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On the Semilinear Equation with an Exponential Decay Memory Term

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Abstract. The main purpose of this paper is to study the non-existence of weak global solutions to the following semi-linear equation with exponential decay memory term, namely

$$\begin{cases} u_{tt} + (-\Delta)^{\frac{\gamma}{2}} u_t + u - \displaystyle\int_0^t e^{\tau - t} u(\tau, x) d\tau = |u|^p, \quad x \in \mathbb{R}^n, \quad t > 0, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^n, \end{cases}$$

where $\gamma \in (0,2]$, $\quad p > 1$, and $(-\Delta)^{\frac{\gamma}{2}}$ is the fractional Laplacian operator of order $\frac{\gamma}{2}$. We extend our result to the case of 2×2-system of the same type. Our technique of proof is the so-called method of test function.

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1 Introduction

The main motivation behind this paper is to study the non-existence of weak global solutions to the following semi-linear equation with expo-

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nential decay memory term:

$$\begin{cases} u_{tt} + (-\Delta)^{\frac{\gamma}{2}} u_t + u - \displaystyle\int_0^t e^{\tau - t} u(\tau, x) d\tau = |u|^p, t > 0, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $p > 1, n \geq 1, \ \gamma \in (0,2]$, and $(-\Delta)^{\frac{\gamma}{2}}$ is the fractional Laplacian operator of order $\frac{\gamma}{2}$. Then we extend our analysis to the case of a 2×2 system of the same type, namely

$$\begin{cases} u_{tt} + (-\Delta)^{\frac{\gamma}{2}} u_t + u - \displaystyle\int_0^t e^{\tau - t} u(\tau, x) d\tau = |v|^p, \quad x \in \mathbb{R}^n, \\ v_{tt} + (-\Delta)^{\frac{\gamma}{2}} v_t + v - \displaystyle\int_0^t e^{\tau - t} v(\tau, x) d\tau = |u|^q, \quad x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^n, \\ v(0,x) = v_0(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{2}$$

We mention below some motivations for studying the considered problems.
Recently, in [3], the following Cauchy problem

$$\begin{cases} y_{tt} - \Delta y + (-\Delta)^{\varrho_1} y_t = |z|^p, \quad x \in \mathbb{R}^n, \quad t > 0, \\ z_{tt} - \Delta z + (-\Delta)^{\varrho_2} z_t = |y|^q, \quad x \in \mathbb{R}^n, \quad t > 0, \\ y(0,x) = y_0(x), \quad y_t(0,x) = y_1(x), \\ z(0,x) = z_0(x), \quad z_t(0,x) = z_1(x) \quad x \in \mathbb{R}^n, \end{cases} \tag{3}$$

is considered. It was shown that
If $\varrho_1, \varrho_2 \in \left[0, \frac{1}{2}\right], y_0 = y_1 = 0$ and $y_1, z_1 \in \mathbb{L}^1(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} y_1(x) dx > \varepsilon_1, \quad \int_{\mathbb{R}^n} y_1(x) dx > \varepsilon_2,$$

and

$$\frac{n}{2} \leq \frac{1 + q\frac{1-\varrho_2}{1-\varrho_1} + (pq-1)\varrho_2}{(q-1)\frac{\varrho_1-\varrho_2}{1-\varrho_2} + (pq-1)} \quad \text{if} \quad \varrho_1 \geq \varrho_2,$$

$$\frac{n}{2} \leq \frac{1 + p\frac{1-\varrho_1}{1-\varrho_2} + (pq-1)\varrho_2}{(p-1)\frac{\varrho_2-\varrho_1}{1-\varrho_1} + (pq-1)} \quad \text{if} \quad \varrho_2 \geq \varrho_1.$$

Then, there is no global (in time) Sobolev solution $(y,z) \in \mathcal{C}\left([0,\infty) \times \mathbb{L}^2(\mathbb{R}^n)\right) \times \mathcal{C}\left([0,\infty) \times \mathbb{L}^2(\mathbb{R}^n)\right)$ to (3). Very recently, the following Cauchy problem for the semi linear σ-evolution models with memory term:

$$\begin{cases} y_{tt} + (-\Delta)^\sigma y + y - g * y = |y|^p, \quad x \in \mathbb{R}^n, \quad t > 0, \\ \\ y(0,x) = y_0(x), \quad y_t(0,x) = y_1(x) \quad x \in \mathbb{R}^n, \end{cases} \tag{4}$$

has been studied by Wenhui Chen and Tuan Anh Dao [1]. It was shown in [1] that if

$$1 < p < 1 + \tfrac{2m\sigma}{n}, \ y_1 \in \mathbb{L}^m(\mathbb{R}^n), \ \int_{\mathbb{R}^n} y_1(x)dx > 0 \ (m=1);$$
$$y_1(x) \geq |x|^{-\frac{n}{m}} \Big(\log(1+|x|)\Big)^{-1} \ m \in (1,2),$$

then, there is no global (in time) weak solution to (4). As far as we know that one of the most typically important methods to verify the non-existence of global weak solutions is the well-known test function method. Concretely, this method is used to prove the nonexistence of global solutions by a contradiction argument. However, standard test function method seems difficult to directly apply to (1) containing pseudo-differential operators $(-\Delta)^{\frac{\alpha}{2}}$ for any $\alpha \in (0,2]$, well-known non-local operators. To overcome the difficulty caused by the nonlocal property of the fractional Laplacian operator, D' Abbicco and Reissig [2] investigated the structurally damped wave equation with the power nonlinearity $|u|^p$. The critical exponent has been studied and they proposed to distinguish between (parabolic models) in the case $\sigma \in (0,1]$, the so-called effective damping, and (hyperbolic models) in the remaining case $\sigma \in (1,2]$, the so-called noneffective damping according to expected decay estimates. In the former case, they proved the existence of global (in time) solutions when

$$p > p_c = 1 + \frac{2}{(n-\sigma)_+}$$

for the small initial data and low space dimensions $2 \leq n \leq 4$ by using the energy estimates. Over the last years, the evolution equations with memory terms has been studied by several authors (see, for example, [6], [7], [8], [9]). Furthermore, we want to underline that, the Cauchy problem for the evolution equations with memory terms have caught a lot of attention from many mathematicians due to their wide applications in physics, mechanics and so on but to our knowledge, the equation has not been widely investigated in the case of presence of non-local operators. For other contributions related to the evolution equations with memory terms, see ([9],[10]), for example and the references therein. Motivated by the above contributions, in particular by [1], our goal in this paper is to investigate problems (1) and (2).

Before stating our main results, we introduce some important fundamental definitions that will be needed for obtaining our results in the next sections.

Definition 1.1. ([5],[11]) Let $s \in (0,1)$. Let X be a suitable set of functions defined on $\mathbb{R}^n$. Then, the fractional Laplacian $(-\Delta)^s$ in $\mathbb{R}^n$is a non-local operator given by

$$(-\Delta)^s : f \in X \to (-\Delta)^s f(x) = C_{n,s}\ P.V \int_{\mathbb{R}^n} \frac{f(x)-f(y)}{|x-y|^{n+2s}}\ dy,$$

where $P.V$ stands for the Cauchy's principal value, and

$$C_{n,s} = \frac{4^s \Gamma\left(\frac{n}{2}+s\right)}{\pi^{\frac{n}{2}}\Gamma(-s)}$$

is the normalization constant and Γ denotes the Gamma function.

We are now in a position, to state the main results of this manuscript.

Theorem 1.2. *Let $\gamma \in (0,2]$ and $\tilde{\gamma} = min\{1,\gamma\}$. We suppose that $u_1 \in \mathbb{L}^1(\mathbb{R}^n)$ such that:*

$$\int_{\mathbb{R}^n} u_1(x)dx > 0. \tag{5}$$

If

$$1 < p \leq 1 + \frac{\tilde{\gamma}}{n}, \quad n \geq 1, \tag{6}$$

then, there is no global (in time) weak solution $u \in \mathcal{C}\left([0,+\infty[, \mathbb{L}^2(\mathbb{R}^n)\right)$ to problem (1).

Theorem 1.3. *Let $\gamma, \beta \in (0,2]$, $\tilde{\gamma} = min\{1,\gamma\}$, and $\tilde{\beta} = min\{1,\beta\}$. We assume that $u_1 \in \mathbb{L}^1(\mathbb{R}^n)$ and $v_1 \in \mathbb{L}^1(\mathbb{R}^n)$ satisfying the following conditions:*

$$\int_{\mathbb{R}^n} u_1(x)dx > 0 \quad \textit{and} \quad \int_{\mathbb{R}^n} v_1(x)dx > 0. \tag{7}$$

If

$$n \leq \frac{1}{pq-1} \max\left\{\tilde{\beta} + \tilde{\gamma}p, \tilde{\gamma} + \tilde{\beta}q\right\}, \quad n \geq 1, \tag{8}$$

then, there is no global (in time) weak solution $(u,v) \in \mathcal{C}\left([0,+\infty[, \mathbb{L}^2(\mathbb{R}^n)\right) \times \mathcal{C}\left([0,+\infty[, \mathbb{L}^2(\mathbb{R}^n)\right)$ to (2). Therefore, the blow-up time T_ε is estimated by

$$T_\varepsilon \leq C\varepsilon^{-\frac{\tilde{\gamma}}{\frac{\tilde{\gamma}+\tilde{\beta}q}{pq-1} - n}} \quad \textit{for all small positive constants} \quad \varepsilon. \tag{9}$$

Our main theorems will be proved in the next section.

2 Proofs

In this section, we give the proofs of Theorems 1.2 and 1.3. We shall use the nonlinear capacity method combined with the following pointwise estimate (see Dao and Reissig [4]).

Lemma 2.1. *([4]) Let* $\langle x\rangle = \left(1+|x|^2\right)^{\frac{1}{2}}$. *Let* $s\in(0,1)$ *and* $\chi:\mathbb{R}^n\to\mathbb{R}$ *be the function defined by*

$$\chi(x) = \begin{cases} \langle x\rangle^{-n-2s} & \text{if} \quad |x|\geq 1, \\ \\ 1 & \text{if} \quad |x|\leq 1. \end{cases} \tag{10}$$

Then $\chi\in\mathcal{C}^2(\mathbb{R}^n)$, *and the following estimate holds*

$$|(-\Delta)^s\chi(x)|\leq C\chi(x), x\in\mathbb{R}^n, \tag{11}$$

where C *is a constant independent of* x.

Lemma 2.2. *([4]) Let* $s\in(0,1)$. *Let* ζ *be a smooth function satisfying* $\partial_x^2\zeta\in\mathbb{L}^\infty(\mathbb{R}^n)$. *For any* $R>0$, *let* ζ_R *be a function defined by*

$$\zeta_R(x) = \zeta\left(\frac{x}{R}\right), \quad \text{for all} \quad x\in\mathbb{R}^n.$$

Then, $(-\Delta)^s\zeta_R$ *satisfies the following scaling properties:*

$$(-\Delta)^s(\zeta_R)(x) = R^{-2s}(-\Delta)^s\zeta\left(\frac{x}{R}\right) \quad \text{for all} \quad x\in\mathbb{R}^n.$$

Remark 2.3. Throughout, C denotes a positive constant, whose value may change from line to line.

Proof. [Theorem 1.2] Let u be a global weak solution to (1). First, we introduce the function $\chi=\chi(x)$ as defined in (10) with $s=\frac{\gamma}{2}$ and the function $\xi=\xi(t)$ having the following properties:

1. $\xi\in\mathcal{C}_0^\infty([0,\infty))$ and $\xi(t) = \begin{cases} 1 & \text{if} \quad 0\leq t\leq \frac{1}{2}, \\ \searrow & \text{if} \quad \frac{1}{2}\leq t\leq 1, \\ 0 & \text{if} \quad t\geq 1. \end{cases}$

2. $\xi^{-\frac{p'}{p}}(t)\left(|\xi'(t)|^{p'}+|\xi''(t)|^{p'}+|\xi'''(t)|^{p'}\right)\leq C$ for any $t\in[\frac{1}{2},1]$.

Let R be a large parameter in $[0,\infty)$. We introduce the following test function:

$$\psi_R(x,t)=\xi_R(t)\theta_R(x),$$

where $\xi_R(t)=\xi(R^{-\tilde{\gamma}}t))$ and $\theta_R(x)=\theta(R^{-1}x)$. Moreover, we check easily that $supp(\xi)\subset[0,R^{\tilde{\gamma}}]$. We define the functionals as follows

$$I_1=\int_0^{+\infty}\int_{\mathbb{R}^n}|u(x,t)|^p\psi_R(t,x)dxdt,$$

$$I_2=\int_0^{+\infty}\int_{\mathbb{R}^n}|u(x,t)|^p\partial_t\psi_R(t,x)dxdt.$$

Observing from the exponential memory kernel having the property

$$\frac{\partial}{\partial t}\int_0^t e^{\tau-t}u(\tau,x)d\tau=u(t,x)-\int_0^t e^{\tau-t}u(\tau,x)d\tau. \tag{12}$$

By performing once integration by parts, we obtain

$$\begin{aligned}
I_1-I_2=&\int_0^{+\infty}\int_{\mathbb{R}^n}\Big(u_{tt}(t,x)+(-\Delta)^{\frac{\gamma}{2}}u_t(t,x)+u(t,x)\\
&-\int_0^t e^{\tau-t}u(\tau,x)d\tau\Big)\psi_R(x,t)dxdt\\
&+\int_0^{+\infty}\int_{\mathbb{R}^n}\Big(u_{ttt}(t,x)+(-\Delta)^{\frac{\gamma}{2}}u_{tt}(t,x)+u_t(t,x)\\
&-u(t,x)+\int_0^t e^{\tau-t}u(\tau,x)d\tau\Big)\psi_R(x,t)dxdt\\
&-\int_{\mathbb{R}^n}\Big(u_{tt}(t,x)+(-\Delta)^{\frac{\gamma}{2}}u_t(t,x)+u(t,x)\\
&-\int_0^t e^{\tau-t}u(\tau,x)d\tau\Big)\psi_R(x,t)\Big|_{t=0}^{t=+\infty}dx\\
=&\int_0^{+\infty}\int_{\mathbb{R}^n}\Big(u_{ttt}(t,x)+u_{tt}(t,x)+(-\Delta)^{\frac{\gamma}{2}}u_{tt}(t,x)\\
&+(-\Delta)^{\frac{\gamma}{2}}u_t(t,x)+u_t(t,x)\Big)\psi_R(x,t)dxdt.
\end{aligned}$$

Applying several times integration by parts in the above identity, one has

$$\begin{aligned} I_1 - I_2 &= -\int_{\mathbb{R}^n} u_1(x)\theta_R(x)dx \\ &- \int_0^{+\infty}\int_{\mathbb{R}^n} u(t,x)\left(\partial_t^3\psi_R(x,t) - \partial_t^2\psi_R(x,t) + \partial_t\psi_R(x,t)\right)dxdt \\ &+ \int_0^{+\infty}\int_{\mathbb{R}^n} (-\Delta)^{\frac{\gamma}{2}}u(t,x)\left(\partial_t^2\psi_R(x,t) - \partial_t\psi_R(x,t)\right)dxdt \\ &= -\int_{\mathbb{R}^n} u_1(x)\theta_R(x)dx + J_1 + J_2. \end{aligned} \tag{13}$$

Applying Holder's inequality with $\frac{1}{p} + \frac{1}{p'} = 1$, we can proceed the estimate for J_1 as follows:

$$\begin{aligned} |J_1| &\leq C\int_{\frac{R^{\tilde{\gamma}}}{2}}^{R^{\tilde{\gamma}}}\int_{\mathbb{R}^n} |u(x,t)|\left(|\xi_R'''(t)| + |\xi_R''(t)| + |\xi_R'(t)|\right)\theta_R(x)dxdt \\ &\leq \left(\int_{\frac{R^{\tilde{\gamma}}}{2}}^{R^{\tilde{\gamma}}}\int_{\mathbb{R}^n}\left(|u(x,t)|\psi_R^{\frac{1}{p}}(t,x)\right)^p\right)^{\frac{1}{p}} \times \\ &\left(\int_{\frac{R^{\tilde{\gamma}}}{2}}^{R^{\tilde{\gamma}}}\int_{\mathbb{R}^n}\left((|\xi_R'''(t)| + |\xi_R''(t)| + |\xi_R'(t)|)\theta_R(x)\psi_R^{-\frac{1}{p}}(t,x)\right)^{p'}\right)^{\frac{1}{p'}} \\ &\leq CI_1^{\frac{1}{p}}\left(\int_{\frac{R^{\tilde{\gamma}}}{2}}^{R^{\tilde{\gamma}}}\int_{\mathbb{R}^n}\eta_R^{-\frac{p'}{p}}(t)\left(|\xi_R'''(t)|^{p'} + |\xi_R''(t)|^{p'} + |\xi_R'(t)|^{p'}\right)\theta_R(x)dxdt\right)^{\frac{1}{p'}} \end{aligned}$$

Using change of variables $\tilde{t} = R^{-\tilde{\gamma}}t$ and $\tilde{x} = R^{-1}x$, we get

$$|J_1| \leq CI_1^{\frac{1}{p}}R^{-\tilde{\gamma}+\frac{n+\tilde{\gamma}}{p'}}\left(\int_{\mathbb{R}^n}\langle\tilde{x}\rangle^{-n-\gamma}\right)^{\frac{1}{p'}} \leq CI_1^{\frac{1}{p}}R^{-\tilde{\gamma}+\frac{n+\tilde{\gamma}}{p'}}. \tag{14}$$

Now let us turn to estimate J_2.

$$\begin{aligned} &\int_0^{+\infty}\int_{\mathbb{R}^n}(-\Delta)^{\frac{\gamma}{2}}u(t,x)\left(\partial_t^2\psi_R(x,t) - \partial_t\psi_R(x,t)\right)dxdt = \\ &\int_0^{+\infty}\int_{\mathbb{R}^n} u(t,x)\left((-\Delta)^{\frac{\gamma}{2}}\partial_t^2\psi_R(x,t) - (-\Delta)^{\frac{\gamma}{2}}\partial_t\psi_R(x,t)\right)dxdt \\ &= \int_0^{+\infty}\int_{\mathbb{R}^n} u(t,x)(-\Delta)^{\frac{\gamma}{2}}\theta_R(x)\left(\xi_R''(t) - \xi_R'(t)\right)dxdt. \end{aligned}$$

Applying Holder's inequality again as we estimated J_1 leads to

$$
\begin{aligned}
|J_2| &\leq C I_1^{\frac{1}{p}} \Big(\int_{\frac{R^{\tilde{\gamma}}}{2}}^{R^{\tilde{\gamma}}} \int_{\mathbb{R}^n} \xi_R^{-\frac{p'}{p}}(t) \left(|\xi_R''(t)|^{p'} + |\xi_R'(t)|^{p'} \right) \\
&\quad |(-\Delta)^{\frac{\gamma}{2}} \theta_R(x)|^{p'}) (\theta_R(x))^{-\frac{p'}{p}} dx dt \Big)^{\frac{1}{p'}} \qquad (15) \\
&\leq C I_1^{\frac{1}{p}} R^{-\tilde{\gamma} + \frac{n+\tilde{\gamma}}{p'}} \left(\int_{\mathbb{R}^n} \langle \tilde{x} \rangle^{-n-\gamma} \right)^{\frac{1}{p'}} \leq C I_1^{\frac{1}{p}} R^{-\tilde{\gamma} + \frac{n+\tilde{\gamma}}{p'}}.
\end{aligned}
$$

Combining the estimates from (13) to (15) we may arrive at

$$
I_1 - I_2 + \int_{\mathbb{R}^n} u_1(x) \theta_R(x) dx \leq C I_1^{\frac{1}{p}} R^{-\tilde{\gamma} + \frac{n+\tilde{\gamma}}{p'}}. \qquad (16)
$$

Moreover, it is clear by applying Young's inequality, that

$$
\frac{1}{p'} I_1 - I_2 + \int_{\mathbb{R}^n} u_1(x) \theta_R(x) dx \leq C R^{-\tilde{\gamma} p' + n + \tilde{\gamma}}. \qquad (17)
$$

Due to the setting that the test function $\xi(t)$ is a non-increasing function, one has $\xi'(t) \leq 0$. In other words, it holds that $I_2 \geq 0$. Which follows from (17) that

$$
I_1 \leq C R^{-\tilde{\gamma} p' + n + \tilde{\gamma}}, \qquad (18)
$$

and

$$
\int_{\mathbb{R}^n} u_1(x) \theta_R(x) dx \leq C R^{-\tilde{\gamma} p' + n + \tilde{\gamma}}. \qquad (19)
$$

It is clear that the assumption (6) is equivalent to $-\tilde{\gamma} p' + n + \tilde{\gamma} \leq 0$. For this reason, we will split our consideration into two cases.

Case 1: In the subcritical case $-\tilde{\gamma} p' + n + \tilde{\gamma} < 0$, letting $R \to \infty$ in (19) we easily deduce

$$
\int_{\mathbb{R}^n} u_1(x) dx \leq 0,
$$

which contradicts the assumption (5).

Case 2: For the critical case $-\tilde{\gamma}p' + n + \tilde{\gamma} = 0$, from (18) we can see that $I_1 \leq C$. By using again (16) we may conclude

$$
\begin{aligned}
0 &< \int_{\mathbb{R}^n} u_1(x)\theta_R(x)dx \\
&\leq C\left(\int_{\frac{R^{\tilde{\gamma}}}{2}}^{R^{\tilde{\gamma}}} \int_{\{x\in\mathbb{R}^n;\ |x|\leq R\}} |u(x,t)|^p \psi_R(t,x)dxdt\right)^{\frac{1}{p}} \\
&- \int_0^{R^{\tilde{\gamma}}} \int_{\{x\in\mathbb{R}^n;\ \frac{R}{2}\leq|x|\leq R\}} |u(x,t)|^p \psi_R(t,x)dxdt.
\end{aligned}
\tag{20}
$$

Letting $R \to +\infty$ in (20), we get again a contradiction to the assumption (5). Summarizing, the proof of the Theorem 1.2 is completed. □

Proof.[Theorem 1.3] First, we introduce the same test function as in Theorem 1.1. Let us assume that (u, v) is the global solution to (2). We define the functionals

$$I_1 = \int_0^{+\infty} \int_{\mathbb{R}^n} |u(x,t)|^q \psi_R(t,x)dxdt,$$

$$I_2 = \int_0^{+\infty} \int_{\mathbb{R}^n} |u(x,t)|^q \partial_t\psi_R(t,x)dxdt,$$

$$J_1 = \int_0^{+\infty} \int_{\mathbb{R}^n} |v(x,t)|^p \psi_R(t,x)dxdt,$$

$$J_2 = \int_0^{+\infty} \int_{\mathbb{R}^n} |v(x,t)|^p \partial_t\psi_R(t,x)dxdt.$$

Repeating the steps of the proof from (14) to (20) we may conclude the following estimates:

$$I_1 - I_2 + \int_{\mathbb{R}^n} v_1(x)\theta_R(x)dx \leq CJ_1^{\frac{1}{p}} R^{-\tilde{\gamma}+\frac{n+\tilde{\gamma}}{p'}}. \tag{21}$$

In the analogous way, one obtains

$$J_1 - J_2 + \int_{\mathbb{R}^n} u_1(x)\theta_R(x)dx \leq CI_1^{\frac{1}{q}} R^{-\tilde{\beta}+\frac{n+\tilde{\beta}}{q'}}. \tag{22}$$

From (21) and (22) we obtain

$$I_1^{\frac{pq-1}{pq}} \le R^{\left(-\tilde{\beta}+\frac{n+\tilde{\beta}}{p'}\right)\frac{1}{q}-\tilde{\gamma}+\frac{n+\tilde{\gamma}}{q'}} = R^{\lambda_1}, \tag{23}$$

$$J_1^{\frac{pq-1}{pq}} \le R^{\left(-\tilde{\gamma}+\frac{n+\tilde{\gamma}}{q'}\right)\frac{1}{p}-\tilde{\beta}+\frac{n+\tilde{\beta}}{p'}} = R^{\lambda_2}. \tag{24}$$

It is clear that the assumption (8) is equivalent to $\max\{\lambda_1,\lambda_2\} \le 0$. For this reason, we will split our consideration into two cases.

Case 1: In the subcritical case $\max\{\lambda_1,\lambda_2\} < 0$, letting $R \to \infty$ in (23)and (24) we easily deduce

$$\int_{\mathbb{R}^n} v_1(x)dx \le 0 \quad \text{and} \quad \int_{\mathbb{R}^n} u_1(x)dx \le 0,$$

which contradicts the assumption (7).

Case 2: For the critical case $\max\{\lambda_1,\lambda_2\} = 0$, from (24) we can see that $J_1 \le C$. Using Beppo Levi's theorem on monotone convergence, one obtains

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^n} |v(x,t)|^p dxdt &= \lim_{R\to\infty} \int_0^{R^{\tilde{\gamma}}} \int_{\mathbb{R}^n} |v(x,t)|^p \psi_R(x,t)dxdt \\ &= \lim_{R\to\infty} J_1 \le C.\end{aligned} \tag{25}$$

We deduce easily that

$$\int_{\frac{R^{\tilde{\gamma}}}{2}}^{R^{\tilde{\gamma}}} \int_{\{x\in\mathbb{R}^n;\ \frac{R}{2}\le|x|\le R\}} |v(x,t)|^p \psi_R(x,t)dxdt \to 0 \quad \text{as} \quad R \to +\infty.$$

Repeating the steps of the proof from (16) to (20) we may conclude the following estimates:

$$\begin{aligned}&J_1 - J_2 + \int_{\mathbb{R}^n} u_1(x)\theta_R(x)dx \\ &\quad\le C\left(\int_{\frac{R^{\tilde{\gamma}}}{2}}^{R^{\tilde{\gamma}}} \int_{\{x\in\mathbb{R}^n;\ \frac{R}{2}\le|x|\le R\}} |u(x,t)|^q \psi_R(t,x)dxdt\right)^{\frac{1}{q}} R^{-\tilde{\beta}+\frac{n+\tilde{\beta}}{q'}},\end{aligned}$$

and

$$
\begin{aligned}
& I_1 - I_2 + \int_{\mathbb{R}^n} v_1(x)\theta_R(x)dxdx \\
& \quad \leq C\left(\int_{\frac{R^{\tilde{\gamma}}}{2}}^{R^{\tilde{\gamma}}} \int_{\{x\in\mathbb{R}^n;\ \frac{R}{2}\leq|x|\leq R\}} |v(x,t)|^p \psi_R(t,x)dxdt\right)^{\frac{1}{p}} R^{-\tilde{\gamma}+\frac{n+\tilde{\gamma}}{p'}}.
\end{aligned}
$$

Considering all the above estimates, and since $\max\{\lambda_1,\lambda_2\}=0$ we easily conclude that

$$
\begin{aligned}
& J_1 - J_2 + \int_{\mathbb{R}^n} u_1(x)\theta_R(x)dx \\
& \quad \leq C\left(\int_{\frac{R^{\tilde{\gamma}}}{2}}^{R^{\tilde{\gamma}}} \int_{\{x\in\mathbb{R}^n;\ \frac{R}{2}\leq|x|\leq R\}} |v(x,t)|^p \psi_R(t,x)dxdt\right)^{\frac{1}{pq}}.
\end{aligned}
\tag{26}
$$

Taking the limit as $R\to\infty$ in (26), one obtains

$$
\int_0^{+\infty}\int_{\mathbb{R}^n} |v(x,t)|^q dxdt + \int_{\mathbb{R}^n} u_1(x)\theta_R(x)dx = 0,
$$

which is a contradiction to (7). Let us now consider the case of subcritical exponent to prove the estimate for lifespan T_ε of solutions to (2). We assume that $(u,v)=(u(x,t),v(x,t)$, is a local solution to (2). In order to prove the lifespan estimate, we replace the initial data $(0,u_1,u_2),(0,v_1,v_2)$ by $(0,\varepsilon f_1,\varepsilon f_2),(0,\varepsilon g_1,\varepsilon g_2)$ with a small constant $\varepsilon>0$, where $(f_1,f_2),(g_1,g_2)\in H^1(\mathbb{R}^n)\times\mathbb{L}^2(\mathbb{R}^n)$ satisfy the assumption (7). Repeating the steps in the above proofs we arrive at the following estimate:

$$
I_1 - I_2 + c\varepsilon \leq J_1^{\frac{1}{q}} R^{-\tilde{\gamma}+\frac{n+\tilde{\gamma}}{q'}}, \tag{27}
$$

and

$$
J_1 - J_2 + c\varepsilon \leq I_1^{\frac{1}{p}} R^{-\tilde{\beta}+\frac{n+\tilde{\beta}}{p'}}. \tag{28}
$$

If we plug (27) in (28), we find

$$
J_1 + C\varepsilon \leq C J_1^{\frac{1}{pq}} R^{\left(-\tilde{\beta}+\frac{n+\tilde{\beta}}{p'}\right)+\left(-\tilde{\gamma}+\frac{n+\tilde{\gamma}}{q'}\right)\frac{1}{p}}. \tag{29}
$$

We easily obtains that

$$C\varepsilon \leq CJ_1^{\frac{1}{pq}} R^{\left(-\tilde{\beta}+\frac{n+\tilde{\beta}}{p'}\right)+\left(-\tilde{\gamma}+\frac{n+\tilde{\gamma}}{q'}\right)\frac{1}{p}} - J_1,$$

which leads to

$$\varepsilon \leq CR^{-\left[\frac{\tilde{\gamma}+\tilde{\beta}q}{pq-1}-n\right]}.$$

Let $R = T^{\frac{1}{\tilde{\gamma}}}$, then with a standard calculation, one has

$$T_\varepsilon \leq \varepsilon^{-\frac{\tilde{\gamma}(pq-1)}{\tilde{\gamma}+\tilde{\beta}q-n(pq-1)}}.$$

This completes the proof of Theorem 1.3. $\square$

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