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# An Efficient Numerical Scheme for a Class of Integro-Differential Equations with it's Convergence and Error Analysis

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**Abstract.**In this article, the current work suggests an efficient numerical approximation based on an operational matrix of Bernstein Polynomials to obtain numerical solution of high-order of integro-differential equations. At first, we present the integral and differential operator matrix of Bernstein Polynomials, and then apply this operator to the governing equation to transform it into an algebraic equations. Solving this system yields an approximate solution for the equation under study. Also, the convergence and error analysis for this method are study. To demonstrate the effectiveness of this scheme, we provide several numerical examples and compare the results with the exact solution and one of the other well-known method such as collocation Bernoulli method..

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## 1 Introduction

Integral equations can be used to model various scientific and engineering problems, including quantum mechanics' scattering, epidemic growth in spatial-temporal dimensions and water-related phenomena, [10, 6, 22]. Additionally, they are used in solving parabolic boundary value problems and constructing physical and biological models [24, 1] as well as in addressing mixed problems of mechanics with continuous media [4].

A variety of numerical methods have been developed to solve integral equations, including the collocation method utilizing radial basis functions [20].

The investigation of integro-differential equations both linear and non-linear has garnered significant interest as they play a crucial role in engineering and sciences [7]. These equations are often used to model physical events, but their solutions cannot be obtained explicitly. However, finding an analytic solution for linear and non-linear integro-differential equations is generally difficult and thus, various numerical methods have been proposed. Therefore, numerical methods that combine interpolation and numerical integration are necessary.

Orthogonal functions and spectral methods have become important tools for solving these types of equations since they can convert the equations into a system of algebraic equations. Maleknejad et al. [16] used the Haar functions method to represent the approximate solution of the integro-differential equation, while Nemati [18] studied the Legendre collocation method. Pour-Mahmoud et al. [21] obtained the approximate solution of this equations by using the Tau method. Loh and coworkers [14, 15] used shifted Genocchi polynomials to fractional type of integro-differential equations. The authors of [13] constructed an operational matrix of fractional integration using the Laplace transform and fractional-order Lagrange polynomials to numerical solving of fractional-order pantograph delay and Riccati differential equations. Also, in [11] the authors focused on a numerical technique that utilizes fractional-order Lagrange polynomials to solve a specific class of fractional-order non-linear Volterra-Fredholm integro-differential equations. The Caputo type of fractional derivative is considered in this analysis. A novel approach for solving non-local boundary value problems of fractional order that commonly arise in the field of chemical reactor theory is presented in [12] . This method utilizes an operational matrix method based on fractional-order Lagrange polynomials. The other efficient numerical methods which were used include the Legendre matrix method [23], the Lagrange method [26], the differential transformation method [3], the Chebyshev polynomial method [2] and the Taylor expansion method [9].

The objective of this study is to investigate the numerical solution of integro-differential equation as follows:

$$\sum_{j=0}^{N} c_j v^j(y) = g(y) + \int_0^y k(y,t)v(t)dt, \tag{1}$$

with inital conditions:

$$v^j(0) = \alpha_j,\ j = 0, 1, 2, \dots, N-1, \tag{2}$$

where $g(y)$ and $v(y)$ are continuous differentiable functions of the desired order and $k(y,t)$ is a differentiable function of the desired order and separable kernel. Also, $c_j, j = 0, \dots, N-1$ are constant coefficients and $v^j(y) = \frac{d^j}{dy^j}v(y)$. In this work, we study a numerical method based on an operational matrix of Bernstein polynomials for solving this integro-differential equations.

The Bernstein polynomials have many applications to numerical solving of differential equations, the authors of [19] used this method to solve non-linear Fredholm integro equations. Yüzbaşı [27] applied these polynomials and generate a collocation scheme to solve non linear Fredhom-Volterra integro-differential equations. To see the other applications of Bernstein polynomials, the readers can see [8] and their references.

In the presented scheme, at first the derivatives are discreticized and then we approximate the solutions by using the Bernstein Polynomials. The rest of paper is as follows: Section 2, is devoted to study of the Bernstein polynomials and their applications. In Section 3, we introduce the approximation of functions and their operational matrix. In Section 4, the convergence and error analysis of the method will be study. Some numerical examples are presented to check the efficiency of proposed numerical method in Section 5. In the end, a brief conclusion is given in Section 6.

## 2 Description of the Bernstein Polynomials

The Bernstein Polynomials of degree $h$ are defined on interval $[0,1]$ and displayed by [25]:

$$B_{j,h}(y) = \begin{pmatrix} h \\ j \end{pmatrix} y^j(1-y)^{h-j},\ j = 0, \ldots, h. \tag{3}$$

Applying the binomial expansion of $(1-y)^{h-j}$, definition (3) may be written as follows:

$$B_{j,h}(x) = \sum_{n=0}^{h-j} (-1)^n \begin{pmatrix} h \\ j \end{pmatrix} \begin{pmatrix} h-j \\ n \end{pmatrix} y^{n+j}.$$

These polynomials are orthogonal on the interval $[0,1]$ and their orthogonality property is given by:

$$\int_0^1 B_{n,j}(y) B_{l,j}(y) dy = \delta_{ln},\ 0 \le l, n \le j,$$

where $\delta_{ln}$ is the Kronecker function. We can expressed Bernstein Polynomials in the following matrix representation:

$$\Theta(y) = \mathbb{A}\mathbb{T}_h(y), \tag{4}$$

where

$$\Theta(y) = [B_{0,h}(y), B_{1,h}(y), \ldots, B_{h,h}(y)]^T,$$
$$\mathbb{T}_h(y) = [1, y, \ldots, y^h]^T.$$

Also, the matrix components of $\mathbb{A}$ is defined by:

$$\mathbb{A} = \begin{pmatrix} (-1)^0 \begin{pmatrix} h \\ 0 \end{pmatrix} & (-1)^1 \begin{pmatrix} h \\ 0 \end{pmatrix} \begin{pmatrix} h-0 \\ 1-0 \end{pmatrix} & \ldots & (-1)^{h-0} \begin{pmatrix} h \\ 0 \end{pmatrix} \begin{pmatrix} h-0 \\ h-0 \end{pmatrix} \\ \ddots & \vdots & \vdots & \vdots \\ 0 & (-1)^0 \begin{pmatrix} h \\ j \end{pmatrix} & \ldots & (-1)^{h-j} \begin{pmatrix} h \\ j \end{pmatrix} \begin{pmatrix} h-j \\ h-j \end{pmatrix} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & (-1)^h \begin{pmatrix} h \\ h \end{pmatrix} \end{pmatrix},$$

in which $det(A) = \prod_{j=0}^{h} (-1)^h \begin{pmatrix} h \\ j \end{pmatrix}$. So, $A$ is an invertible matrix.

# 3 Approximation of Function and Operational Matrix

Let $v(y) \in L^2[0,1]$ be a continuous function, it can be approximated by the Bernstein Polynomials as:

$$v(y) \simeq \sum_{j=0}^{h} v_j B_{j,h}(y) = V^T \Theta(y), \tag{5}$$

in which Bernstein coefficients $V^T = [v_0, v_1, \ldots, v_h]$ may be calculated as follows:

$$V^T = \mathbb{Q}^{-1} \int_0^1 v(y) \Theta^T(y) dy. \tag{6}$$

Also, $\mathbb{Q}_{(h+1)\times(h+1)}$ is the dual matrix of $\Theta(y)$ which is given by:

$$\begin{aligned} \mathbb{Q} &= \int_0^1 \Theta(y)\Theta(y)^T dy = \int_0^1 \mathbb{A}\mathbb{T}_h(y)\big(\mathbb{A}\mathbb{T}_h(y)\big)^T dy \\ &= \mathbb{A}\Big(\int_0^1 \ T_h(y)\ T_h(y)dy\Big)\mathbb{A}^T = \mathbb{A}\mathbb{H}\mathbb{A}^T, \end{aligned}$$

where $\mathbb{H}$ is the Hilbert matrix as follows:

$$\mathbb{H} = \begin{pmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{h+1} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{h+2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{h+1} & \frac{1}{h+2} & \cdots & \frac{1}{2h+1} \end{pmatrix}.$$

## 3.1 First order derivative operational matrix of $\frac{dv}{dy}$ in terms of the Bernstein Polynomials

The differentiation of function $v(y)$ may be expressed as:

$$\begin{aligned} \frac{dv}{dy} &= V^T \frac{d}{dy}\Theta(y) = V^T\big(\mathbb{A}\mathbb{T}_h(y)\big)' \\ &= V^T \mathbb{A} \begin{bmatrix} (1)' \\ (y)' \\ \vdots \\ (y^h)' \end{bmatrix} = V^T \mathbb{A} \begin{bmatrix} 0 \\ 1 \\ \vdots \\ hy^{h-1} \end{bmatrix} = V^T \mathbb{A} \mathbf{V}_{(h+1)\times h} \Theta^*(y), \end{aligned}$$

where the matrices $\mathbf{V}_{(h+1)\times h}$ and $\Theta^*(y)$ are obtained by:

$$\mathbf{V}_{(h+1)\times h}\begin{bmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & h \end{bmatrix} \quad \text{and} \quad \Theta^*(y) = \begin{bmatrix} 1 \\ y \\ \vdots \\ y^{h-1} \end{bmatrix}_{h\times 1}.$$

Using relation (4), $\Theta^*(y)$ is calculated as:

$$\Theta^*(y) = \mathbf{B}^*\Theta(y), \tag{7}$$

where

$$\mathbf{B}^* = \begin{bmatrix} \mathbb{A}^{-1}_{[1]} \\ \mathbb{A}^{-1}_{[2]} \\ \vdots \\ \mathbb{A}^{-1}_{[h]} \end{bmatrix}, \tag{8}$$

and $\mathbb{A}^{-1}_{[i]}$ is the $i$th row of $\mathbb{A}^{-1}$, $i = 1, 2, 3, \dots, h$. Due to relations (7) and (8), $\frac{dv}{dy}$ can be rewritten as:

$$\frac{dv}{dy} = V^T \mathbb{A}\mathbf{V}_{(h+1)\times h}\mathbf{B}^*\Theta(y).$$

In addition the differential operator matrices of higher order in terms of the Bernstein Polynomials are obtained by using the mathematical induction as follows:

$$\frac{d^j v}{dy^j} = V^T\left(\mathbb{A}\mathbf{V}_{(h+1)\times h}\mathbf{B}^*\right)^j\Theta(y),\ j \geq 2.$$

Applying (5) and (6), results in:

$$\begin{aligned}
\int_0^y k(y,t)\big[v(t)\big]dt &= \int_0^y \Theta^T(y)\mathbf{K}\Theta(t)\Theta^T(t)V^T dt \\
&= \Theta^T(y)\mathbf{K}\mathbb{A}\Big(\int_0^y \mathbb{T}_h(t)\mathbb{T}_h^T(t)dt\Big)\mathbb{A}^T V^T \\
&= \Theta^T(y)\mathbf{K}\mathbb{A}\int_0^y \begin{pmatrix} 1 & t & \dots & t^h \\ t & t^2 & \dots & t^{h+1} \\ \vdots & \vdots & \ddots & \vdots \\ t^h & t^{h+1} & \dots & t^{2h} \end{pmatrix} dt\mathbb{A}^T V^T \\
&= \Theta^T(y)\mathbf{K}\mathbb{A}\mathbb{R}\mathbb{A}^T V^T,
\end{aligned}$$

where $k(y,t) = \Theta^T(y)\mathbf{K}\Theta(t)$ and

$$\mathbb{R} = \begin{pmatrix} y & \frac{y^2}{2} & \dots & \frac{y^{h+1}}{h+1} \\ \frac{y^2}{2} & \frac{y^3}{3} & \dots & \frac{y^{h+2}}{h+2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{y^{h+1}}{h+1} & \frac{y^{h+2}}{h+2} & \dots & \frac{y^{2h+1}}{2h+1} \end{pmatrix}.$$

Also, the function $g(y)$ can be approximated by the Bernstein Polynomials as:

$$g(y) \simeq \sum_{j=0}^{h} f_j B_{j,h}(y) = \mathbb{F}^T\Theta(y).$$

Substituting this relation in Eq. (1), yields

$$\begin{aligned}
\textstyle\sum_{j=0}^N c_j V^T \Big(\mathbb{A}\mathbf{V}_{(h+1)\times h}\mathbf{B}^*\Big)^j \Theta(y) &= \mathbb{F}^T\Theta(y) \\
&+\Theta^T(y)\mathbf{K}\mathbb{A}\mathbb{R}\mathbb{A}^T V^T. \qquad (9)
\end{aligned}$$

Therefore, a system of algebraic equations will be obtained. By using the collocation points $y_k = \frac{(2k-1)}{2(N+1)}$, $k = 0,\dots,N$ in (9) the unknown coefficients will be determined. In this case, the approximate solution of Eq. (1) is found and our method is completed.

## 4 Error and Convergence Analysis

In this part, we analyse the accuracy of the suggested scheme. To check the error analysis, we consider the Sobolev space $\mathbb{H}^l(c,d)$ as follows:

$$\mathbb{H}^l(c,d) = \{v : (c,d) \to \mathbb{R} \ \text{ such that } \ v^{(i)} \in L^2(c,d) \ \text{ for } \ i = 0,1,\dots,l\},$$

in which the function $v^{(i)}$ denotes the i-th order derivative of $v$. Moreover, we consider the Sobolev space with the following norm:

$$\| v \|_{\mathbb{H}^l(c,d)} = \sqrt{\sum_{k=0}^{h} \| v^{(k)} \|^2_{L^2(c,d)}}.$$

**Lemma 4.1.** *Assume that $v \in \mathbb{H}^l(-1,1)$ for $l = 0,1,\dots$. Also, suppose that $P_n(v) = \sum_{j=0}^{h} v_j B_{j,h}(y) = V^T\Theta(y)$ be the truncated Bernstein series of $v$. Then, we have:*

$$\| v - P_n(v) \|_{L^2(-1,1)} \leq K n^{-l} |v|_{\mathbb{H}^{l;n}(-1,1)},$$

*where*

$$|g|_{\mathbb{H}^{l;n}(-1,1)} = \sqrt{\sum_{m=\min\{l,n+1\}}^{l} \| v^{(m)} \|^2_{L^2(-1,1)}},$$

*and $K > 0$ is a constant which is independent of function $v$ and $n$.*

**Proof.** The proof is similar to Ref. [5]. □

**Lemma 4.2.** *Let $v \in \mathbb{H}^l(0,T)$ and $\bar{v} : (-1,1) \to \mathbb{R}$ is defined by $\bar{v}(t) = v\left(\frac{T(t+1)}{2}\right)$ for all $-1 < t < 1$. Then, we have:*

$$\| \bar{v}^{(h)} \|_{L^2(-1,1)} = 2^{\frac{1}{2}-h} T^{m-\frac{1}{2}} \| v^{(h)} \|_{L^2(0,T)}, \; h = 0,1,\dots,l. \tag{10}$$

**Proof.** Considering the variable $y = \frac{T(t+1)}{2}$, one obtains

$$\begin{aligned}\| \bar{v}^{(h)} \|^2_{L^2(-1,1)} &= \int_{-1}^{1} |\bar{v}^{(h)}(t)|^2 dt \\ &= \int_{-1}^{1} \Big| v^{(h)}\Big(\frac{T(t+1)}{2}\Big)\Big|^2 dt \\ &= 2^{1-2h} T^{2h-1} \int_0^T \Big| v^{(h)}(x)|^2 dx \\ &= 2^{1-2h} T^{2h-1} \| v^{(h)} \|^2_{L^2(0,T)} . \end{aligned} \tag{11}$$

By taking the square root from both sides of Eq. (11), the relation (10) will be obtained. □

**Theorem 4.3.** *Let $v(t) \in \mathbb{H}^{l+1}(0,T)$ be the exact solution of Eqs. (1) with (2). Also, let $v_n(t)$ be the approximate solution computed by using the proposed method. Then, we have:*

$$\| v - v_n(t) \|_{L^2(0,T)} \leq j n^{-l} |v|_{\mathbb{H}^{l;n}(0,T)},$$

*where*

$$|v|_{\mathbb{H}^{l;n}(0,T)} = \sqrt{\sum_{h=\min\{l,n+1\}}^{l} T^{2h} 2^{-2h} \| v^{(h+1)} \|^2_{L^2(0,T)}}.$$

**Proof.** Let the function $P_n(v)(t)$ defined by $P_n(v) = \sum_{j=0}^{h} v_j B_{j,h}(y)$ be the best approximation of $v(t)$. Then,

$$\begin{aligned}\| v - v_n(t) \|^2_{L^2(0,T)} &\leq \frac{T}{2} j n^{-2l} \sum_{h=\min\{l,n+1\}}^{l} \| \bar{v}^{(h)} \|^2_{L^2(-1,1)} \\ &= j n^{-2l} \sum_{h=\min\{l,n+1\}}^{l} 2^{-2h} T^{2h} \| v^{(h)} \|^2_{L^2(0,T)} \\ &= j n^{-2l} \sum_{m=\min\{l,n+1\}}^{l} 2^{-2m} T^{2m} \| v^{(m+1)} \|^2_{L^2(0,T)} .\end{aligned}$$

Applying the Lemmas 4.1 and 4.2, one obtains

$$\begin{aligned}
\| v - v_n(t) \|^2_{L^2(0,T)} &\leq \frac{T}{2} jn^{-2l} \sum_{h=\min\{l,n+1\}}^{l} \| \bar{v}^{(h)} \|^2_{L^2(-1,1)} \\
&= jn^{-2l} \sum_{h=\min\{l,n+1\}}^{l} 2^{-2h} T^{2h} \| v^{(h)} \|^2_{L^2(0,T)} \\
&= jn^{-2l} \sum_{m=\min\{l,n+1\}}^{l} 2^{-2m} T^{2m} \| v^{(m+1)} \|^2_{L^2(0,T)} . \quad (12)
\end{aligned}$$

By taking the squared root from both sides of Eq. (12), the proof is completed. □

**Theorem 4.4.** *(Convergence analysis) Let $v(y)$ be in $H^{l+1}(0,T)$ where $|v| \leq M$, $M > 0$. Also, suppose that $\bar{v}(y)$ be the best approximation of $v(y)$ out of $\prod = Span\{B_{0,h}(y), B_{1,h}(y), \ldots, B_{h,h}(y)\}$ with respect to the norm of $L^2[c,d]$, Then we have*

$$\| v - \bar{v} \|_2 \to 0.$$

**Proof.** Let $z(y) \in \prod$. Then, the $N$-th Taylor polynomial of $v(y)$ at $y = c$ is as follows:

$$z(y) = \sum_{i=0}^{N} \frac{v^{(i)}(c)(y-c)^i}{i!}.$$

Therefore, there exists $\eta \in (c,d)$, such that:

$$\big|v - z\big| \leq \frac{(y-c)^{N+1}}{(N+1)!} \big|v^{(N+1)}(\eta)\big|.$$

Since $\bar{v}$ is the best approximation for $v$ on the interval $[c,d]$, so

$$\begin{aligned}
\| v - \bar{v} \|^2_2 &\leq \| v - z \|^2_2 \\
&= \int_c^d (v-z)^2 dy \\
&\leq \int_c^d \Big[ \frac{(y-c)^{N+1}}{(N+1)!} \big|v^{(N+1)}(\eta)\big| \Big]^2 dy \\
&\leq \frac{M^2 (d-c)^{2N+3}}{(N+1)!^2 (2N+3)},
\end{aligned}$$

if $N \to \infty$, we get

$$\lim_{N\to\infty} \frac{M(d-c)^{N+\frac{3}{2}}}{(N+1)!\sqrt{2N+3}} = 0.$$

These results show that $\| v - \bar{v} \|_2 \to 0$. Therefore $\bar{v}$ is converges to $v$ and the proof is completed. □

## 5 Numerical Examples

In this section, some numerical examples are presented to show the effectiveness and capability of the proposed method. To measure the accuracy of the method, the maximum absolute error is calculated by the following formula

$$\| E(y) \| = \| v(y) - v_n(y) \|,$$

where $v(y)$ and $v_n(y)$ are the exact solution and numerical solution computed by the presented scheme.

**Example 5.1.** We consider the following Volterra integro-differential equation of the forth order

$$v^{(iv)}(y) - v(y) = y(1+e^y) + 3e^y - \int_0^y v(t)dt \ , y \in [0,1],$$

with initial conditions

$$v(0) = 1, v^{'}(y) = 1, v^{''}(y) = 2, v^{'''}(y) = 3,$$

and the exact solution $v(y) = 1 + ye^y$ [17].

**Table 1:** Comparison of the absolute error with different values of $h$ in Example 5.1.

| $y$ | Bernoulli [17]($h=4$) | $h=4$ | $h=6$ | $h=8$ | $h=9$ |
|---|---|---|---|---|---|
| 0.2 | $5.4322E-05$ | $3.8214E-08$ | $1.1537e-10$ | $2.0384E-13$ | $7.2997E-15$ |
| 0.4 | $1.5088E-04$ | $3.4643E-07$ | $9.7906e-10$ | $1.6953E-12$ | $6.0396E-14$ |
| 0.6 | $4.4668E-04$ | $1.1474E-06$ | $3.3130e-09$ | $5.7467E-12$ | $2.0495E-13$ |
| 0.8 | $2.2000E-03$ | $2.7497E-06$ | $7.8394e-09$ | $1.3630E-11$ | $4.8694E-13$ |

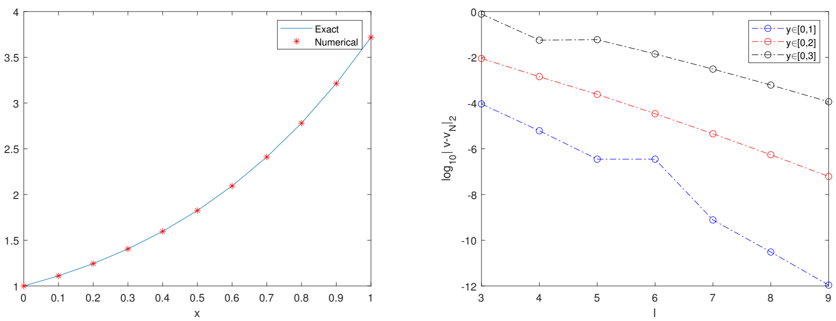


**Figure 1:** Numerical and the exact solutions with $h = 7$(left) and it's related absolute error(right) with some values of $h$ in Example 1.

Table 1 shows the results between the proposed method and exact solution. Also, column one of this table show the errors from [17] which is the Bernoulli collocation method. According to the results of this table, one can conclude that our scheme is more accurate than the method in [17]. Also, by increasing the number of Bernstein polynomials, the error is decreased. Figure 1 displays the absolute error between the exact and numerical solutions for different values of $h$ which shows that our scheme is efficient to obtain numerical solution for this example. A comparison of numerical solution and analytical solution is given in Figure 1. Also, Figure 1 (right) display the logarithm of absolute errors in other intervals $[0, 1], [0, 2], [0, 3]$. From the results of table 1 and Figure 1, one can conclude that our scheme is efficient and valid to obtain numerical solution of this example.

**Example 5.2.** We consider the following Volterra integro-differential equation

$$v^{'}(y) = 1 - \int_0^y v(t)dt \, , y \in [0, 1],$$

with the initial condition

$$v(0) = 0,$$

and the exact solution $v(y) = sin(y)$ [17]. Similar to the Example 1, the errors for y= 0.2, 0.4, 0.6 and 0.8 are listed in Table 2. From this

**Table 2:** Comparison of the absolute errors with different values of $h$ for Example 5.2.

| $y$ | Bernoulli [17]($h=4$) | $h=4$ | $h=6$ | $h=8$ | $h=9$ |
|---|---|---|---|---|---|
| 0.2 | $5.4422E-06$ | $8.9767E-07$ | $1.9358E-09$ | $3.0902E-12$ | $2.1677E-13$ |
| 0.4 | $7.9628E-06$ | $7.6589E-07$ | $1.9956E-09$ | $2.9596E-12$ | $2.0495E-13$ |
| 0.6 | $9.6856E-06$ | $6.7687E-07$ | $1.7914E-09$ | $2.6463E-12$ | $1.8374E-13$ |
| 0.8 | $3.6086E-06$ | $6.6712E-07$ | $1.3048E-09$ | $2.1551E-12$ | $1.5654E-13$ |

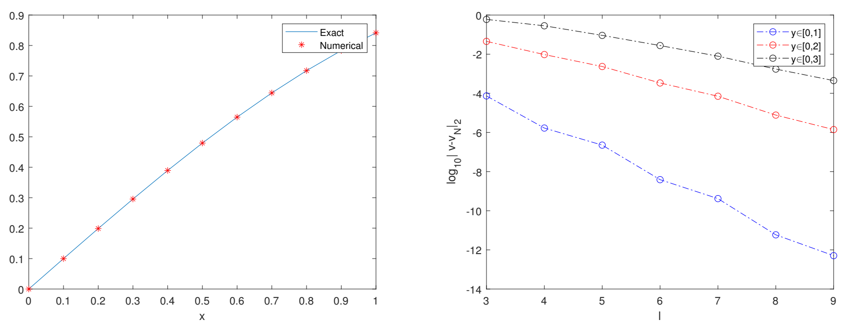


**Figure 2:** Numerical and the exact solutions with $h = 7$(left) and it's related absolute error(right) with some values of $h$ in Example 2.

table, one can observe that by increasing the number of Bernstein polynomials(h), the error is decreased. Also, column two of this table shows the absolute errors with the Bernoulli polynomials which show that our scheme is more accurate than the mentioned method[17]. Figure 2(left) shows the numerical and exact solutions in [0,1]. Also, we increase the length of the domain interval and report the results in Figure 2(right). According to the results of this Figure, the presented algorithm has a good accuracy in the mentioned intervals.

**Example 5.3.** For the last example, we consider the following Volterra integro-differential equation

$$v'(y) + v(y) = \int_0^y e^{t-y} v(t)dt, \ \ , y \in [0, 1],$$

with initial condition

$$v(0) = 1,$$

which has the exact solution $v(y) = e^{-y} \cosh y$[17].

**Table 3:** Comparison of the absolute error with different values of $h$ in Example 5.3.

| $y$ | Bernouli [17]($h = 4$) | $h = 4$ | $h = 6$ | $h = 8$ | $h = 9$ |
|---|---|---|---|---|---|
| 0.1 | $7.8212E-03$ | $3.7360E-05$ | $2.9887e-07$ | $1.5169E-09$ | $9.5288E-11$ |
| 0.2 | $1.1000E-03$ | $2.4953E-05$ | $1.9820e-07$ | $1.2199E-09$ | $8.2503E-11$ |
| 0.3 | $8.4639E-04$ | $1.7802E-05$ | $2.0083e-07$ | $1.1828E-09$ | $7.7246E-11$ |
| 0.4 | $8.5470E-03$ | $1.9953E-05$ | $1.9133e-07$ | $1.0822E-09$ | $7.2141E-11$ |

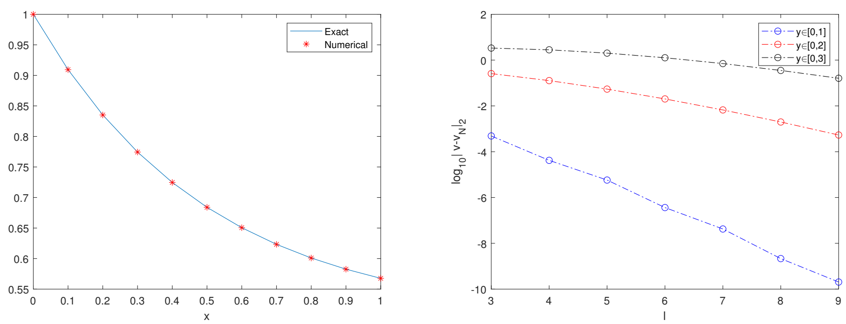


**Figure 3:** Numerical and The exact solutions with $h = 7$(left) and it's related absolute error(right) with some values of $h$ in Example 3.

In this problem, similar to the previous examples, we report the error between the exact solution and the solution with the current method for several points. Also, a comparison with the Bernoulli method is given in this table. The shape of the exact solution and the numerical solution are depicted in Figure 3 for this example. In Figure 3 (right), errors are also given for different intervals. According to Figure3 and Table 3, the presented method has a good performance to obtain an approximate values for the exact solution.

## 6 Conclusion

The focus of this study was solving high order integro-differential equations by using the Bernstein Polynomials matrix method. An operational matrix of derivatives based on these polynomials was used to developed a fast numerical method for solving these types of equations. Also, the convergence and error bounds for this algorithm was given.

Moreover the effectiveness of the presented numerical scheme was illustrated through some examples. In addition, the results are compared with the modified Bernoulli method and the exact solution, to show the capability of presented algorithm for solving integro-differential equations.

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