Journal of Mathematical Extension
Vol. 18, No. 8, (2024) (5)1-22
URL: https://doi.org/10.30495/JME.2024.2923
ISSN: 1735-8299
Original Research Paper

The Concept of r-Efficiency and Decomposed Location Problem

D. Foroutannia*
Vali-e-Asr University of Rafsanjan

A. R. Salajegheh
Vali-e-Asr University of Rafsanjan

Abstract. In this paper, we introduce the concept of r-efficiency and classify the weakly efficient solutions by this concept, where $r = 1, 2, \dots, m$ and m is the number of objective functions. Afterward, we decompose the multi-objective optimization problem into the collection of subproblems with cardinality r, for two reasons. First, we apply the scalarization techniques for these subproblems to generate r-efficient solutions. Second, in order to present the new mean-equity models for solving the location problem, we employ the mean and inequality measures on these subproblems. Moreover, by introducing the consistency property for the inequality measures and stating the sufficient conditions to maintain this property, we investigate the relationship between r-efficient solutions of the new mean-equity models and efficient solutions of the location problem.

AMS Subject Classification: 90C29; 90B80; 90B50
Keywords and Phrases: r-Efficiency, Decomposition, Inequality measure, Location, Multi-objective optimization

Received: November 2023; Accepted: December 2024
*Corresponding Author

1 Introduction and Preliminaries

Multi-objective optimization deals with mathematical optimization problems involving more than one objective function to be optimized simultaneously. These problems can be expressed mathematically as:

$$\begin{aligned} &\min\left(f_1(x), f_2(x), \ldots, f_m(x)\right), \\ &\textit{subject to } x \in X \end{aligned} \tag{1}$$

where x denotes a vector of decision variables selected from the feasible set X and $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$ is a vector function that maps the feasible set X into the objective (criterion) space $\mathbb{R}^m$, where $\mathbb{R}^m$ is the Euclidean vector space. We refer to the elements of the objective space as outcome vectors. An outcome vector y is attainable if it expresses outcomes of a feasible solution, i.e., $y = f(x)$ for some $x \in X$. The set of all attainable outcome vectors will be denoted by $Y = f(X)$.

Let $y', y'' \in \mathbb{R}^m$. The notation $y' \leqq y''$ means that $y'_i \leq y''_i$ for $i = 1, \ldots, m$. Moreover, the symbol $y' < y''$ denotes $y'_i < y''_i$ for $i = 1, \ldots, m$, also the notation $y' \leq y''$ denotes $y' \leqq y''$ but $y' \neq y''$. Due to the existence of conflicting objectives, there is not a single optimal solution for a multi-objective problem but rather a set of Pareto optimal solutions exists.

Definition 1.1. *A feasible solution $\hat{x} \in X$ is called efficient or Pareto optimal, if there is no other $x \in X$ such that $f(x) \leq f(\hat{x})$. If $\hat{x}$ is efficient, $f(\hat{x})$ is called nondominated point. The set of all efficient solutions is denoted by X_E and called efficient set, or Pareto optimal set. The set of all nondominated points $\hat{y} = f(\hat{x}) \in Y$, where $\hat{x} \in X_E$, is denoted by Y_N and called the nondominated set.*

Definition 1.2. *A feasible solution $\hat{x} \in X$ is called weakly efficient, if there is no other $x \in X$ such that $f(x) < f(\hat{x})$. The set of all weakly efficient solutions is denoted by X_{WE} and called weakly efficient set. If $\hat{x}$ is weakly efficient, $f(\hat{x})$ is called weakly nondominated. The set of all weakly nondominated points $\hat{y} = f(\hat{x}) \in Y$, where $\hat{x} \in X_{WE}$, is denoted by Y_{WN} and called the weakly nondominated set.*

In practical applications, decision makers are generally more focused on efficient solutions rather than weakly efficient ones. However, there are some reasons why identifying weak efficiency is necessary. For example, the set of efficient solutions is unstable, while the set of weakly efficient solutions is stable. In fact, the limit of a convergent sequence of efficient solutions may not be efficient, but it will always be weakly efficient. Due to this fact, Luc in [7] proposed a method for generating the set of weakly efficient solutions for a nonconvex multi-objective optimization problem.

Scalarization is a common method for solving a multi-objective problem. Scalarizing functions are used to transform a given multi-objective problem into a single-objective optimization problem by aggregating the objectives of a multi-objective problem into a single objective. There are a wide variety of scalarization methods in the literature, such as the weighted sum method, Benson's method, etc. The relationships between optimal solutions of these scalarization problems and (weakly) efficient solutions of multi-objective problems are investigated in [2].

The study tries to identify the elements in $X_{WE} - X_E$, by introducing the concept of r-efficiency, where $r = 1, 2, \dots, m$. We generalize the Pareto preference relation and decompose the multi-objective optimization problem into a collection of subproblems with cardinality r. After that, we apply some scalarization techniques to generate solutions that are r-efficient for these subproblems.

The problem (1) can be considered as the generic location problem from a multi-criteria perspective, where X denotes the feasible set of location patterns (location decisions). There is given a set $M = \{1, 2, \dots, m\}$ of m clients (service recipients). Each client is represented by a specific point in the geographical space. The real value of the function $f_i(x)$ measures the outcome $y_i = f_i(x)$ of the location pattern x for client i. The outcomes can be measured by distance, travel time, the levels of clients dissatisfaction of locations, etc.

The issue of spatial equity in the location of public facilities is both interesting and important. The concept of equity, which implies fairness and justice, is usually quantified with the so-called inequality measures to be minimized. Inequality measures have primarily been studied in the field of economics [15]. Marsh and Schilling in [11] compiled twenty different measures proposed in the literature to gauge the level of equity in facility location alternatives. Among these measures, the simplest ones are based on the absolute measurement of the spread of outcomes, as the maximum absolute difference

$$S(y) = \max_{i,j \in M} |y_i - y_j|, \tag{2}$$

and the mean absolute difference (the Gini's mean difference)

$$D(y) = \frac{1}{2m^2} \sum_{i=1}^{m} \sum_{j=1}^{m} |y_i - y_j|. \tag{3}$$

Many inequality measures related to the deviations from the mean outcome, such as the maximum absolute deviation

$$R(y) = \max_{i \in M} |y_i - \mu(y)|, \tag{4}$$

and the mean absolute deviation

$$\delta(y) = \frac{1}{m} \sum_{i=1}^{m} |y_i - \mu(y)|, \tag{5}$$

where $\mu(y) = \frac{1}{m} \sum_{i=1}^{m} y_i$. The standard deviation

$$\sigma(y) = \sqrt{\frac{1}{m} \sum_{i=1}^{m} (y_i - \mu(y))^2} = \sqrt{\frac{1}{2m^2} \sum_{i=1}^{m} \sum_{j=1}^{m} (y_i - y_j)^2}, \tag{6}$$

and the variance σ^2 consider both the deviations and the spread measurement. Several inequality measures have focused on the upper semideviations from the mean outcome such as the maximum upper semideviation

$$\Delta(y) = \max_{i \in M} (y_i - \mu(y)), \tag{7}$$

and the mean upper semideviation

$$\bar{\delta}(y) = \frac{1}{m} \sum_{i=1}^{m} (y_i - \mu(y))_+, \tag{8}$$

and the standard upper semideviation

$$\overline{\sigma}(y) = \sqrt{\frac{1}{m}\sum_{i=1}^{m}(y_i - \mu(y))_+^2}, \tag{9}$$

where $(.)_+$ denotes the non-negative part of a number. It is important to note that the inequality measures used in economics are typically normalized by dividing by the mean outcome. One typical example of a relative inequality measure is the Gini coefficient $D(y)/\mu(y)$, which has been analyzed in the context of location. For more details on the inequality measures in location problems, we refer the reader to [5, 8, 9, 12, 13].

Let us denote an arbitrary inequality measure by the symbol ρ. It can be easily verified that directly minimizing the inequality measures, i.e.

$$\begin{aligned} &\min \rho(f(x)),\\ &\textit{subject to } x \in X, \end{aligned}$$

contradicts the optimization of individual outcomes. To overcome this flaw, Mandell [10] introduced the following bicriteria mean-equity model

$$\begin{aligned} &\min \left(\mu(f(x)), \rho(f(x))\right),\\ &\textit{subject to } x \in X. \end{aligned}$$

While this model takes into account both efficiency by minimizing the mean outcome $\mu(f(x))$ and equity through the minimization of an inequality measure $\rho(f(x))$, it does not entirely resolve contradictions related to minimizing individual outcomes. Ogryczak [14] used the idea of combining the inequality measures with the mean itself into optimization criteria and proposed the following bicriteria problem

$$\begin{aligned} &\min \left(\mu(f(x)), \mu(f(x)) + \rho(f(x))\right),\\ &\textit{subject to } x \in X. \end{aligned} \tag{10}$$

The model (10) effectively resolves the contradiction in minimizing individual outcomes, while maintaining consistency with both inequality minimization and the minimization of distances. Moreover, Ogryczak introduced the concept of equitably consistent and stated sufficient conditions for the inequality measures to keep this concept. Through this concept, he demonstrated that every efficient solution of the bicriteria problem (10) is an equitably efficient location.

Compared to the study by Ogryczak [14], we decompose the location problem into a collection of subproblems with cardinality r. We then apply mean and inequality measures to these subproblems and present new models for mean-equity. Additionally, we present the consistency property for inequality measures and examine the relationship between r-efficient solutions of the new mean-equity models and efficient solutions of the location problem.

The paper is organized as follows. In Section 2, we generalize the Pareto preference relation to define the concept of r-efficiency for $r = 1, 2, \ldots, m$, and we investigate the relationships among them. Additionally, some scalarization techniques have been developed to generate r-efficient solutions. In Section 3, the concept of consistency is introduced, and sufficient conditions are presented for the inequality measures to maintain this consistency property. Finally, the last section presents some conclusions.

2 The Concept of r-Efficiency

Let $M = \{1, 2, \ldots, m\}$ and $r \in M$. The following notations and definitions are useful in this text.

$$\mathbb{R}^m_{\geq_r} = \{d \in \mathbb{R}^m \colon d_j \geqslant 0 \textit{ for all } j \in M \textit{ and } d_{j_k} > 0 \textit{ for some } j_1, \ldots, j_r \in M\},$$
$$\mathbb{R}^m_{\geqq} = \{d \in \mathbb{R}^m \colon d_j \geqslant 0 \textit{ for all } j \in M\}.$$

Definition 2.1. *Let $y', y'' \in Y$. We say that y' r-dominates y'' and write $y' <_r y''$ if and only if $y'_j \leqslant y''_j$ for all $j \in M$ and there exist $j_1, j_2, \ldots, j_r \in M$ such that $y'_{j_k} < y''_{j_k}$ for $k = 1, 2, \ldots, r$. A feasible solution $\hat{x} \in X$ is called r-efficient, if there is no other $x \in X$ such that $f(x) <_r f(\hat{x})$. If $\hat{x}$ is an r-efficient solution, $f(\hat{x})$ is called an r-nondominated point. The set of all r-efficient solutions and the set of all r-nondominated points are denoted by X_{rE} and Y_{rN}, respectively. These sets are called the r-efficient and r-nondominated sets, respectively.*

Recently, Huerga et al. introduced the general concept of quasi-efficiency that unifies the most well-known notions of efficiency in multi-objective optimization, [6]. The concept of r-efficiency can be seen as a special case of (C, h)-quasi efficiency when $C = \mathbb{R}^m_{\geq_r}$ and h is the constant function of 1. For more details, the reader can refer to Definition 2 from [6].

The concept of r-efficiency becomes the efficiency and weak efficiency concepts, when $r = 1$ and $r = m$, respectively. Hence $X_{1E} = X_E$ and $X_{mE} = X_{WE}$. It is worth to mention the feasible solution $\hat{x} \in X$ is r-efficient if and only if

$$(f(\hat{x}) - \mathbb{R}^m_{\geq_r}) \cap f(X) = \emptyset.$$

The inclusion relations between the sets of X_{rE}, X_{sE} are as follows.

Proposition 2.2. *If $r, s \in M$ and $r \leqslant s$, then $X_E \subset X_{rE} \subset X_{sE} \subset X_{WE}$. In particular, we have*

$$X_E \subset X_{2E} \subset \ldots \subset X_{(m-1)E} \subset X_{WE}.$$

Proof. By Definition 2.1, the proof is trivial. □

The following example is given to illustrate Proposition 2.2.

Example 2.3. Let

$$X = [0, 1] \times [0, 1] \times [0, 1],$$

and $f(x) = x$ and $Y = X$. Since

$$\mathbb{R}^3_{\geq_2} = \mathbb{R}^3_{\geqq} - \{\{(d_1, 0, 0) \colon d_1 \geqslant 0\} \cup \{(0, d_2, 0) \colon d_2 \geqslant 0\} \cup \{(0, 0, d_3) \colon d_3 \geqslant 0\}\},$$

we obtain

$$\begin{aligned}
X_{1E} &= X_E = \{(0, 0, 0)\}, \\
X_{2E} &= \{(x_1, 0, 0) \colon 0 \leqslant x_1 \leqslant 1\} \cup \{(0, x_2, 0) \colon 0 \leqslant x_2 \leqslant 1\} \\
&\quad \cup \{(0, 0, x_3) \colon 0 \leqslant x_3 \leqslant 1\}, \\
X_{3E} &= X_{WE} = \{(x_1, x_2, 0) \colon 0 \leqslant x_1, x_2 \leqslant 1\} \\
&\quad \cup \{(x_1, 0, x_3) \colon 0 \leqslant x_1, x_3 \leqslant 1\} \cup \{(0, x_2, x_3) \colon 0 \leqslant x_2, x_3 \leqslant 1\}.
\end{aligned}$$

Thus $X_{1E} \subset X_{2E} \subset X_{3E}$.

Throughout this paper, we assume that $R_1, R_2, \ldots, R_\alpha$ are the all subsets of M with cardinality r, i.e. $|R_k| = r$ for $k = 1, 2, \ldots, \alpha$, where $\alpha = \binom{m}{r}$. The collection $\{R_k \subset M\colon k = 1, 2, \ldots, \alpha\}$ is called a decomposition of M and the collection of all subproblems

$$\begin{aligned} &\min f_{R_k}(x) = (f_j(x))_{j\in R_k}, \qquad (k = 1, 2, \ldots, \alpha) \\ &\textit{subject to } x \in X, \end{aligned}$$

is called the decomposed multi-objective optimization problem. The decomposition and coordination methods are considered in several studies e.g., [3, 4]. In these studies, the relationships between efficient solutions of the subproblems and the original problem are discussed. Also, the original optimization problem is decomposed into a number of single-objective problems by scalarizing functions.

In general, scalarization means converting a multi-objective optimization problem into a suitable single optimization problem. In the following, we offer to the decision maker to apply his favorite scalarization functions for $f_{R_k}(x) = (f_j(x))_{j\in R_k}$ and introduce the problem

$$\min_{x\in X} \left(s_1(f_{R_1}(x)), s_2(f_{R_2}(x)), \ldots, s_\alpha(f_{R_\alpha}(x))\right), \tag{11}$$

where $s_i : \mathbb{R}^r \to \mathbb{R}$ $(i = 1, 2, \ldots, \alpha)$ is a scalarization function. For $\alpha = 1$ or $r = m$, the problem (11) becomes the problem

$$\min_{x\in X} s(f(x)). \tag{12}$$

We define the particular classes of scalarization functions and investigate the relationship between the efficient solutions of the problem (11) and the r-efficient solutions of the original problem.

Definition 2.4. *Let $s : \mathbb{R}^k \to \mathbb{R}$ be a scalarization function and $y^1, y^2 \in \mathbb{R}^k$.*

1. *We say that the scalarization function s is increasing, if*

$$y^1 \leqq y^2 \Longrightarrow s(y^1) \leqslant s(y^2).$$

2. *An increasing scalarization function s is called strictly increasing, if*

$$y^1 < y^2 \Longrightarrow s(y^1) < s(y^2).$$

3. *An increasing scalarization function s is called strongly increasing, if*

$$y^1 \leq y^2 \Longrightarrow s(y^1) < s(y^2).$$

Proposition 2.5. *(i) Suppose that the functions s_k are strictly increasing and $\hat{x} \in X$. If $\hat{x}$ is an efficient solution of the problem (11), then it is an r-efficient of the problem (1). In particular, every optimal solution of the problem (12) is a weakly efficient solution of the problem (1).*

(ii) Suppose that the functions s_k are strongly increasing and $\hat{x} \in X$. If $\hat{x}$ is an β-efficient solution of the problem (11), then it is a β-efficient of the problem (1), where $\beta = \binom{m-1}{r-1}$. In particular, every optimal solution of the problem (12) is an efficient solution of the problem (1).

Proof. (i) Let $\hat{x}$ be an efficient solution of the problem (11) and $\hat{x} \notin X_{rE}$. There are some $x \in X$ such that $f_k(x) \leqslant f_k(\hat{x})$ for all $k \in R_i$, $i = 1, 2, \ldots, \alpha$ and there exists $i' \in \{1, 2, \ldots, \alpha\}$ such that $f_k(x) < f_k(\hat{x})$ for all $k \in R_{i'}$. Since the functions s_i are strictly increasing, we have

$$s_i(f_{R_i}(x)) \leqslant s_i(f_{R_i}(\hat{x})), \quad \textit{and} \quad s_i(f_{R_{i'}}(x)) < s_i(f_{R_{i'}}(\hat{x})),$$

but this contradicts the assumption that $\hat{x}$ is an efficient solution of the problem (11).

(ii) Let $\hat{x} \in X$ be a β-efficient solution of the problem (11). If $\hat{x} \notin X_E$, then there is a feasible solution $x \in X$ such that $f_i(x) \leqslant f_i(\hat{x})$ for each $i \in M$ and $f_{i'}(x) < f_{i'}(\hat{x})$ for some $i' \in M$. Since $R_1, R_2, \ldots, R_\alpha$ are the all subsets of M with $|R_k| = r$ for $k = 1, 2, \ldots, \alpha$, We can easily conclude that $i' \in R_k$ for $k = i_1, i_2, \ldots, i_\beta$, where $\beta = \binom{m-1}{r-1}$. Now, Since the functions s_i are strongly increasing, we obtain

$$s_j(f_{R_j}(x)) \leqslant s_j(f_{R_j}(\hat{x})) \quad (j = 1, 2, \ldots, \alpha),$$
$$s_{i_k}(f_{R_{i_k}}(x)) < s_{i_k}(f_{R_{i_k}}(\hat{x})) \quad (k = 1, 2, \ldots, \beta),$$

which contradicts $\hat{x} \in X_{\beta E}$. $\square$

By applying the weighting method to problem (11), we get the following problem

$$\min_{x\in X} \sum_{k=1}^{\alpha} \lambda_k s_k(f_{R_k}(x)). \tag{13}$$

Since every optimal solution of the problem (13) is an efficient solution of the problem (11) and $X_E \subset X_{\beta E}$, the following result is concluded directly from the above proposition, where $\beta = \binom{m-1}{r-1}$.

Corollary 2.6. *Let $\lambda_k > 0$ for $k = 1, 2, \ldots, \alpha$ and $\hat{x} \in X$ be an optimal solution of the problem (13).*

(i) If the functions s_k are strictly increasing, then $\hat{x}$ is an r-efficient solution of the original problem.

(ii) If the functions s_k are strongly increasing, then $\hat{x}$ is an efficient solution of the original problem.

One of the commonly used scalarization techniques is Benson's Scalarization method, [1]. Similar to Benson's method, we use an initial feasible solution $x^\circ \in X$ and nonnegative deviation variables $l_i = f_i(x^\circ) - f_i(x)$. We will now introduce the following problem

$$\begin{aligned} &\max \left\{ \sum_{k=1}^{\alpha} \min_{i\in R_k} l_i \right\}, \\ &\textit{subject to} \\ &l_i = f_i(x^\circ) - f_i(x) \geqslant 0 \quad (i = 1, 2, \ldots, m), \\ &x \in X. \end{aligned} \tag{14}$$

which is converted to Benson's method, for $r = 1$ or $\alpha = m$.

Proposition 2.7. *The feasible solution $x^\circ \in X$ is r-efficient if and only if the optimal objective value of the problem (14) is* 0.

Proof. We have $x^\circ \notin X_{rE}$, if and only if there are a feasible solution (x, l) of (14) and a subset $R_k \subset M$ such that $l_i \geqslant 0$ for all $i \in M$ and

$l_i > 0$ for all $i \in R_k$. Hence

$$\sum_{j=1}^{\alpha} \min_{i \in R_j} l_i > 0,$$

which implies that the optimal objective value of the problem (14) is greater than zero. $\square$

Proposition 2.8. *If problem (14) has an optimal solution $(\hat{x}, \hat{l})$ and the optimal objective value is finite, then $\hat{x} \in X_{rE}$.*

Proof. If $\hat{x} \notin X_{rE}$, then there are some $x' \in X$ and $k' \in \{1, 2, \ldots, \alpha\}$ such that $f_i(x') \leqslant f_i(\hat{x})$ for all $i \in R_k$, $k = 1, 2, \ldots, \alpha$ and $f_i(x') < f_i(\hat{x})$ for all $i \in R_{k'}$. We define $l'_i = f_i(x^\circ) - f_i(x')$. Then (x', l') is a feasible solution of (14) because

$$l'_i = f_i(x^\circ) - f_i(x') \geqslant f_i(x^\circ) - f_i(\hat{x}) = \hat{l}_i \geqslant 0.$$

Furthermore, $\sum_{k=1}^{\alpha} \min_{i \in R_k} l'_i > \sum_{k=1}^{\alpha} \min_{i \in R_k} \hat{l}_i$. This is impossible because $(\hat{x}, \hat{l})$ is an optimal solution of (14). $\square$

3 Inequality Measures and Efficient Locations

According to the conventions of the previous section, let $R_1, R_2, \ldots, R_\alpha$ be all subsets of M with $|R_k| = r$ and $f_{R_k}(x) = (f_i(x))_{i \in R_k}$, for $k = 1, 2, \ldots, \alpha$. In this section, we are interested in the issue of equity by minimization of the inequality measures of subproblems. For this purpose, we decompose the original problem into α subproblems with cardinality r, and introduce the following problem

$$\begin{aligned} &\min \left(\rho(f_{R_1}(x)), \rho(f_{R_2}(x)), \ldots, \rho(f_{R_\alpha}(x))\right), \\ &\textit{subject to } x \in X. \end{aligned} \tag{15}$$

Note that by considering ρ as a scalarization function, the above problem becomes the problem (11). Unfortunately, we can easily verify that the minimization of (15) contradicts the minimization of individual outcomes in (1). This can be illustrated by the simple example of a discrete location problem.

Example 3.1. Let us consider a single facility location problem with three clients (C_1, C_2 and C_3) and three potential locations (P_1, P_2 and P_3). Assume that

$$C_1 = (6, 8), C_2 = (7.2, 6.94), C_3 = (10, 0), P_1 = (7.2, 9.6), P_2 = (4, 8), P_3 = (0, 0).$$

represent the position of clients and potential locations in the Cartesian coordinate system. The distances between several clients and potential locations, in terms of kilometers, are as follows:

	C_1	C_2	C_3
P_1	2	2.66	10
P_2	2	$\sqrt{11.3636}$	10
P_3	10	10	10

Hence, the potential locations generate the outcome vectors $y^1 = (2, 2.66, 10)$, $y^2 = (2, \sqrt{11.3636}, 10)$ and $y^3 = (10, 10, 10)$, respectively. Note that $y^1 \leq y^2 \leq y^3$, $y^1 \leq_2 y^3$ and $y^2 \leq_2 y^3$, so $Y_N = \{y^1\}$ and $Y_{2N} = \{y^1, y^2\}$. Since

$$\begin{aligned} &y^1_{R_1} = (2, 2.66), y^1_{R_2} = (2, 10), y^1_{R_3} = (2.66, 10), \\ &y^2_{R_1} = (2, \sqrt{11.3636}), y^2_{R_2} = (2, 10), y^2_{R_3} = (\sqrt{11.3636}, 10), \\ &y^3_{R_1} = (10, 10), y^3_{R_2} = (10, 10), y^3_{R_3} = (10, 10), \end{aligned}$$

we obtain $\rho(y^i_{R_k}) > 0$ $(i = 1, 2)$ and $\rho(y^3_{R_k}) = 0$ $(k = 1, 2, 3)$ for any inequality measures ρ defined by (2)-(9). Hence, the third location pattern, y^3, is nondominated and 2-nondominated for the problem (15).

Similar to the idea proposed by Ogryczak in [14] for equitable efficiency, in order to overcome the flaws of direct minimization of inequality measures of subproblems, we present the following problem

$$\begin{aligned} &\min \left(\mu(f(x)), \mu(f_{R_1}(x)) + \rho(f_{R_1}(x)), \ldots, \mu(f_{R_\alpha}(x)) + \rho(f_{R_\alpha}(x))\right), \\ &\textit{subject to } x \in X. \end{aligned} \tag{16}$$

The model takes into account both the efficiency with minimization of the mean outcome $\mu(f(x))$ and the equity with minimization of the sum of the mean outcome and the inequality measure for all subproblems with cardinality r, i.e. $\mu(f_{R_k}(x)) + \rho(f_{R_k}(x))$, for $k = 1, 2, \ldots, \alpha$.

For $\alpha = 1$, (16) becomes the model (10) which is introduced by Ogryczak, [14]. Ogryczak worked on location problems and developed bicriteria mean-equity models as simplified approaches. These models deal with the equity concern by adapting the inequality measures to the location framework and trying to minimize them. Also, he discussed different ways to find efficient solutions to these bicriteria models.

In the following, we state the consistency concept for $\rho(y)$ whereby inequality measures can be used together with the means in the optimization problem (16) to maintain the r-efficiency of selected locations.

Definition 3.2. *1. The inequality measure $\rho(y)$ is called mean-complementary consistent, if*

$$y' \leqq y'' \Longrightarrow \mu(y') + \rho(y') \leqslant \mu(y'') + \rho(y''). \tag{17}$$

2. The inequality measure $\rho(y)$ is called mean-complementary strongly consistent if, in addition to (17), the following relation holds

$$y' \leq y'' \Longrightarrow \mu(y') + \rho(y') < \mu(y'') + \rho(y'').$$

To simplify, we use the words "consistent" and "strongly consistent" instead of "mean-complementary consistent" and "mean-complementary strongly consistent", respectively.

Theorem 3.3. *(i) If the inequality measure $\rho(y)$ is consistent, then every efficient solution of the problem (16) is an efficient location. In particular, every efficient solution of the problem (10) is an efficient location.*

(ii) If the inequality measure $\rho(y)$ is strongly consistent, then every β-efficient solution of the problem (16) is an efficient location, where $\beta = \frac{r\alpha}{m} + 1$. In particular, every weakly efficient solution of the problem (10) is an efficient location.

Proof. (i) Suppose that $\hat{x} \in X$ is an efficient solution of the problem (16) and it is not an efficient location. There exists a feasible solution $x \in X$ such that $f(x) \leq f(\hat{x})$, so $\mu(f(x)) < \mu(f(\hat{x}))$ and $\mu(f_{R_k}(x)) + \rho(f_{R_k}(x)) \leqslant \mu(f_{R_k}(\hat{x})) + \rho(f_{R_k}(\hat{x}))$ for $k = 1, 2, \ldots, \alpha$. Hence $\hat{x}$ cannot be an efficient solution of the problem (16), which is a contradiction.

(*ii*) Let $\hat{x} \in X$ be a β-efficient solution of the problem (16). If $\hat{x} \notin X_E$, then there is a feasible solution $x \in X$ such that $f_i(x) \leqslant f_i(\hat{x})$ for each $i \in M$ and $f_{i'}(x) < f_{i'}(\hat{x})$ for some $i' \in M$. Since $R_1, R_2, \ldots, R_\alpha$ are the all subsets of M with $|R_k| = r$ for $k = 1, 2, \ldots, \alpha$, We can easily conclude that $i' \in R_k$ for $k = i_1, i_2, \ldots, i_\beta$, where $\beta_1 = \binom{m-1}{r-1}$. Hence the property of strongly consistent implies that

$$\mu(f_{R_{i_k}}(x)) + \rho(f_{R_{i_k}}(x)) < \mu(f_{R_{i_k}}(\hat{x})) + \rho(f_{R_{i_k}}(\hat{x})),$$

for $k = 1, 2, \ldots, \beta_1$. On the other hand, we have $\mu(f(x)) < \mu(f(\hat{x}))$ and $\beta = \beta_1 + 1$. Thus $\hat{x}$ cannot be a β-efficient solution of (16), which it is a contradiction.

□

Since $\sum_{k=1}^{\alpha} \mu(f_{R_k}(x)) = \alpha\mu(f(x))$, by summing from the second criteria onwards in the problem (16), we obtain the bicriteria problem

$$\min \left(\mu(f(x)), \mu(f(x)) + \frac{1}{\alpha} \sum_{k=1}^{\alpha} \rho(f_{R_k}(x)) \right), \quad \textit{subject to } x \in X. \tag{18}$$

It should be noted that, the problem(18) is converted to the problem (10), when $\alpha = 1$. In addition, one can easily show that every (weakly) efficient solution of the problem (18) is (weakly) efficient for the problem (16). Therefore, the following corollary holds.

Corollary 3.4. *(i) If the inequality measure $\rho(y)$ is consistent, then every efficient solution of the bicriteria problem (18) is an efficient location.*

(ii) If the inequality measure $\rho(y)$ is strongly consistent, then every weakly efficient solution of the bicriteria problem (18) is an efficient location.

One of the important advantages of the mean-equity approach (18) is the possibility of trade-off analysis. Let $0 < \lambda < 1$ be the trade-off coefficient between the mean outcome and the mean of inequality measures $\rho(y_{R_k})$, i.e. $\frac{1}{\alpha} \sum_{k=1}^{\alpha} \rho(f_{R_k}(x))$. By

$$(1 - \lambda)\mu(y) + \lambda \left(\mu(y) + \frac{1}{\alpha} \sum_{k=1}^{\alpha} \rho(y_{R_k}) \right) = \mu(y) + \frac{\lambda}{\alpha} \sum_{k=1}^{\alpha} \rho(y_{R_k}), \tag{19}$$

we can directly compare real values of $\mu(y) + \frac{\lambda}{\alpha}\sum_{k=1}^{\alpha}\rho(y_{R_k})$. The relation (19) and Corollary 3.3 allow us to express the following assertion.

Corollary 3.5. *If the inequality measure $\rho(y)$ is consistent and $0 < \lambda < 1$, then every optimal solution of the problem*

$$\min\left\{\mu(f(x)) + \frac{\lambda}{\alpha}\sum_{k=1}^{\alpha}\rho(f_{R_k}(x)) \colon x \in X\right\}. \tag{20}$$

is an efficient location.

Ogryczak [14] introduced sufficient conditions for the inequality measures to keep equitable consistency property. In continuation, we will recall some of these conditions which guarantee the property of consistency.

Definition 3.6. *(i) We say the inequality measure ρ is convex iff*

$$\rho(\lambda y' + (1-\lambda)y'') \leqslant \lambda\rho(y') + (1-\lambda)\rho(y''),$$

for any $y', y'' \in \mathbb{R}^m$ and $0 \leqslant \lambda \leqslant 1$.

(ii) The inequality measure ρ is positively homogeneous iff $\rho(\lambda y) = \lambda\rho(y)$ for positive real number λ.

(iii) We say that inequality measure $\rho(y) \geqslant 0$ is Δ-bounded iff $\rho(y) \leqslant \Delta(y)$ for any $y \in \mathbb{R}^m$. This means that $\rho(y)$ is upper bounded by the maximum upper deviation. Moreover, we say that $\rho(y) \geqslant 0$ is strictly Δ-bounded if $\rho(y) < \Delta(y)$ for any y with $\Delta(y) > 0$.

Proposition 3.7. *Let $\rho(y) \geqslant 0$ be a convex, positively homogeneous inequality measure.*

(i) If ρ is Δ-bounded, then it is consistent.

(ii) If ρ is strictly Δ-bounded, then it is strongly consistent.

Proof. The proof follows from Theorem 4 of [14]. $\square$

By applying Theorem 3.3 and Proposition 3.7, the next assertion is valid.

Remark 3.8. Let $\rho(y) \geqslant 0$ be a convex, positively homogeneous inequality measure. If the measure ρ is Δ-bounded(or strictly Δ-bounded), then the results obtained in Theorem 3.3 and Corollaries 3.4, 3.5 are valid.

It can be easily checked that the typical inequality measures (2)-(9) are convex and positively homogeneous. As discussed in [14], we have

$$D(y) = \frac{1}{m^2}\sum_{i=1}^{m}\sum_{j=1}^{m}(\max\{y_i, y_j\} - \mu(y)) \leqslant \Delta(y), \tag{21}$$

$$\overline{\sigma}(y) \leqslant \sqrt{\Delta(y)^2} = \Delta(y), \tag{22}$$

$$\overline{\delta}(y) \leqslant \frac{1}{m}\sum_{i=1}^{m}\Delta(y) = \Delta(y). \tag{23}$$

The first equality holds because $|y_i - y_j| = 2\max\{y_i, y_j\} - y_i - y_j$. Hence, the measures D, $\bar{\sigma}$ and $\overline{\delta}$ are Δ-bounded. For any outcome vector y with $\Delta(y) > 0$, it concludes that at least one outcome y_i must be below the mean. Thus, we can deduce that the above inequalities are strict. This means that the above inequality measures are strictly Δ-bounded.

It is also obvious that the maximum absolute upper deviation $\Delta(y)$ is Δ-bounded but it is not strictly Δ-bounded. On the other hand, again by [14], we have

$$\frac{1}{m}S(y) \leqslant \Delta(y), \tag{24}$$

$$\frac{1}{m-1}R(y) \leqslant \Delta(y), \tag{25}$$

$$\frac{1}{2}\delta(y) \leqslant \Delta(y), \tag{26}$$

$$\frac{1}{\sqrt{m-1}}\sigma(y) \leqslant \Delta(y). \tag{27}$$

Thus, the measures S, R, δ and σ are not Δ-bounded, and thereby the validity of Remark 3.8 is questionable. To overcome this problem, we use the concept of consistency for the inequality measure $\xi\rho(y)$ instead of $\rho(y)$ in problem (16). Therefore, we obtain

$$\min\left(\mu(f(x)), \mu(f_{R_1}(x)) + \xi\rho(f_{R_1}(x)), \ldots, \mu(f_{R_\alpha}(x)) + \xi\rho(f_{R_\alpha}(x))\right), \quad \textit{subject to } x \in X. \tag{28}$$

Definition 3.9. *We say that inequality measure $\rho(y)$ is ξ-consistent if*

$$y' \leqq y'' \Longrightarrow \mu(y') + \xi\rho(y') \leqslant \mu(y'') + \xi\rho(y''). \tag{29}$$

Also, we say that inequality measure $\rho(y)$ is strongly ξ-consistent if, in addition to (29), the following holds

$$y' \leq y'' \Longrightarrow \mu(y') + \xi\rho(y') < \mu(y'') + \xi\rho(y'').$$

Note that the concept of ξ-consistency becomes consistency when $\xi = 1$. Moreover, the inequality measure $\rho(y)$ is ξ-consistent if and only if the inequality measure $\xi\rho(y)$ is consistent. By Theorem 3.3, the next assertion is true.

Corollary 3.10. *(i) If the inequality measure $\rho(y)$ is ξ-consistent, then every efficient solution of the problem (28) is an efficient location.*

(ii) If the inequality measure $\rho(y)$ is strongly ξ-consistent, then every β-efficient solution of the problem (28) is an efficient location, where $\beta = \frac{r\alpha}{m} + 1$.

Similar to the problem (18), by summing from the second criteria onwards in the problem (28), we obtain the bicriteria problem

$$\begin{aligned} &\min \left(\mu(f(x)), \mu(f(x)) + \frac{\xi}{\alpha} \sum_{k=1}^{\alpha} \rho(f_{R_k}(x)) \right), \\ &\textit{subject to } x \in X. \end{aligned} \tag{30}$$

It is obvious that every (weakly) efficient solution of the problem (30) is (weakly) efficient for the problem (28).

Corollary 3.11. *(i) If the inequality measure $\rho(y)$ is ξ-consistent, then every efficient solution of the bicriteria problem (30) is an efficient location.*

(ii) If the inequality measure $\rho(y)$ is strongly ξ-consistent, then every weakly efficient solution of the bicriteria problem (30) is an efficient location.

For $0 < \lambda < \xi$, this fact that

$$(1-\frac{\lambda}{\xi})\mu(y)+\frac{\lambda}{\xi}\left(\mu(y)+\frac{\xi}{\alpha}\sum_{k=1}^{\alpha}\rho(y_{R_k})\right)=\mu(y)+\frac{\lambda}{\alpha}\sum_{k=1}^{\alpha}\rho(y_{R_k}),$$

leads us to the next result.

Corollary 3.12. *If the inequality measure $\rho(y)$ is ξ-consistent and $0 < \lambda < \xi$, then every optimal solution of the problem (20) is an efficient location.*

For $\xi > 0$, the inequality measure $\xi\rho(y)$ satisfies the convexity and positive homogeneity conditions if these conditions hold for $\rho(y)$. By applying Proposition 3.7 for $\xi\rho(y)$, we obtain sufficient conditions for inequality measures to keep the property of ξ-consistency.

Corollary 3.13. *Let $\rho(y) \geqslant 0$ be a convex, positively homogeneous inequality measure.*

(i) If the measure $\xi\rho(y)$ is Δ-bounded, then it is ξ-consistent.

(ii) If the measure $\xi\rho(y)$ is strictly Δ-bounded, then it is strongly ξ-consistent

According to the above corollary, the condition Δ-bounded of the measure $\xi\rho(y)$ is one of the sufficient conditions for ξ-consistency. Hence, by inequalities (21)-(27), we can determine the intervals of ξ-consistency for typical inequality measures. For example, inequality (21) implies that $\xi S(y) \leqslant \Delta(y)$, when $0 < \xi \leqslant \frac{1}{m}$. Therefore, the interval of ξ-consistency for the maximum absolute difference is equal to $(0, \frac{1}{m}]$. The consistency results for typical inequality measures (2)-(9) are summarized in Table 1. To illustrate further these results, let us consider Example 3.1. Recall that $Y_N = \{y^1\}$ and the outcome vector y^1 is a non-dominated point of the location problem. We have calculated the values of inequality measures in Table 2 for y^1, y^2, y^3, and $y^i_{R_k}$ for $i, k = 1, 2, 3$.

Table 1: consistency results

Measure			ξ-consistency	interval of ξ-consistency
Maximum absolute difference	$S(y)$	(2)	$\frac{1}{m}$	$(0, \frac{1}{m}]$
Mean absolute difference	$D(y)$	(3)	1	$(0, 1]$
Maximum absolute deviation	$R(y)$	(4)	$\frac{1}{m-1}$	$(0, \frac{1}{m-1}]$
Mean absolute deviation	$\delta(y)$	(5)	$\frac{1}{2}$	$(0, \frac{1}{2}]$
Standard deviation	$\sigma(y)$	(6)	$\frac{1}{\sqrt{m-1}}$	$(0, \frac{1}{\sqrt{m-1}}]$
Maximum upper semideviation	$\Delta(y)$	(7)	1	$(0, 1]$
Mean absolute semideviation	$\bar{\delta}(y)$	(8)	1	$(0, 1]$
Standard upper semideviation	$\bar{\sigma}(y)$	(9)	1	$(0, 1]$

Table 2: Values of inequality measures for Example 3.1

y	$\mu(y)$	$s(y)$	$D(y)$	$R(y)$	$\delta(y)$	$\sigma(y)$	$\Delta(y)$	$\bar{\delta}(y)$	$\bar{\sigma}(y)$
y^1	4.887	8	1.777	5.113	3.409	3.626	5.113	1.704	2.951
$y^1_{R_1}$	2.33	0.66	0.165	0.33	0.33	0.33	0.33	0.165	0.233
$y^1_{R_2}$	6	8	2	4	4	4	4	2	2.828
$y^1_{R_3}$	6.33	7.34	1.835	3.67	3.67	3.67	3.67	1.835	2.594
y^2	5.124	8	1.777	4.876	3.251	3.508	4.876	1.625	2.815
$y^2_{R_1}$	2.685	1.371	0.343	0.685	0.685	0.685	0.685	0.343	0.485
$y^2_{R_2}$	6	8	2	4	4	4	4	2	2.828
$y^2_{R_3}$	6.685	6.69	1.675	3.314	3.314	3.314	3.314	1.657	2.344
y^3	10	0	0	0	0	0	0	0	0
$y^3_{R_1}$	10	0	0	0	0	0	0	0	0
$y^3_{R_2}$	10	0	0	0	0	0	0	0	0
$y^3_{R_3}$	10	0	0	0	0	0	0	0	0

Since the inequality measures $D, \overline{\sigma}, \overline{\delta}$ are strongly consistent, one can easily check that

$$\begin{aligned}&\left(\mu(y^1), \mu(y^1_{R_1}) + \rho(y^1_{R_k}), \mu(y^1_{R_2}) + \rho(y^1_{R_2}), \mu(y^1_{R_3}) + \rho(y^1_{R_3})\right)\\ &\leq_3 \left(\mu(y^2), \mu(y^2_{R_1}) + \rho(y^2_{R_k}), \mu(y^2_{R_2}) + \rho(y^2_{R_2}), \mu(y^2_{R_3}) + \rho(y^2_{R_3})\right)\\ &\leq_3 \left(\mu(y^3), \mu(y^3_{R_1}) + \rho(y^3_{R_k}), \mu(y^3_{R_2}) + \rho(y^3_{R_2}), \mu(y^3_{R_3}) + \rho(y^3_{R_3})\right),\end{aligned}$$

for these inequality measures. Thus y^1 is a 3-nondominated point of the problem (16), which confirms the validity of Theorem 3.3. On the other hand, the outcome vectors y^2, y^3 are two nondominated points of the problem

$$\begin{aligned}&\min\left(\mu(f(x)), \mu(f_{R_1}(x)) + s(f_{R_1}(x)), \mu(f_{R_2}(x)) + s(f_{R_2}(x)), \mu(f_{R_3}(x)) + s(f_{R_3}(x))\right),\\ &\textit{subject to } x \in X,\end{aligned}$$

because the inequality measure s is not consistent. However, for ξ and values smaller than ξ, according to Table 2, we have

$$\begin{aligned}&\left(\mu(y^1), \mu(y^1_{R_1}) + \xi\rho(y^1_{R_k}), \mu(y^1_{R_2}) + \xi\rho(y^1_{R_2}), \mu(y^1_{R_3}) + \xi\rho(y^1_{R_3})\right)\\ &\leq_3 \left(\mu(y^2), \mu(y^2_{R_1}) + \xi\rho(y^2_{R_k}), \mu(y^2_{R_2}) + \xi\rho(y^2_{R_2}), \mu(y^2_{R_3}) + \xi\rho(y^2_{R_3})\right)\\ &\leq_3 \left(\mu(y^3), \mu(y^3_{R_1}) + \xi\rho(y^3_{R_k}), \mu(y^3_{R_2}) + \xi\rho(y^3_{R_2}), \mu(y^3_{R_3}) + \xi\rho(y^3_{R_3})\right),\end{aligned}$$

for the inequality measures s, R, δ, σ. Thus, the outcome vector y^1 is a 3-nondominated point of the problem (28). This increases the validity of our results in Corollary 3.4.

4 Conclusion

We decomposed the multi-objective location problem into the collection of subproblems and applied the mean and inequality measures to these subproblems. The new mean-equity models introduced by this paper take into account both the efficiency with minimization of the mean outcome and the equity with minimization of the sum of the mean outcome and the inequality measure for all subproblems. Moreover, the consistency property of inequality measures enables us to investigate the relationship between r-efficient solutions of the new mean-equity models and efficient solutions of the location problem.

References

[1] H. P. Benson, An improved definition of proper efficiency for vector maximization with respect to cones, *Journal of Mathematical Analysis and Applications*, 71(1979), 232-241.

[2] M. Ehrgott, *Multi-Criteria Optimization*, Springer Verlag, Berlin, 2005.

[3] A. Engau, *Domination and Decomposition in Multi-Objective Programming*, Ph.D. thesis, Department of Mathematical Sciences, Clemson University, 2007.

[4] A. Engau, M. M. Wiecek, Interactive coordination of objective decompositions in multi-objective programming, *Management Science*, 54(2008), 1350-1363.

[5] E. Erkut, Inequality measures for location problems, *Location Science*, 1(1993), 199-217.

[6] L. Huerga, B. Jimenez, D. T. Luc, V. Novo, A unified concept of approximate and quasi efficient solutions and associated subdifferentials in multiobjective optimization, *Mathematical Programming*, 189(2021), 379-407.

[7] D. Gourion, D. T. Luc, Generating the weakly efficient set of nonconvex multi-objective problems, *Journal of Global Optimization*, 41(4)(2008), 517-538.

[8] M. C. López-de-los-Mozos, J. A. Mesa, The maximum absolute deviation measure in location problems on networks, *European Journal of Operational Research*, 135(2001), 184-194.

[9] M. C. López-de-los-Mozos, J. A. Mesa, The sum of absolute differences on a network: algorithm and comparison with other equality measures, *Information Systems and Operational Research*, 41(2003), 195-210.

[10] M. B. Mandell, Modelling effectiveness-equity trade-offs in public service delivery systems, *Management Science*, 37(1991), 467-482.

[11] M. T. Marsh, D. A. Schilling, Equity measurement in facility location analysis: A review and framework, *European Journal of Operational Research*, 74(1994), 1-17.

[12] G. F. Mulligan, Equity measures and facility location, *Regional Science*, 70(1991), 345-365.

[13] W. Ogryczak, Inequality measures and equitable approaches to location problems, *European Journal of Operational Research*, 122(2000), 374-391.

[14] W. Ogryczak, Inequality measures and equitable locations, *Annals of Operations Research*, 167(2009), 61-86.

[15] A. Sen, *On Economic Inequality*, Clarendon Press, Oxford, 1973.

Davoud Foroutannia
Associate Professor of Mathematics
Department of Mathematics, Vali-e-Asr University of Rafsanjan
Rafsanjan, Iran
E-mail: foroutan@vru.ac.ir

Ali Reza Salajegheh
Ph. D. Candidate
Department of Mathematics, Vali-e-Asr University of Rafsanjan
Rafsanjan, Iran
E-mail: Salajegheh@stu.vru.ac.ir