$.*

Proof. Suppose R is injective and $(x, s) \in (\sigma_R)_X(\Delta_X)$. So we have:

$$(x, s) \in (\sigma_R)_X(\Delta_X) \Leftrightarrow \text{ there exists } x_1 \in \overrightarrow{R}(X) \text{ such that } \begin{array}{ccc} & s & \\ & {}_{R}\searrow & \\ x & & x_1 \\ {}_{R}\searrow & & \nearrow_{\Delta_X} \\ & x_1 & \end{array}.$$

Since R is injective, $x = s$ and we have $(\sigma_R)_X(\Delta_X) \subseteq \Delta_X$. Conversely, suppose $(x, s), (x', s) \in R$, so we have $\begin{array}{ccc} & x' & \\ & {\scriptstyle R}\searrow & \\ x & & s. \\ {\scriptstyle R}\searrow & & \nearrow{\scriptstyle \Delta_X} \\ & s & \end{array}$ Then $(x, x') \in (\sigma_R)_X(\Delta_X)$. Since $(\sigma_R)_X(\Delta_X) \subseteq \Delta_X$, $x = x'$. Thus R is injective. $\square$

Proposition 4.14. *Suppose X is a nominal set and $R \in \mathcal{R}(X, X)$ is an equivariant symmetric relation. If $(\sigma_R)_X(R)$ is well-defined, then*
(i) *R is injective.*
(ii) *R is well-defined.*

Proof. (i) Suppose $(x', x), (x'', x) \in R$, so we have $\begin{array}{ccc} & x & \\ & {\scriptstyle R}\searrow & \\ x' & & x' \\ {\scriptstyle R}\searrow & & \nearrow{\scriptstyle R} \\ & x & \end{array}$ and $\begin{array}{ccc} & x' & \\ & {\scriptstyle R}\searrow & \\ x & & x \\ {\scriptstyle R}\searrow & & \nearrow{\scriptstyle R} \\ & x' & \end{array}$ and $\begin{array}{ccc} & x & \\ & {\scriptstyle R}\searrow & \\ x'' & & x'' \\ {\scriptstyle R}\searrow & & \nearrow{\scriptstyle R} \\ & x & \end{array}$ and $\begin{array}{ccc} & x'' & \\ & {\scriptstyle R}\searrow & \\ x & & x\,. \\ {\scriptstyle R}\searrow & & \nearrow{\scriptstyle R} \\ & x'' & \end{array}$ Then $(x', x), (x, x'), (x'', x), (x, x'') \in (\sigma_R)_X(R)$. Since $(\sigma_R)_X(R)$ is well-defined, $x' = x''$. Thus R is injective.

(ii) The proof is similar to part (i). $\square$

Corollary 4.15. *Suppose X is a nominal set and $R \in \mathcal{R}(X, X)$ is an equivariant symmetric relation. Then*
(i) *$(\sigma_R)_X(R)$ is well-defined if and only if R is injective.*
(ii) *If $(\sigma_R)_X(R)$ is well-defined and $(x, y) \in R$, then $\operatorname{supp} x = \operatorname{supp} y$.*

Proof. (i) Suppose R is symmetric and $(\sigma_R)_X(R)$ is well-defined. Then, by Proposition 4.14(i), R is injective. Conversely, suppose R is injective. Since R is symmetric, it is well-defined. Now, Proposition 4.11(i) implies that $(\sigma_R)_X(R)$ is well-defined.

(ii) Suppose R is an equivariant symmetric relation and $(\sigma_R)_X(R)$ is well-defined and $(x, y) \in R$. By Proposition 4.14(i), R is injective. Then Proposition 2.20(i) implies that $\operatorname{supp} x = \operatorname{supp} y$. $\square$

Proposition 4.16. *Suppose X is a nominal set and $R \in \mathcal{R}(X, X)$ is equivariant. If $\Delta_X \subseteq R$, then $\Delta_X \subseteq (\sigma_R)_X(R)$.*

Proof. Since $(x,x) \in R$, for every $x \in X$, we have $\begin{array}{ccc} & x & \searrow^{R} \\ x & & x \\ {}_{R}\searrow & x & \nearrow_{R} \end{array}$. Thus $(x,x) \in (\sigma_R)_X(R)$, for every $x \in X$, and we get the desired result. □

Corollary 4.17. *Suppose X is a nominal set and $R \in \mathcal{R}(X,X)$ is equivariant. Then R is total injective if and only if $(\sigma_R)_X(\Delta_X) = \Delta_X$.*

Proof. Suppose R is total injective. Then, $(\sigma_R)_X(\Delta_X) \subseteq \Delta_X$ follows from Proposition 4.13. Since $\mathrm{Dom}R = X$, $\Delta_X \subseteq (\sigma_R)_X(\Delta_X)$. Therefore $(\sigma_R)_X(\Delta_X) = \Delta_X$. Conversely, suppose $(\sigma_R)_X(\Delta_X) = \Delta_X$. Since $(\sigma_R)_X(\Delta_X) \subseteq \Delta_X$, applying Proposition 4.13, R is injective. Since $\Delta_X \subseteq (\sigma_R)_X(\Delta_X)$, $\mathrm{Dom}R = X$. Therefore R is total injective. □

Theorem 4.18. *Let X be a nominal set, $R \in \mathcal{R}(X,X)$ be equivariant.*
(i) *If $\rho \in \mathcal{R}_{\mathrm{fs}}(X,X)$ is coreflexive, then $(\sigma_R)_X(\rho)$ is symmetric.*
(ii) *If R is reflexive, then $(\sigma_R)_X(R) \subseteq R$ and so $(\sigma_R)_X(\Delta_X) \subseteq R$.*
(iii) *If R is symmetric and transitive, then $(\sigma_R)_X(\Delta_X) \subseteq R$.*

Proof. (i) Let $(x,y) \in (\sigma_R)_X(\rho)$. Then we have:

$$(x,y) \in (\sigma_R)_X(\rho) \Leftrightarrow \text{ there exist } x_1 \in X, x_2 \in \overrightarrow{R}(X) \text{ such that } \begin{array}{ccc} & y & \searrow^{R} \\ x & & x_2 \\ {}_{R}\searrow & x_1 & \nearrow_{\rho} \end{array}.$$

Since ρ is coreflexive, $x_1 = x_2$ and we have $\begin{array}{ccc} & x & \searrow^{R} \\ y & & x_1 \\ {}_{R}\searrow & x_1 & \nearrow_{\rho} \end{array}$. So $(y,x) \in (\sigma_R)_X(\rho)$.

(ii) Since R is reflexive, $\Delta_X \subseteq R$. So, $x \in \overrightarrow{R}(X)$, for all $x \in X$. Consider an arbitrary $(x,y) \in (\sigma_R)_X(R)$. Then we have:

$$(x,y) \in (\sigma_R)_X(R) \Leftrightarrow \text{ there exist } x \in X, y \in \overrightarrow{R}(X) \text{ such that } \begin{array}{ccc} & y & \searrow^{R} \\ x & & y \\ {}_{R}\searrow & x & \nearrow_{R} \end{array}$$

$$\Rightarrow (x,y) \in R.$$

Now, since $(\sigma_R)_X$ is a map and $\Delta_X \subseteq R$, we have $(\sigma_R)_X(\Delta_X) \subseteq (\sigma_R)_X(R) \subseteq R$.

(iii) For every $(x, s) \in (\sigma_R)_X(\Delta_X)$, we have:

$$
\begin{aligned}
(x,s) \in (\sigma_R)_X(\Delta_X) &\Leftrightarrow \exists\, y_1 \in \overrightarrow{R}(X) \text{ such that } \begin{array}{ccc} & s \overset{R}{\searrow} & \\ x & & y_1 \\ & {}^{R}\!\searrow y_1 \nearrow^{\Delta_X} & \end{array} \\
&\Leftrightarrow \exists\, y_1 \in \overrightarrow{R}(X) \text{ such that } (x, y_1) \in R, (s, y_1) \in R \\
&\Leftrightarrow \exists\, y_1 \in \overrightarrow{R}(X) \text{ such that } (x, y_1) \in R, (y_1, s) \in R \\
&\Rightarrow (x, s) \in R.
\end{aligned}
$$

So $(\sigma_R)_X(\Delta_X) \subseteq R$. □

Proposition 4.19. *Suppose X is a nominal set with $\mathcal{Z}(X) = \emptyset$ and $R \in \mathcal{R}(X, X)$ is a partial surjective equivariant map. If $(x, y) \in (\sigma_R)_X(\Delta_X)$, then $(x, y) \notin \sharp_X$.*

Proof. Suppose $(x, y) \in (\sigma_R)_X(\Delta_X)$. Then we have:

$$
(x,y) \in (\sigma_R)_X(\Delta_X) \Leftrightarrow \text{ there exists } x_1 \in \overrightarrow{R}(X) \text{ such that } \begin{array}{ccc} & y \overset{R}{\searrow} & \\ x & & x_1 \\ & {}^{R}\!\searrow x_1 \nearrow^{\Delta_X} & \end{array} .
$$

Since R is a partial surjective equivariant map , Corollary 2.30(ii) implies $\operatorname{supp} x_1 \subseteq \operatorname{supp} y$ and $\operatorname{supp} x_1 \subseteq \operatorname{supp} x$. Since $\mathcal{Z}(X) = \emptyset$, $\operatorname{supp} x \cap \operatorname{supp} y \neq \emptyset$. Thus $(x, y) \notin \sharp_X$. □

Corollary 4.20. *Suppose X and $\mathbb{D}$ are nominal sets and $R \in \mathcal{R}(X, \mathbb{D})$ is a partial surjective equivariant map. If $(x, y) \in (\sigma_R)_X(\Delta_{\mathbb{D}})$, then*

(i) $(x, y) \notin \sharp_X$.

(ii) $x, y \notin \mathcal{Z}(X)$.

Proof. This follows from Proposition 4.19. □

4.2 Some concrete examples for natural deterministic morphisms

Here, we try to provide a better understanding of the concept of natural deterministic morphism by giving some examples.

Definition 4.21. [3] *Let X be a nominal set. A binding operator on X is an equivariant map $l : X \longrightarrow \mathcal{P}_{\mathrm{f}}(\mathbb{D})$. Each l gives rise to a relation $\equiv_l$ on X, defined as*

$x_1 \equiv_l x_2 \Leftrightarrow$ *there exists* $\pi \in \mathrm{Perm}(\mathbb{D})$, *such that* $\pi \sharp \mathrm{supp}(x_1) \setminus l(x_1)$ *and* $x_2 = \pi x_1$.

Remark 4.22. [3]*Let X be a nominal set endowed with a binding operator l. Then*

(i) *the relation $\equiv_l$ is an equivariant equivalence relation.*

(ii) *if $(x, y) \in \equiv_l$, then $\mathrm{supp}\, x \setminus l(x) = \mathrm{supp}\, y \setminus l(y)$.*

Corollary 4.23. *Let X be a nominal set endowed with a binding operator l. Then,*

(i) $(\sigma_{\equiv_l})_X(\Delta_X) \subseteq \equiv_l$.

(ii) *There exists $\pi \in \mathrm{Perm}(\mathbb{D})$ such that $\pi \sharp \mathrm{supp}\, x \setminus l(x)$ and $s = \pi x$, for $(x, s) \in (\sigma_{\equiv_l})_X(\Delta_X)$.*

(iii) $\mathrm{supp}\, x \setminus l(x) = \mathrm{supp}\, s \setminus l(s)$, *for* $(x, s) \in (\sigma_{\equiv_l})_X(\Delta_X)$.

Proof. (i) Follows from Theorem 4.18(iii).

(ii) Follows from (i).

(iii) Follows from (ii) and Remark 4.22(ii). □

Proposition 4.24. *Suppose X is a nominal set endowed with a binding operator l and $R \in \mathcal{R}(X, X)$ is equivariant. If there exists $x \in X$ with $(x, x) \in (\sigma_{\equiv_l})_X(R)$, then $\equiv_l \cap R \neq \emptyset$.*

Proof. Let $(x, x) \in (\sigma_{\equiv_l})_X(R)$. Then,

$$(x, x) \in (\sigma_{\equiv_l})_X(R) \Leftrightarrow \text{ there exist } y_1 \in X, y_2 \in \overrightarrow{\equiv_l}(X) \text{ such that } \begin{array}{ccc} & x & \\ x & \searrow^{\equiv_l} & y_2 . \\ {}_{\equiv_l}\searrow & y_1 & \nearrow_R \end{array}$$

Then there exist π_1, π_2 with $\pi_1 x = y_1$, $\pi_2 x = y_2$ and $\pi_1, \pi_2 \sharp \operatorname{supp} x \setminus l(x)$. Also, $\operatorname{supp} x \setminus l(x) = \operatorname{supp} y_1 \setminus l(y_1) = \operatorname{supp} y_2 \setminus l(y_2)$. Thus, $y_2 = \pi_2 x = \pi_2 \pi_1^{-1} y_1$.

$$\begin{aligned} \pi_2 {\pi_1}^{-1}(\operatorname{supp} y_1 \setminus l(y_1)) &= \pi_2 {\pi_1}^{-1}(\operatorname{supp} x \setminus l(x)) \\ &= \pi_2(\operatorname{supp} x \setminus l(x)) \\ &= \operatorname{supp} x \setminus l(x) \\ &= \operatorname{supp} y_1 \setminus l(y_1). \end{aligned}$$

Therefore $(y_1, y_2) \in \equiv_l$ and $\equiv_l \cap R \neq \emptyset$. □

Corollary 4.25. *Suppose X is a nominal set endowed with a binding operator l and $R \in \mathcal{R}(X, X)$ is equivariant. If $\Delta_X \cap (\sigma_{\equiv_l})_X(R) \neq \emptyset$, then $\equiv_l \cap R \neq \emptyset$.*

Proof. This follows from Proposition 4.24. □

Proposition 4.26. *Suppose X is a nominal set endowed with a binding operator l. If $R = \{(x, x') : \operatorname{supp} x \setminus l(x) = \operatorname{supp} x' \setminus l(x')\}$, then*

(i) $\equiv_l \subseteq R$.

(ii) *if R is injective, then $(\sigma_R)_S(\equiv_l \cap (X \times \overrightarrow{R}(S)) \subseteq \equiv_l$.*

Proof. (i) Let $(x, x') \in \equiv_l$. Then, $\operatorname{supp} x \setminus l(x) = \operatorname{supp} x' \setminus l(x')$ and so $(x, x') \in R$.

(ii) Let $(x, s) \in (\sigma_R)_S(\equiv_l \cap (X \times \overrightarrow{R}(S)))$. Then,

$$(x, s) \in (\sigma_R)_S(\equiv_l \cap (X \times \overrightarrow{R}(S))) \Leftrightarrow \exists y_1 \in X, y_2 \in \overrightarrow{R}(S),$$

such that

$$\begin{array}{ccc} & s & \\ & \searrow^{R} & \\ x & & y_2. \\ {}_{R}\searrow & & \nearrow_{\equiv_l \cap (X \times \overrightarrow{R}(S)))} \\ & y_1 & \end{array}$$

Since $(y_1, y_2) \in \equiv_l \cap (X \times \overrightarrow{R}(S)))$, there exists $\pi \in \operatorname{Perm}(\mathbb{D})$ with $y_2 = \pi y_1$ and $\pi \sharp \operatorname{supp} y_1 \setminus l(y_1)$. Also, since $(x, y_1) \in R$, $\operatorname{supp} y_1 \setminus l(y_1) = \operatorname{supp} x \setminus l(x)$. Thus $\pi \sharp \operatorname{supp} x \setminus l(x)$. Since $(x, y_1) \in R$ and R is equivariant, $(\pi x, y_2) = (\pi x, \pi y_1) \in R$. Now, since $(\pi x, y_2), (s, y_2) \in R$ and R is injetive, we have $\pi x = s$. Therefore, $\pi x = s$ and $\pi \sharp \operatorname{supp} x \setminus l(x)$ and so $(x, s) \in \equiv_l$. $\square$

We recall from [16] that an equivalence relation ρ over a nominal set X is called *an equivariant equivalence relation (or congruence)* on X, whenever ρ is equivariant as a subset of $X \times X$. We also recall that if X and Y are nominal sets, then $(X \times Y)/\approx$ is a nominal set where $\approx$ is a congruence on $X \times Y$ defined by

$$(x, y) \approx (x', y') \iff \pi(x, y) = (x', y'),$$

for some $\pi \in \mathrm{Perm}(\mathbb{D})$ with $\pi \sharp(\operatorname{supp} y \setminus \operatorname{supp} x)$. We also have $\operatorname{supp} y \setminus \operatorname{supp} x = \operatorname{supp} y' \setminus \operatorname{supp} x'$ and $\operatorname{supp}(x, y)/\approx = \operatorname{supp} y \setminus \operatorname{supp} x$.

Example 4.27. For every $X \in \mathbf{Nom}$ and $i : Y \hookrightarrow Y' \in \mathrm{Eqsub}(X)$, the assignment $\epsilon_X^{eq} : \mathrm{Eqsub}(X) \to \mathbf{Nom}$ defined by $\epsilon_X^{eq}(Y) = (X \times Y)/\approx$ and $\epsilon_X^{eq}(i)[(x, y)/\approx] = (x, i(y))/\approx$ is a functor.

Remark 4.28. *Suppose X is a nominal set and $T \in \mathrm{Eqsub}(X)$. If $R \in \mathcal{R}(X, X)$ is an equivariant relation, then $\overrightarrow{R}(T)$ is a nominal set.*

Proposition 4.29. *Let X be a nominal set. Then, each injective equivariant relation $R \in \mathcal{R}(X, X)$ determines a natural deterministic morphism.*

Proof. Let T be an equivariant subset of X and $z \in \overrightarrow{R}(T)$. Since R is injective, there exists a unique $t \in T$ with $(t, z) \in R$. Define $((\sigma_R)_T)_{T \in \mathrm{Eqsub}(X)}$ in which $(\sigma_R)_T : \epsilon_X^{eq}(\overrightarrow{R}(T)) \to \epsilon_X^{eq}(T)$ assigns every $(x, z)/\approx \in (X \times \overrightarrow{R}(T))/\approx$ to $(x, t)/\approx \in (X \times T)/\approx$. We show that $(\sigma_R)_T$'s are well-defined. To do so, let $(x, z)/\approx = (x', z')/\approx$ where $z, z' \in \overrightarrow{R}(T)$. Then, there exists $\pi \in \mathrm{Perm}(\mathbb{D})$ with $\pi(x, t) = (x', t')$ and $\pi \sharp \operatorname{supp} t \setminus \operatorname{supp} x$. Indeed, the assumption $(x, z)/\approx = (x', z')/\approx$ implies that there exists $\pi \in \mathrm{Perm}(\mathbb{D})$ with $\pi(x, z) = (x', z')$ and $\pi \sharp(\operatorname{supp} z - \operatorname{supp} x)$. Since R is equivariant and injective, we get

$$(t, z) \in R \Rightarrow (\pi t, z') = (\pi t, \pi z) \in R, \quad (t', z'), (\pi t, z') \in R \Rightarrow \pi t = t'.$$

Thus, $\pi(x, t) = (x', t')$. Also, since R is injective, by Corollary 2.15(ii), $\operatorname{supp} t \subseteq \operatorname{supp} z$. Thus, $\operatorname{supp} t - \operatorname{supp} x \subseteq \operatorname{supp} z - \operatorname{supp} x$ and so $\pi \sharp(\operatorname{supp} t - \operatorname{supp} x)$.

Now, we show that $((\sigma_R)_T)_{T\in \mathrm{Eqsub}(X)}$'s are equivariant. Let $\pi \in \mathrm{Perm}(\mathbb{D})$ and $\pi(\sigma_R)_T(x,z)/{\approx} = (\pi x, \pi t)/{\approx}$. We claim that $(\sigma_R)_T(\pi x, \pi z)/{\approx} = (\pi x, \pi t)/{\approx}$. Notice that, $z \in \overrightarrow{R}(T)$ and $(t,z) \in R$. Since $\pi z \in \overrightarrow{R}(T)$ and R is injective, there exists a unique $t' \in T$ with $(t', \pi z) \in R$. Take $(\sigma_R)_T(\pi x, \pi z)/{\approx} = (\pi x, t')/{\approx}$. On the other hand, since $(t,z) \in R$ and R is equivariant, $(\pi t, \pi z) \in R$. Now, since R is injective and $(t', \pi z), (\pi t, \pi z) \in R$, $\pi t = t'$. Thus, $\pi(\sigma_R)_T(x,z)/{\approx} = (\pi x, \pi t)/{\approx} = (\pi x, t')/{\approx} = (\sigma_R)_T(\pi x, \pi z)/{\approx}$. The naturality of σ_R is obtained easily. □

4.3 Stochastic maps between nominal sets

In this subsection, we introduce another morphism in the category **Rel**(**Nom**), see the following definition.

Definition 4.30. *Let X and Y be two nominal sets. A stochastic map is a natural deterministic morphism (f,σ) in which $f : X \longrightarrow Y$ is a finitely supported map and $\sigma : \epsilon_Y \circ f \to \epsilon_X$ is a natural transformation.*

It is worth noting that, by Proposition 4.5, every equivariant map $f : X \longrightarrow Y$ determines the stochastic map (f, σ_f). In this subsection, we focus on a stochastic map.

Example 4.31. Given each nominal set X, the support map supp : $X \longrightarrow \mathcal{P}_{\mathrm{f}}(\mathbb{D})$ gives rise a stochastic map $(\mathrm{supp}, \sigma_{\mathrm{supp}})$.

Theorem 4.32. *Given a non-discrete nominal set X, then*
(i) $x \sharp y$ *if and only if* $(x,y) \notin (\sigma_{\mathrm{supp}})_X(\Delta_{\mathbb{D}})$.
(ii) $(\sigma_{\mathrm{supp}})_X(\Delta_{\mathbb{D}}) = X \times X \setminus \sharp_X$.

Proof. Let $f = \mathrm{supp}$. Then, applying Proposition 4.5, we have:

$$(x,y) \in (\sigma_f)_X(\Delta_{\mathbb{D}}) \Leftrightarrow \text{ there exists } d_1 \in \overrightarrow{R}(X) \text{ such that } \begin{array}{ccc} & y & \overset{f}{\searrow} \\ x & & d_1. \\ {}_{f}\searrow & d_1 & \nearrow_{\Delta_{\mathbb{D}}} \end{array}$$

So, $(x,y) \in (\sigma_f)_X(\Delta_{\mathbb{D}}) \Leftrightarrow d_1 \in \mathrm{supp}\, x \cap \mathrm{supp}\, y$.

(ii) We have:

$$\begin{aligned}(\sigma_{\mathrm{supp}})_X(\Delta_{\mathbb{D}}) &= \{(x,y) : \operatorname{supp} x \cap \operatorname{supp} y \neq \emptyset\}\\ &= \{(x,y) : (x,y) \notin \sharp\}\\ &= X \times X \setminus \sharp_X.\end{aligned}$$

So, $(\sigma_{\mathrm{supp}})_X(\Delta_{\mathbb{D}}) = X \times X \setminus \sharp_X$. $\square$

Corollary 4.33. *Let $f : X \longrightarrow X$ be in* **Nom**. *Then f is injective if and only if $(\sigma_f)_X(\Delta_X) = \Delta_X$.*

Proof. This follows from Corollary 4.17. $\square$

Proposition 4.34. *Let $f : X \longrightarrow Y$ be an equivariant surjective map between nominal sets. If $(\sigma_f)_A$ maps every reflexive relation $\rho \in \mathcal{R}_{\mathrm{fs}}(Y,Y)$ to a subset of Δ_A, for every $A \in \mathcal{P}_{\mathrm{fs}}(X)$, then f is an isomorphism.*

Proof. To show that f is injective, we note that $(\sigma_f)_X(\rho) \subseteq \Delta_A$. If $(x,y),(x',y) \in f$ then, by the diagram $\begin{array}{ccc} & x' \overset{f}{\searrow} & \\ x & & y \\ & {}_f\searrow\, y \,\nearrow_\rho & \end{array}$, we have $(x,x') \in (\sigma_f)_X(\rho) \subseteq \Delta_A$ and $x = x'$. Therefore f is an isomorphism. $\square$

Proposition 4.35. *Let $f : X \longrightarrow X$ be in* **Nom**, *such that $f|_A$ is bijective where $A \in \mathcal{P}_{\mathrm{fs}}(X)$. Then $(\sigma_{f|_A})_A(\rho) \cap \Delta_A \neq \emptyset$ if and only if $\rho \cap \Delta_Y \neq \emptyset$, for every $\rho \in \mathcal{R}_{\mathrm{fs}}(Y,Y)$.*

Proof. Suppose $(\sigma_{f|_A})_A(\rho) \cap \Delta_A \neq \emptyset$. Then there exsits $(x,x) \in (\sigma_{f|_A})_A(\rho) \cap \Delta_A$. So we have $\begin{array}{ccc} & x \overset{f|_A}{\searrow} & \\ x & & f(x) \\ & {}_{f|_A}\searrow\, f(x) \,\nearrow_\rho & \end{array}$. Thus $(f(x), f(x)) \in \rho \cap \Delta_Y \neq \emptyset$. Conversely, suppose $(y,y) \in \rho \cap \Delta_Y$. Since $f|_A$ is bijective, there exists a unique $x \in A$, such that, we have $\begin{array}{ccc} & x \overset{f|_A}{\searrow} & \\ x & & y \\ & {}_{f|_A}\searrow\, y \,\nearrow_\rho & \end{array}$. Thus $(x,x) \in (\sigma_{f|_A})_A(\rho)$ and $(\sigma_{f|_A})_A(\rho) \cap \Delta_A \neq \emptyset$. $\square$

Theorem 4.36. *Suppose $f : X \longrightarrow Y$ is an isomorphism in* **Nom**. *Then the stochastic morphism (f, σ_f)*

(i) *preserves and reflects well-defined relations.*

(ii) *preserves and reflects injectivity.*

(iii) *preserves and reflects constant relations.*

Proof. (i) Suppose ρ is well-defined and $(x, y), (x, y') \in (\sigma_f)_A(\rho)$. So we have: $x \xrightarrow{f} f(x) \xrightarrow{\rho} f(y) \xleftarrow{f} y$ and $x \xrightarrow{f} f(x) \xrightarrow{\rho} f(y') \xleftarrow{f} y'$. Since ρ is well-defined, $f(y) = f(y')$. Since f is injective, $y = y'$. Thus $(\sigma_f)_A(\rho)$ is well-defined. Conversely, suppose $(\sigma_f)_A(\rho)$ is well-defined and $(y, y'), (y, y'') \in \rho$. Since f is bijective, we have $f^{-1}(y) \xrightarrow{f} y \xrightarrow{\rho} y' \xleftarrow{f} f^{-1}(y')$ and $f^{-1}(y) \xrightarrow{f} y \xrightarrow{\rho} y'' \xleftarrow{f} f^{-1}(y'')$. Then $(f^{-1}(y), f^{-1}(y')), (f^{-1}(y), f^{-1}(y'')) \in (\sigma_f)_A(\rho)$. Since $(\sigma_f)_A(\rho)$ and f are well-defined $f^{-1}(y') = f^{-1}(y'')$ and $y' = y''$. Thus ρ is well-defined.

(ii) Suppose ρ is injective and $(x, y), (x', y) \in (\sigma_f)_A(\rho)$. So we have $x \xrightarrow{f} f(x) \xrightarrow{\rho} f(y) \xleftarrow{f} y$ and $x' \xrightarrow{f} f(x') \xrightarrow{\rho} f(y) \xleftarrow{f} y$. Since ρ and f are injective, $f(x) = f(x')$ and $x = x'$. Thus $(\sigma_f)_A(\rho)$ is injective. Conversely, suppose $(\sigma_f)_A(\rho)$ is injective and $(y', y), (y'', y) \in \rho$. Since f is bijective, we have $f^{-1}(y') \xrightarrow{f} y' \xrightarrow{\rho} y \xleftarrow{f} f^{-1}(y)$ and $f^{-1}(y'') \xrightarrow{f} y'' \xrightarrow{\rho} y \xleftarrow{f} f^{-1}(y)$. Then $(f^{-1}(y'), f^{-1}(y)), (f^{-1}(y''), f^{-1}(y)) \in (\sigma_f)_A(\rho)$. Since $(\sigma_f)_A(\rho)$ is injective, $f^{-1}(y') = f^{-1}(y'')$. Since f is well-defined, $y' = y''$. Thus ρ is injective.

(iii) Suppose ρ is constant but $(\sigma_f)_A(\rho)$ is not constant, that is, there

exist $(x,y),(x',y') \in (\sigma_f)_A(\rho)$, where $y \neq y'$. So we have $\begin{array}{ccc} & y & \searrow^{f} \\ x & \searrow_{f} & f(y) \\ & f(x) & \nearrow^{\rho} \end{array}$ and $\begin{array}{ccc} & y' & \searrow^{f} \\ x' & \searrow_{f} & f(y') \\ & f(x') & \nearrow^{\rho} \end{array}$. Since ρ is constant, $f(y) = f(y')$. Since f is injective, $y = y'$, which is a contradiction. Conversely, suppose $(\sigma_f)_A(\rho)$ is constant ρ is not constant, that is, there exist $(y_1,y_2),(y_3,y_4) \in \rho$, where $y_2 \neq y_4$. Since f is bijective, we have $\begin{array}{ccc} & f^{-1}(y_2) & \searrow^{f} \\ f^{-1}(y_1) & \searrow_{f} & y_2 \\ & y_1 & \nearrow^{\rho} \end{array}$ and $\begin{array}{ccc} & f^{-1}(y_4) & \searrow^{f} \\ f^{-1}(y_3) & \searrow_{f} & y_4 \\ & y_3 & \nearrow^{\rho} \end{array}$.
Then $(f^{-1}(y_1), f^{-1}(y_2)),(f^{-1}(y_3), f^{-1}(y_4)) \in (\sigma_f)_A(\rho)$. Since $(\sigma_f)_A(\rho)$ is constant and f is well-defined, $f^{-1}(y_2) = f^{-1}(y_4)$ and $y_2 = y_4$, which is a contradiction. $\square$

Proposition 4.37. *Let* $f : X \longrightarrow X$ *be in* **Nom**. *Then*

(i) *the assignment* $(\sigma_f)_X$ *preserves reflexive relations.*

(ii) *the assignment* $(\sigma_f)_X$ *preserves symmetric relations.*

(iii) *the assignment* $(\sigma_f)_X$ *preserves transitive relations.*

Proof. (i) Suppose $\rho \in \mathcal{R}_{\mathrm{fs}}(X, f(X))$ is reflexive. So, for every $x \in X$, we have $\begin{array}{ccc} & x & \searrow^{f} \\ x & \searrow_{f} & f(x) \\ & f(x) & \nearrow^{\rho} \end{array}$. Thus $(x,x) \in (\sigma_f)_X(\rho)$.

(ii) Suppose $\rho \in \mathcal{R}_{\mathrm{fs}}(X, f(X))$ is symmetric and $(x,y) \in (\sigma_f)_X(\rho)$. So we have $\begin{array}{ccc} & y & \searrow^{f} \\ x & \searrow_{f} & f(y) \\ & f(x) & \nearrow^{\rho} \end{array}$. Since ρ is symmetric, we have $\begin{array}{ccc} & x & \searrow^{f} \\ y & \searrow_{f} & f(x) \\ & f(y) & \nearrow^{\rho} \end{array}$.
Therefore $(y,x) \in (\sigma_f)_X(\rho)$.

(iii) Suppose $(x, y), (y, z) \in (\sigma_f)_X(\rho)$, and ρ is transitive in $\mathcal{R}_{\rm fs}(X, f(X))$, so we have $\begin{array}{ccc} & y & \\ & & \searrow^{f} \\ x & & f(y) \\ {}_{f}\searrow & & \nearrow_{\rho} \\ & f(x) & \end{array}$ and $\begin{array}{ccc} & z & \\ & & \searrow^{f} \\ y & & f(z) \\ {}_{f}\searrow & & \nearrow_{\rho} \\ & f(y) & \end{array}$. Then we have $\begin{array}{ccc} & z & \\ & & \searrow^{f} \\ x & & f(z) \\ {}_{f}\searrow & & \nearrow_{\rho} \\ & f(x) & \end{array}$. Thus $(x, z) \in (\sigma_f)_X(\rho)$. $\square$

Corollary 4.38. *Let $f : X \longrightarrow X$ be in* **Nom**.

(i) *Then the stochastic morphism (f, σ_f) preserves equivalence relations.*

(ii) *If $\rho \in \mathcal{R}_{\rm fs}(X, X)$ is a congruence, then $(\sigma_f)_X(\rho)$ is a congruence.*

Proof. (i) Follows from Proposition 4.37.

(ii) Follows from (i) and Proposition 4.8. $\square$

5 Conclusion

The category of nominal sets and equivariant maps between them atractted a lot of interest of computer scientists due to their unique properties. In this paper we replace equivariant relations rather than equivariant maps and consider the category **Rel**(**Nom**), because this category not only contains the category **Nom** and is more expressive than **Nom** but also because of the kind of morphisms in this category, one can allows to work various structures that are not functions. For example **Rel**(**Nom**) can be used to model dependent types, which are types that depend on several values, or data that can be correlated by relations. A deterministic morphism, which gives each input data set a specific output of the same type, are also introduced. On the other hand, each input can be given a set of outputs by using stochastic maps, which there is a given likelihood that each will occur. We also introduce and examin stochastic maps.

This paper consists of four sections. The needed foundational concepts are covered in the first section. In the second section, we introduce the category **Rel**(**Nom**) consisting of nominal sets and equivariant relations between them and examine some properties of this category. In the third section, we define two functors $\mathcal{P}_{\mathrm{fs}*}$ and ${\mathcal{P}_{\mathrm{fs}}}^*$. In Theorem 3.3, we show that ${\mathcal{P}_{\mathrm{fs}}}^* \dashv \mathcal{P}_{\mathrm{fs}*}$, and hence the functor $\mathcal{P}_{\mathrm{fs}*}$ is the functor asigning each nominal set in **Rel**(**Nom**) to its sheaf representation. Finally, in Section 4, we introduce the notion of a deterministic and stochastic morphism. In Proposition 4.5, we show that the notion of a every equivariant relation determines a natural deterministic morphism. Also, we investigate the support of a deterministic morphism in Proposition 4.8 and also the property of a $(\sigma_f)_X$, where f is an equivariant map, in Proposition 4.37.

For further work in the future, one can focus on free, indecomposable, cyclic and injective objects in the category **Rel**(**Nom**) with stochastic and deterministic morphisms. Also, one can study some categorical properties in this category, for example, the existence of monad, the Kleisli and Eilenberg-Moore categories, filtered category and sheaf representation of nominal sets in the category **Rel**(**Nom**) with stochastic and deterministic morphisms.

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