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Screen Locally Conformal Lightlike Hypersurfaces of an Almost Para-Hyperhermitian Statistical Manifold

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Abstract. In this paper, we investigate lightlike hypersurfaces of an almost para-hyperhermitian statistical manifold of constant holomorphic sectional curvature. We obtain some relations between the induced objects of such lightlike hypersurfaces with a conformal shape operator on the screen distribution. Further, we give an example of a lightlike hypersurface of an almost para-hyperhermitian statistical manifold.

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1 Introduction

Information geometry is a branch of science to extract features from objects which has fascinating applications in machine learning and evolutionary biology. A statistical manifold is a differential manifold that specifies each point with a probability distribution. In 1987, Lauritzen defined the notion of statistical manifold as a generalization of a statistical model with the Fisher metric and the Amari-Chenstov tensor [2]. Statistical manifolds are geometric objects viewed as a Riemannian manifold which admits a torsion-free affine connection $\bar{\nabla}$ and it's dual connection $\bar{\nabla}^*$ with respect to the metric $\bar{\mathfrak{g}}$.

In lightlike submanifolds, the normal vector bundle and the tangent bundle intersects each other which does not occur in non-degenerate submanifolds [1, 8, 11].
The theory of lightlike hypersurfaces has been subject of interest by many of authors [5, 6, 12]. C. Atindogbe and K.L. Duggal investigated screen locally conformal lightlike hypersurfaces and derived some classification theorems [3]. Lightlike hypersurfaces of a statistical manifold were discussed by O. Bahadir, M.M. Tripathi [4].
However, the conception of Sasakian structures was presented by Shigeo Sasaki and furthermore H. Furuhata developed this idea for statistical manifolds [9]. K.L. Duggal and B. Sahin surveyed real lightlike hypersurfaces of an indefinite quaternion Kaehler manifolds [7, 14]. The main properties of a para-quaternionic hermitian manifold were given in [10, 13]. We intend to use these conceptions to achieve equivalent results for lightlike hypersurfaces of a para-hyperhermitian statistical manifold whose holomorphic sectional curvature is constant.

The present work is organized as follows: Section 2, contains some basic definitions about statistical manifolds and manifolds with mixed 3-structures. In Section 3 we study lightlike hypersurfaces of an almost para-hyperhermitian manifold. The relations between the induced objects of such lightlike hypersurfaces are obtained in Section 3. In particular, we review screen locally conformal lightlike hypersurfaces of an almost para-hyperhermitian statistical manifold of constant holomorphic sectional curvature. Also, the last section is concluded with an example.

2 Preliminaries

2.1 Statistical manifolds

Suppose that $(\bar{\mathcal{M}}, \bar{\mathfrak{g}})$ is a semi-Riemannian manifold and $\bar{\nabla}$ is an affine connection on $\bar{\mathcal{M}}$ associated with the semi-Riemannian metric $\bar{\mathfrak{g}}$. We will review some main definitions about statistical manifolds based on [9].

Definition 2.1. The triple $(\bar{\mathcal{M}}, \bar{\nabla}, \bar{\mathfrak{g}})$ is termed as a statistical manifold if $\bar{\nabla}$ is torsion free and the equalization

$$(\bar{\nabla}_E \bar{\mathfrak{g}})(F, C) = (\bar{\nabla}_F \bar{\mathfrak{g}})(E, C) \tag{1}$$

is satisfied for all $E, F, C \in \Gamma(T\bar{\mathcal{M}})$.

The dual affine connection $\bar{\nabla}^*$ of $\bar{\nabla}$ is indicated by

$$E\bar{\mathfrak{g}}(F, C) = \bar{\mathfrak{g}}(\bar{\nabla}_E F, C) + \bar{\mathfrak{g}}(F, \bar{\nabla}^*_E C), \tag{2}$$

Denote by $\tilde{\nabla} = \frac{1}{2}(\bar{\nabla} + \bar{\nabla}^*)$ the Levi-Civita connection associated with the metric $\bar{\mathfrak{g}}$.

Remark 2.2. For a statistical manifold $(\bar{\mathcal{M}}, \bar{\nabla}, \bar{\mathfrak{g}})$, we define a tensor $\mathcal{K} \in \Gamma(T\bar{\mathcal{M}}^{(1,2)})$ by $\mathcal{K}_E F = \frac{1}{2}(\bar{\nabla}_E F - \bar{\nabla}^*_E F)$ which satisfies

$$\mathcal{K}_E F = \mathcal{K}_F E, \qquad \bar{\mathfrak{g}}(\mathcal{K}_E F, C) = \bar{\mathfrak{g}}(F, \mathcal{K}_E C). \tag{3}$$

for all $E, F, C \in \Gamma(T\bar{\mathcal{M}})$.

Let $\bar{\mathcal{R}}, \bar{\mathcal{R}}^*$ be the curvature tensor fields of $\bar{\nabla}$ and $\bar{\nabla}^*$, respectively. Then the statistical curvature tensor field of the manifold $(\bar{\mathcal{M}}, \bar{\nabla}, \bar{\mathfrak{g}})$ is characterized by $\bar{\mathcal{S}}(E, F)C = \frac{1}{2}\{\bar{\mathcal{R}}(E, F)C + \bar{\mathcal{R}}^*(E, F)C\}$ for all $E, F, C \in \Gamma(T\bar{\mathcal{M}})$.

A $(1, 1)$-tensor field $\mathfrak{X}$ which satisfies $\mathfrak{X}^2 = -\mathcal{I}d$ is called an almost complex structure on $\bar{\mathcal{M}}$. Let $\mathfrak{X} \in \Gamma(T\bar{\mathcal{M}}^{(1,1)})$ be an almost complex structure such that $\bar{\mathfrak{g}}(\mathfrak{X}E, \mathfrak{X}F) = \bar{\mathfrak{g}}(E, F)$. We put θ as a 2-form on $\bar{\mathcal{M}}$ defined by $\theta(E, F) = \bar{\mathfrak{g}}(E, \mathfrak{X}F)$. A statistical manifold $(\bar{\mathcal{M}}, \bar{\mathfrak{g}}, \bar{\nabla})$

furnished by an almost complex structure $\mathfrak{X}$ satisfying $\bar{\nabla}\theta = 0$, is called a holomorphic statistical manifold. In addition, the following relations $\bar{\nabla}_E \mathfrak{X} F = \mathfrak{X} \bar{\nabla}^*_E F$ and $\bar{\mathcal{R}}(E,F)\mathfrak{X}C = \mathfrak{X}\bar{\mathcal{R}}^*(E,F)C$ are deducible for all $E, F, C \in \Gamma(T\mathcal{M})$.

Definition 2.3. A holomorphic statistical manifold $(\bar{\mathcal{M}}, \bar{\mathfrak{g}}, \bar{\nabla}, \mathfrak{X})$ is supposed to be of constant holomorphic sectional curvature $\bar{c} \in \mathbb{R}$ if

$$\begin{aligned} \bar{\mathcal{S}}(E,F)C = \frac{\bar{c}}{4}\{&\bar{\mathfrak{g}}(F,C)E - \bar{\mathfrak{g}}(E,C)F \\ &+ \bar{\mathfrak{g}}(\mathfrak{X}F,C)\mathfrak{X}E - \bar{\mathfrak{g}}(\mathfrak{X}E,C)\mathfrak{X}E + 2\bar{\mathfrak{g}}(E,\mathfrak{X}F)\mathfrak{X}C\} \end{aligned} \tag{4}$$

holds for all $E, F, C \in \Gamma(T\bar{\mathcal{M}})$.

2.2 Mixed 3-structure manifolds

An almost product structure $\mathfrak{X}$ on a smooth semi-Riemannian manifold $\bar{\mathcal{M}}$ is a $(1,1)$-tensor field satisfying $\mathfrak{X}^2 = \mathcal{I}d$, where $\mathcal{I}d$ indicates the identity tensor field on $\bar{\mathcal{M}}$.

Definition 2.4. [10] Let $H = (\mathfrak{X}_i)_{i=1,2,3}$ be a local basis of subbundle of $End(T\bar{\mathcal{N}})$ of rank 3. Then, $(\bar{\mathcal{M}}, \bar{\mathfrak{g}}, \mathfrak{X} = (\mathfrak{X}_i)_{i=1,2,3})$ is called an almost para-hypercomplex manifold if $\mathfrak{X}_1, \mathfrak{X}_2$ are almost product structure on $\bar{\mathcal{M}}$ and $\mathfrak{X}_3$ is an almost complex structure on $\bar{\mathcal{M}}$ which satisfies $\mathfrak{X}_1\mathfrak{X}_2 = -\mathfrak{X}_2\mathfrak{X}_1 = \mathfrak{X}_3$.

A semi-Riemannian metric $\bar{\mathfrak{g}}$ on $\bar{\mathcal{M}}$ satisfying

$$\bar{\mathfrak{g}}(\mathfrak{X}_1E, \mathfrak{X}_1F) = \bar{\mathfrak{g}}(\mathfrak{X}_2E, \mathfrak{X}_2F) = -\bar{\mathfrak{g}}(\mathfrak{X}_3E, \mathfrak{X}_3F) = -\bar{\mathfrak{g}}(E,F), \tag{5}$$

is called compatible to the almost para-hypercomplex structure $H = (\mathfrak{X}_i)_{i=1,2,3}$ for all $E, F \in \Gamma(T\bar{\mathcal{M}})$.

Definition 2.5. [10] A triple $(\psi_i, \zeta_i, \eta_i)_{i=1,2,3}$ of structures on $(\bar{\mathcal{M}}, \bar{\nabla}, \bar{\mathfrak{g}})$ satisfying

$$\psi_i^2 = \tau_i(-\mathcal{I} + \eta_i \otimes \mathfrak{z}_i), \qquad \eta_i(\zeta_i) = 1 \qquad \tau_1 = \tau_2 = -\tau_3 = -1 \tag{6}$$

is said to be a mixed 3-structure if $(\psi_1, \zeta_1, \eta_1)$ and $(\psi_2, \zeta_2, \eta_2)$ are almost paracontact structures with $\tau_i = 1$, and $(\psi_3, \zeta_3, \eta_3)$ is an almost contact structure, that is $\tau_i = -1$. Here, $\zeta_i^{'}$s indicate the structure vector fields, $\eta_i^{'}$s are 1-forms on $\bar{\mathcal{M}}$ and $\psi_i^{'}s$ are $(1,1)$-tensor fields. Moreover, (6) yields

$$\begin{aligned} &\eta_i(\zeta_j) = 0, \qquad \psi_i(\eta_j) = \tau_j\zeta_k, \\ &\psi_j(\zeta_i) = -\tau_j\zeta_k \\ &\eta_i o\psi_j = -\eta_j o\psi_i = \tau_k\eta_k, \\ &\psi_i\psi_j - \tau_i\eta_j \otimes \zeta_i = -\psi_j\psi_i + \tau_j\eta_i \otimes \zeta_j = \tau_k\psi_k. \end{aligned} \tag{7}$$

where (i,j,k) is an even permutation of $(1,2,3)$.

A semi-Riemannian metric $\bar{\mathfrak{g}}$ on the smooth manifold $\bar{\mathcal{M}}$ is called compatible to the mixed 3-structure $(\psi_i, \zeta_i, \eta_i)_{i=1,2,3}$, if the relation

$$\bar{\mathfrak{g}}(\psi_i E, \psi_i F) = \tau_i[\bar{\mathfrak{g}}(E,F) - \varepsilon_i\eta_i(E)\eta_i(F)] \tag{8}$$

holds for any $E, F \in \Gamma(T\bar{\mathcal{M}})$, where $\varepsilon_i = \bar{\mathfrak{g}}(\zeta_i, \zeta_i) = \pm 1$, $i = 1,2,3$.

2.3 Lightlike real hypersurfaces

Let $(\bar{\mathcal{M}}, \bar{\mathfrak{g}})$ be an $(n+1)$-dimensional semi-Riemannian manifold and $(\mathcal{M}, \mathfrak{g})$ be a hypersurface of $\bar{\mathcal{M}}$. If $\bar{\mathfrak{g}}$ is degenerate then the normal vector bundle $T^{\perp}(\mathcal{M})$ and tangent vector bundle $T\mathcal{M}$ have an intersection along a non-zero differentiable distribution $\mathfrak{r}ad(T\mathcal{M})$ indicated by radical distribution. For a lightlike hypersurface $\mathcal{M}$ of a semi-Riemannian manifold $(\bar{\mathcal{M}}, \bar{\mathfrak{g}})$, we have $\mathfrak{r}ad(T\mathcal{M}) = T^{\perp}(\mathcal{M})$. Denoted by $\mathfrak{s}(T\mathcal{M})$ the complementary subbundle of $T\mathcal{M}$ to $\mathfrak{r}ad(T\mathcal{M})$, we have $T\mathcal{M} = T^{\perp}(\mathcal{M}) \oplus \mathfrak{s}(T\mathcal{M})$ [7]. Denote by $tr(T\mathcal{M})$ the complementary (but not orthogonal) vector bundle to $T\mathcal{M}$ in $T\bar{\mathcal{M}}$. We have the decomposition

$$T\bar{\mathcal{M}} = \mathfrak{s}(T\mathcal{M}) \perp (T^{\perp}(\mathcal{M}) \oplus tr(T\mathcal{M})) = T\mathcal{M} \oplus tr(T\mathcal{M}), \tag{9}$$

The Gauss-Weingarten formulas for a lightlike hypersurface $\mathcal{M}$ of $(\bar{\mathcal{M}}, \bar{\mathfrak{g}})$, are given by

$$\bar{\nabla}_E F = \nabla_E F + \omega(E,F), \qquad \bar{\nabla}_E \mathcal{L} = -A_{\mathcal{L}}E + \nabla^{\perp}_F \mathcal{L}. \tag{10}$$

Here, $\{\nabla_E F, A_{\mathcal{L}}E\}$ belong to $\Gamma(T\mathcal{M})$ and $\{\omega, \nabla^{\perp}_F \mathcal{L}\} \in \Gamma(tr(T\mathcal{M}))$. We set $\sigma(E,F) = \bar{\mathfrak{g}}(\omega(E,F), \zeta)$ and $s(E) = \bar{\mathfrak{g}}(\bar{\nabla}^{\perp}_E \mathcal{L}, \zeta)$. From (10), we have the following formulas

$$\bar{\nabla}_E F = \nabla_E F + \sigma(E,F)\mathcal{L}, \qquad \bar{\nabla}_E \mathcal{L} = -A_{\mathcal{L}}E + s(E)\mathcal{L} \tag{11}$$

for all $E, F \in \Gamma(T\mathcal{M}), \mathcal{L} \in \Gamma(tr(T\mathcal{M}))$ and $\zeta \in \Gamma(\mathfrak{r}ad(T\mathcal{M}))$. Here, σ denotes the second fundamental form associated with $\bar{\nabla}$ and $A_{\mathcal{L}}$ is the shape operator on $\mathcal{M}$. Let denote by $\mathcal{P}$ the projection morphism of $T\mathcal{M}$ on $\mathfrak{s}(T\mathcal{M})$. Then, the Gauss-Weingarten formulas for $\mathfrak{s}(T\mathcal{M})$ are given by

$$\nabla_E \mathcal{P}F = \nabla'_E \mathcal{P}F + \rho(E, \mathcal{P}F)\zeta, \qquad \nabla_E \zeta = -A'_\zeta E + s'(E)\zeta \quad (12)$$

Here, $\{\nabla'_E \mathcal{P}F, A'_\zeta E\}$ belong to $\Gamma(\mathfrak{s}(T\mathcal{M}))$ and we have

$$\rho(E, \mathcal{P}F) = \bar{\mathfrak{g}}(\nabla_E \mathcal{P}F, \mathcal{L}), \qquad s'(E) = \bar{\mathfrak{g}}(\nabla_E \zeta, \mathcal{L}), \qquad (13)$$
$$s(E) = -s'(E),$$

for all $E, F \in \Gamma(T\mathcal{M})$, $\zeta \in \Gamma(\mathfrak{r}ad(T\mathcal{M}))$ and $\mathcal{L} \in \Gamma(tr(T\mathcal{M}))$.

Moreover, we have

$$\bar{\mathfrak{g}}(\mathfrak{X}_i\zeta, \zeta) = \bar{\mathfrak{g}}(\mathfrak{X}_i\mathcal{L}, \mathcal{L}) = \bar{\mathfrak{g}}(\mathfrak{X}_i\zeta, \mathcal{L}) = 0, \qquad \bar{\mathfrak{g}}(\mathfrak{X}_i\zeta, \mathfrak{X}_i\mathcal{L}) = \tau_i, \quad (14)$$

$\mathfrak{X}_iT^\perp(\mathcal{M})$ and $\mathfrak{X}_itr(T\mathcal{M})$ are distributions on $\mathcal{M}$ of rank 3 such that $\mathfrak{X}_iT^\perp(\mathcal{M}) \cap T^\perp(\mathcal{M}) = 0$ and $\mathcal{M}_itr(T\mathcal{M}) \cap T^\perp(\mathcal{M}) = 0$, $i = 1, 2, 3$. Thus, $\mathfrak{X}_iT\mathcal{M}\perp \oplus \mathfrak{X}_itr(T\mathcal{M})$ is a vector subbundle of $\mathfrak{s}(T\mathcal{M})$, where $\{e, f, g\}$ denotes an even permutation of $\{1, 2, 3\}$. Besides, we have $\bar{\mathfrak{g}}(\mathfrak{X}_i\zeta, \mathfrak{X}_j\mathcal{L}) = 0$, which consequently implies that $\mathfrak{X}_iT^\perp(\mathcal{M}) \oplus \mathfrak{X}_itr(T\mathcal{M})$ is a vector subbundle of $\mathfrak{s}(T\mathcal{M})$ of rank 6. Thus, there exists a non-degenerate distribution Δ_0 on $\mathcal{M}$ such that $\mathfrak{s}(T\mathcal{M}) = \{\Delta_1 \oplus \Delta_2\} \perp \Delta_0$, where $\Delta_1 = \mathfrak{X}_1\zeta \oplus \mathfrak{X}_2\zeta \oplus \mathfrak{X}_3\zeta$ and $\Delta_2 = \mathfrak{X}_1\mathcal{L} \oplus \mathfrak{X}_2\mathcal{L} \oplus \mathfrak{X}_3\mathcal{L}$. Thus, the following decomposition

$$T\mathcal{M} = \{T^\perp(\mathcal{M}) \oplus_{orth} \Delta_0 \oplus (\Delta_1 \oplus \Delta_2)\}, \ \Delta = \{T^\perp(\mathcal{M}) \oplus \Delta_1\} \oplus \Delta_0 \quad (15)$$

are obtained. Considering $\mathfrak{X}_i\zeta = \zeta_i$ and $\mathfrak{X}_i\mathcal{L} = \xi_i$, we define $\mu_i, \nu_i \in \Gamma(T\mathfrak{X}^{(0,1)})$ by

$$\mu_i(E) = \bar{\mathfrak{g}}(E, \zeta_i), \qquad \nu_i(E) = \bar{\mathfrak{g}}(E, \xi_i). \qquad (16)$$

Let $\tilde{S}$ be the projection morphism of $T\mathcal{M}$ on Δ. Consequently, we may write

$$E = \tilde{S}E + \mu_i(E)\xi_i, \qquad (17)$$

and

$$\mathfrak{X}_i E = \psi_i E + \mu_i(E)\mathcal{L} \tag{18}$$

for any $E \in \Gamma(T\mathcal{M})$, where $\psi_i E$, imply tangent part of $\mathfrak{X}_i E$. Applying $\mathfrak{X}_i$ to (18) and using the fact that $\mathfrak{X}_i^2 = -\tau_i \mathcal{I}$, we have

$$\psi_i^2 E = \tau_i(-E + \mu_i(E)\xi_i). \tag{19}$$

From (16), (18) and (19), it concludes that

$$\nu_i(\xi_i) = \nu_i(\zeta_j) = \mu_i(\xi_j) = 0, \;\; \psi_i(\zeta_j) = \tau_j \zeta_k,$$
$$\mu_i o \psi_j = -\mu_j o \psi_i = \tau_k \mu_k, \psi_i \psi_j - \tau_i \mu_j \otimes \mu_i = -\psi_j \psi_i + \tau_j \mu_i \otimes \mu_j = \tau_k \psi_k$$

where (e, f, g) is regarded as an even permutation of $(1, 2, 3)$ and $\tau_1 = \tau_2 = -\tau_3 = -1$. Then, the triple $(\psi_i, \zeta_i, \mu_i)_{i=1,2,3}$ is indicated as an almost contact mixed 3-structure on $\mathcal{M}$ [7].

3 Lightlike Real Hypersurfaces of an Almost Para-Hyperhermitian Statistical Manifold

Consider $(\bar{\mathcal{M}}, \bar{\mathfrak{g}}, \mathfrak{X} = (\mathfrak{X}_i)_{i=1,2,3})$ as an almost para-hyperhermitian manifold furnished by a statistical structure $(\bar{\mathfrak{g}}, \bar{\nabla})$ on $\bar{\mathcal{M}}$. Supposing $(\mathcal{M}, \mathfrak{g})$ as a lightlike hypersurface of an almost para-hyperhermitian statistical manifold $(\bar{\mathcal{M}}, \bar{\mathfrak{g}}, \bar{\nabla}, \mathfrak{X} = (\mathfrak{X}_i)_{i=1,2,3})$, we have the Gauss-Weingarten formulas as follow

$$\begin{aligned} \bar{\nabla}_E F &= \nabla_E F + \sigma(E, F)\mathcal{L}, & \bar{\nabla}^*_E F &= \nabla^*_E F + \sigma^*(E, F)\mathcal{L}, \\ \bar{\nabla}_E \mathcal{L} &= -A^*_{\mathcal{L}} E + s(E)\mathcal{L}, & \bar{\nabla}^*_E \mathcal{L} &= -A_{\mathcal{L}} E + s^*(E)\mathcal{L}, \end{aligned} \tag{20}$$

respectively. Here, the induced connections on $\mathcal{M}$ are indicated by ∇, ∇^* and σ, σ^* denote the second fundamental forms associated with $\bar{\nabla}, \bar{\nabla}^*$. Taking $\mathcal{P}$ as the projection morphism of $T\mathcal{M}$ on $\mathfrak{s}(T\mathcal{M})$, the Gauss and Weingarten formulas for $\mathfrak{s}(T\mathcal{M})$ are given by

$$\begin{aligned} \nabla_E \mathcal{P}F &= \nabla'_E \mathcal{P}F + \rho(E, \mathcal{P}F)\zeta, & \nabla^*_E \mathcal{P}F &= \nabla'^*_E \mathcal{P}F + \rho^*(E, \mathcal{P}F)\zeta, \\ \nabla_E \zeta &= -A'^*_{\zeta} E + s'(E)\zeta, & \nabla^*_E \zeta &= -A'_{\zeta} E + s'^*(E)\zeta, \end{aligned} \tag{21}$$

$\{A'_\zeta, A'^*_\zeta\}$ are shape operators on $\mathfrak{s}(T\mathcal{M})$ and $\nabla'_E\mathcal{P}F, \nabla'^*_E\mathcal{P}F, A'_\zeta E, A'^*_\zeta E$ belong to $\Gamma(\mathfrak{s}(T\mathcal{M}))$. The induced geometric objects are related to each other in this way

$$\rho(E,\mathcal{P}F) = \bar{\mathfrak{g}}(A^*_\mathcal{L}E,\mathcal{P}F), \qquad \rho^*(E,\mathcal{P}F) = \bar{\mathfrak{g}}(A_\mathcal{L}E,\mathcal{P}F) \tag{22}$$
$$\sigma(E,F) = \bar{\mathfrak{g}}(A'^*_\zeta E,F), \qquad \sigma^*(E,F) = \bar{\mathfrak{g}}(A'_\zeta E,F),$$

for any $E,F \in \Gamma(T\mathcal{M}), \zeta \in \Gamma(\mathfrak{r}ad(T\mathcal{M}))$ and $\mathcal{L} \in \Gamma(tr(T\mathcal{M}))$.

Remark 3.1. Note that the induced connection on a non-degenerate submanifold of a statistical manifold is statistical which is not true for a lightlike submanifold of a statistical manifold.

Using (20) and the relation (2), it yields

$$\begin{aligned}\nabla_E\bar{\mathfrak{g}}(F,C) + \nabla^*_E\bar{\mathfrak{g}}(F,C) &= \sigma(E,F)u(C) + \sigma(E,C)u(F) \\ &\quad + \sigma^*(E,F)u(C) + \sigma^*(E,C)u(F)\end{aligned} \tag{23}$$

for all $E,F,C \in \Gamma(T\mathcal{M})$ where u is a 1-form such that $u(E) = \bar{\mathfrak{g}}(E,\mathcal{L})$.

Using (20) and (21), we have the following formulas for the statistical curvature tensor fields

$$\begin{aligned}2\bar{\mathcal{S}}(E,F)C &= 2\mathcal{S}(E,F)C - \sigma(F,C)A^*_\mathcal{L}E + \sigma(E,C)A^*_\mathcal{L}C \\ &\quad -\sigma^*(F,C)A_\mathcal{L}E + \sigma^*(E,C)A_\mathcal{L}F \\ &\quad +\{\sigma(F,C)s^*(E) - \sigma(E,C)s^*(F) \\ &\quad +\sigma^*(F,C)s(E) - \sigma^*(E,C)s(F) \\ &\quad +(\nabla_E\sigma)(F,C) - (\nabla_F\sigma)(E,C) \\ &\quad +(\nabla^*_E\sigma^*)(F,C) - (\nabla^*_F\sigma^*)(E,C)\}\mathcal{L},\end{aligned} \tag{24}$$

$$\begin{aligned}2\mathcal{S}(E,F)\mathcal{P}C &= 2\mathcal{S}'(E,F)\mathcal{P}C + \rho(E,\mathcal{P}C)A'^*_\zeta F - \rho(F,\mathcal{P}C)A'^*_\zeta E \\ &\quad +\rho^*(E,\mathcal{P}C)A'_\zeta F - \rho^*(F,\mathcal{P}C)A'_\zeta E \\ &\quad +\{(\nabla_E\rho)(F,\mathcal{P}C) - (\nabla_F\rho)(E,\mathcal{P}C) \\ &\quad +(\nabla^*_E\rho^*)(F,\mathcal{P}C) - (\nabla^*_F\rho^*)(E,\mathcal{P}C)\}\zeta\end{aligned} \tag{25}$$

where

$$\begin{aligned}((\nabla^*_E\sigma)(F,C) &= \nabla_E\sigma(F,C) - \sigma(\nabla_E F,C) - \sigma(F,\nabla_E C), \\ (\nabla^*_E\sigma^*)(F,C) &= \nabla^*_E\sigma(F,C) - \sigma(\nabla^*_E F,C) - \sigma(F,\nabla^*_E C), \\ (\nabla_E\rho)(F,C) &= \nabla_E\rho(F,C) - \rho(\nabla_E F,C) - \rho(F,\nabla_E C), \\ (\nabla^*_E\rho^*)(F,C) &= \nabla^*_E\rho^*(F,C) - \rho^*(\nabla_E F,C) - \rho^*(F,\nabla_E C)\end{aligned} \tag{26}$$

with $\mathcal{S}(E,F)C = \frac{1}{2}\{\mathcal{R}(E,F)C + \mathcal{R}^*(E,F)C)\}$ and $\mathcal{S}'(E,F)C = \frac{1}{2}\{\mathcal{R}'(E,F)C + \mathcal{R}'^*(E,F)C)\}$ for all $E, F, C \in \Gamma(T\mathcal{M})$.

Furthermore, from (20), we derive

$$\begin{aligned} \sigma(E,\xi_i) &= \rho(E,\zeta_i), \qquad i = 1,2,3 \\ \nabla_E \xi_i &= -\psi_i A_{\mathcal{L}} E + s^*(E)\xi_i, \qquad \nabla^*_E \xi_i = -\psi_i A^*_{\mathcal{L}} E + s(E)\xi_i \\ \nabla_E \zeta_i &= -\psi_i A'_\zeta E + s'^*(E)\zeta_i, \qquad \nabla^*_E \zeta_i = -\psi_i A'^*_\zeta E + s'(E)\zeta_i \end{aligned} \tag{27}$$

Definition 3.2. [3, 4] Let $(\mathcal{M}, \mathfrak{g}, \nabla)$ be a lightlike real hypersurface of an almost para-hyperhermitian statistical manifold $(\bar{\mathcal{M}}, \bar{\mathfrak{g}}, \bar{\nabla}, \mathfrak{X} = (\mathfrak{X}_i)_{i=1,2,3})$. It is said that $(\mathcal{M}, \mathfrak{g})$ is

1. totally umbilical with respect to $\bar{\nabla}$ and $\bar{\nabla}^*$ if there exist smooth functions κ and κ^* on a neighborhood $\mathcal{U}$ such that $\sigma(E,F) = \kappa\mathfrak{g}(E,F)$ and $\sigma^*(E,F) = \kappa^*\mathfrak{g}(E,F)$, respectively.

2. totally geodesic with respect to $\bar{\nabla}$ and $\bar{\nabla}^*$ if $\sigma = \sigma^* = 0$

3. screen locally conformal with respect to $\bar{\nabla}$ and $\bar{\nabla}^*$ if the shape operators $\{A_{\mathcal{L}}, A'_\zeta\}$ and $\{A^*_{\mathcal{L}}, A'^*_\zeta\}$ are related by

$$A_{\mathcal{L}} E = \gamma A'_\zeta E, \qquad A^*_{\mathcal{L}} E = \gamma^* A'^*_\zeta E, \tag{28}$$

for all $E, F \in \Gamma(T\mathcal{M})$. Here, γ, γ^* are smooth functions on a neighborhood $\mathcal{U}$ in $\mathcal{M}$ which do not vanish.

Definition 3.3. It is said that $(\bar{\mathcal{M}}, \bar{\mathfrak{g}}, \bar{\nabla}, \mathfrak{X} = (\mathfrak{X}_i)_{i=1,2,3})$ of real dimension $4n \geq 8$ is of constant holomorphic sectional curvature $\bar{c}$ if and only if

$$\begin{aligned} \bar{\mathcal{S}}(E,F)C = \frac{\bar{c}}{4}\{&\bar{\mathfrak{g}}(F,C)E - \bar{\mathfrak{g}}(E,C)F \\ &+ \Sigma_{i=1}^3 \tau_i[\bar{\mathfrak{g}}(\mathfrak{X}_i F, C)\mathfrak{X}_i E - \bar{\mathfrak{g}}(\mathfrak{X}_i E, C)\mathfrak{X}_i F + 2\bar{\mathfrak{g}}(E, \mathfrak{X}_i F)\mathfrak{X}_i C]\} \end{aligned} \tag{29}$$

holds for E, F, C on $\Gamma(\bar{\mathcal{M}})$.

Lemma 3.4. *Let $(\mathcal{M}, \mathfrak{g}, \nabla)$ be a lightlike real hypersurface of an almost para-hyperhermitian statistical manifold $(\bar{\mathcal{M}}, \bar{\mathfrak{g}}, \bar{\nabla}, \mathfrak{X} = (\mathfrak{X}_i)_{i=1,2,3})$ of constant holomorphic sectional curvature $\bar{c}$. Then, we conclude that*

$$\begin{aligned}
&\frac{\bar{c}}{2}\{\mu_i(F)\bar{\mathfrak{g}}(\mathfrak{X}_iE, C) - \mu_i(E)\bar{\mathfrak{g}}(\mathfrak{X}_iF, C) - 2\mu_i(C)\bar{\mathfrak{g}}(E, \mathfrak{X}_iF)\} \\
&= 2\bar{\mathfrak{g}}(\bar{\mathcal{S}}(E, F)C, \zeta) = \sigma(F, C)s^*(E) - \sigma(E, C)s^*(F) \\
&+ \sigma^*(F, C)s(E) - \sigma^*(E, C)s(F) \\
&+ (\nabla_F\sigma)(E, C) - (\nabla_F\sigma)(E, C) \\
&+ (\nabla^*_E\sigma^*)(F, C) - (\nabla^*_F\sigma^*)(E, C)
\end{aligned} \tag{30}$$

and

$$\begin{aligned}
&\frac{\bar{c}}{2}\{\bar{\mathfrak{g}}(F, C)u(E) - \bar{\mathfrak{g}}(E, C)u(F) + \bar{\mathfrak{g}}(\mathfrak{X}_iE, C)\nu_i(F) \\
&- \bar{\mathfrak{g}}(\mathfrak{X}_iF, C)\nu_i(E) - 2\bar{\mathfrak{g}}(E, \mathfrak{X}_iF)\nu_i(C)\} \\
&= 2\bar{\mathfrak{g}}(\bar{\mathcal{S}}(E, F)C, \mathcal{L}) = 2\mathfrak{g}(\bar{\mathcal{S}}(E, F)C, \mathcal{L}) \\
&= -\sigma(F, C)\bar{\mathfrak{g}}(A^*_{\mathcal{L}}E, \mathcal{L}) + \sigma(E, C)\bar{\mathfrak{g}}(A^*_{\mathcal{L}}F, \mathcal{L}) \\
&- \sigma^*(F, C)\bar{\mathfrak{g}}(A_{\mathcal{L}}E, \mathcal{L}) + \sigma^*(E, C)\bar{\mathfrak{g}}(A_{\mathcal{L}}F, \mathcal{L})
\end{aligned} \tag{31}$$

for all $E, F, C \in \Gamma(T\mathcal{M})$, $\mathcal{L} \in ltr(T\mathcal{M})$ and $\zeta \in \Gamma(\mathfrak{r}ad(T\mathcal{M}))$.

Proof. By taking the inner product with ζ and $\mathcal{L}$ to (24) and using (29), we get (30) and (31), respectively. □

Proposition 3.5. *Let $(\mathcal{M}, \mathfrak{g}, \nabla)$ be a lightlike real hypersurface of an almost para-hyperhermitian statistical manifold $(\bar{\mathcal{M}}, \bar{\mathfrak{g}}, \bar{\nabla}, \mathfrak{X} = (\mathfrak{X}_i)_{i=1,2,3})$ of constant holomorphic sectional curvature $\bar{c}$, then we obtain*

$$\begin{aligned}
\frac{\bar{c}}{2}\{\mu_i(F)\nu_i(E) - \mu_i(E)\nu_i(F)\} &= \rho(F, A'_\zeta E) - \rho(E, A'_\zeta F) \\
&+ \rho^*(F, A'^*_\zeta E) - \rho^*(E, A'^*_\zeta F) \\
&- \sigma(F, \zeta)\bar{\mathfrak{g}}(A^*_{\mathcal{L}}E, \mathcal{L}) + \sigma(E, \zeta)\bar{\mathfrak{g}}(A^*_{\mathcal{L}}F, \mathcal{L}) \\
&- \sigma^*(F, \zeta)\bar{\mathfrak{g}}(A_{\mathcal{L}}E, \mathcal{L}) + \sigma^*(E, \zeta)\bar{\mathfrak{g}}(A_{\mathcal{L}}F, \mathcal{L}) \\
&- 2ds(E, F) - 2d^*s^*(E, F)
\end{aligned} \tag{32}$$

for all $E, F \in \Gamma(T\mathcal{M})$.

Proof. Putting $C = \zeta$ into relation (31), we get

$$\begin{aligned}\frac{\bar{c}}{2}\{\mu_i(F)\nu_i(E) - \mu_i(E)\nu_i(F)\} = 2\bar{\mathfrak{g}}(\mathcal{S}(E,F)\zeta,\mathcal{L}) \\ - \sigma(F,\zeta)\bar{\mathfrak{g}}(A^*_{\mathcal{L}}E,\mathcal{L}) + \sigma(E,\zeta)\bar{\mathfrak{g}}(A^*_{\mathcal{L}}F) \\ - \sigma^*(F,\zeta)\bar{\mathfrak{g}}(A_{\mathcal{L}}E,\mathcal{L}) + \sigma^*(E,\zeta)\bar{\mathfrak{g}}(A_{\mathcal{L}}F,\mathcal{L}) \\ - 2ds(E,F) - 2d^*s^*(E,F)\end{aligned} \tag{33}$$

for all $E, F \in \Gamma(T\mathcal{M})$. However, using (20) and (21), we obtain

$$\bar{\mathfrak{g}}(\mathcal{R}(E,F)\zeta,\mathcal{L}) = \bar{\mathfrak{g}}(A'^*_{\zeta}F, A_{\mathcal{L}}E) - \bar{\mathfrak{g}}(A'^*_{\zeta}E, A_{\mathcal{L}}F) - 2ds(E,F) \tag{34}$$

where $ds(E,F) = \nabla_E s(F) - \nabla_F s(E) - s[E,F]$.

Similarly,

$$\bar{\mathfrak{g}}(\mathcal{R}^*(E,F)\zeta,\mathcal{L}) = \bar{\mathfrak{g}}(A'_{\zeta}F, A^*_{\mathcal{L}}E) - \bar{\mathfrak{g}}(A'_{\zeta}E, A^*_{\mathcal{L}}F) - 2d^*s^*(E,F) \tag{35}$$

with $d^*s^*(E,F) = \nabla^*_E s^*(F) - \nabla^*_F s^*(E) - s^*[E,F]$. Then, from (33), (34) and (35), the assertion follows. $\square$

Theorem 3.6. *Let $(\mathcal{M}, \mathfrak{g}, \nabla)$ be a lightlike real hypersurface of an almost para-hyperhermitian statistical manifold $(\bar{\mathcal{M}}, \bar{\mathfrak{g}}, \bar{\nabla}, \mathfrak{X} = (\mathfrak{X}_i)_{i=1,2,3})$ of constant holomorphic sectional curvature $\bar{c}$. If ξ_i, $(i = 1,2,3)$, are eigenvectors of A'_{ζ} and A'^*_{ζ}, then we get*

$$\begin{aligned}&\frac{\bar{c}}{4}\{\mu_i(F)u(E) - \mu_i(E)u(F) - 2\tau_i\bar{\mathfrak{g}}(E,\mathfrak{X}_iF)\} \\ &= E(\beta_i + \beta^*_i)\nu_i(F) - F(\beta_i + \beta^*_i)\nu_i(E) \\ &+ \beta_i\bar{\mathfrak{g}}(E,(\psi_iA_{\mathcal{L}} + A_{\mathcal{L}}\psi_i)F) \\ &+ \beta^*_i\bar{\mathfrak{g}}(E,(\psi_iA^*_{\mathcal{L}} + A^*_{\mathcal{L}}\psi_i)F) \\ &- 2\bar{\mathfrak{g}}(E,(A_{\mathcal{L}}\psi_iA'_{\zeta} + A^*_{\mathcal{L}}\psi_iA'^*_{\zeta})F) \\ &+ 2(\beta_i - \beta^*_i)\{s(E)\nu_i(F) - s(F)\nu_i(E)\}\end{aligned} \tag{36}$$

for all $E, F \in \Gamma(T\mathcal{M})$ where $\beta_i = \tau_i\mu_i(A'_{\zeta}\xi_i)$ and $\beta^*_i = \tau_i\mu_i(A'^*_{\zeta}\xi_i)$.

Proof. From $A'^*_{\zeta}\xi_i = \beta^*_i\xi_i$, $i = 1,2,3$ and using relation (27), we obtain

$$\begin{aligned}\bar{\mathfrak{g}}((\nabla^*_E A'^*_{\zeta})F,\xi_i) = (E\beta^*_i)\nu_i(F) - \beta^*_i\bar{\mathfrak{g}}(F,\psi_iA^*_{\mathcal{L}}E) \\ + \bar{\mathfrak{g}}(A'^*_{\zeta}F,\psi_iA^*_{\mathcal{L}}E),\end{aligned} \tag{37}$$

Similarly, we have

$$\bar{\mathfrak{g}}((\nabla_E A'_\zeta)F,\xi_i) = (E\beta_i)\nu_i(F) - \beta_i\bar{\mathfrak{g}}(F,\psi_i A_{\mathcal{L}}E) + \bar{\mathfrak{g}}(A'_\zeta F,\psi_i A_{\mathcal{L}}E) \tag{38}$$

So, putting $C = \xi_i$, $i = 1,2,3$ into (30) and using equations (37) and (38), the assertion follows. □

Theorem 3.7. *Let $(\mathcal{M},\mathfrak{g},\nabla)$ be a screen locally conformal lightlike real hypersurface of an almost para-hyperhermitian statistical manifold $(\bar{\mathcal{M}},\bar{\mathfrak{g}},\bar{\nabla},\mathfrak{X}=(\mathfrak{X}_i)_{i=1,2,3})$ of constant holomorphic sectional curvature $\bar{c}$. If the vector fields ξ_i,ζ_i $i=1,2,3$ are eigenvalues of the shape operators $A_{\mathcal{L}},A^*_{\mathcal{L}}$ then, we have*

$$\frac{\bar{c}}{2} = \frac{\lambda_i\lambda_i^* - \lambda_i\alpha_i^*}{\gamma} + \frac{\lambda_i\lambda_i^* - \lambda_i^*\alpha_i}{\gamma^*} - 2(ds + d^*s^*)(\zeta_i,\xi_i) \tag{39}$$

where

$$A_{\mathcal{L}}\xi_i = \lambda_i\xi_i, \qquad A^*_{\mathcal{L}}\xi_i = \lambda_i^*\xi_i, \qquad A_{\mathcal{L}}\zeta_i = \alpha_i\zeta_i, \qquad A^*_{\mathcal{L}}\zeta_i = \alpha_i^*\zeta_i. \tag{40}$$

Proof. Putting $F = \xi_i$ and $E = \zeta_i$, $i = 1,2,3$ into relation (32) and using (28), we get

$$\begin{aligned}
\frac{\bar{c}}{2}\{\mu_i(\xi)\nu_i(\zeta_i) - \mu_i(\zeta_i)\nu_i(\xi_i)\} &= \rho(\xi_i,A'_\zeta\zeta_i) - \rho(\zeta_i,A'_\zeta\xi_i)\\
&+ \rho^*(\xi_i,A'^*_\zeta\zeta_i) - \rho^*(\zeta_i,A'^*_\zeta\xi_i),\\
&- \sigma(\xi_i,\zeta)\bar{\mathfrak{g}}(A^*_{\mathcal{L}}\zeta_i,\mathcal{L}) + \sigma(\zeta_i,\zeta)\bar{\mathfrak{g}}(A^*_{\mathcal{L}}\xi_i,\mathcal{L})\\
&- \sigma^*(\xi_i,\zeta)\bar{\mathfrak{g}}(A_{\mathcal{L}}\zeta_i,\mathcal{L}) + \sigma^*(\zeta_i,\zeta)\bar{\mathfrak{g}}(A_{\mathcal{L}}\xi_i,\mathcal{L})\\
&- 2ds(\zeta_i,\xi_i) - 2d^*s^*(\zeta_i,\xi_i)
\end{aligned}$$

Using relations (22) in the above equation and taking into account that $\mathcal{M}$ is screen conformal, we have

$$A'^{*}_{\zeta}\zeta_i = \frac{1}{\gamma^*}A^*_{\mathcal{L}}\zeta_i = \frac{1}{\gamma^*}\alpha^*\zeta_i, \qquad A'_{\zeta}\zeta_i = \frac{1}{\gamma}A_{\mathcal{L}}\zeta_i = \frac{1}{\gamma}\alpha\zeta_i$$

Since ξ_i, ζ_i are eigenvalues of $A_{\mathcal{L}}, A^*_{\mathcal{L}}$. From two last equations and the fact that $\bar{\mathfrak{g}}(\xi_i,\zeta_i) = 1$, $\bar{\mathfrak{g}}(\xi_i,\xi_i) = \bar{\mathfrak{g}}(\mathcal{L},\mathcal{L}) = 0 = \bar{\mathfrak{g}}(\zeta_i,\zeta_i) = \bar{\mathfrak{g}}(\zeta,\zeta)$, we can state

$$\begin{aligned}\frac{\bar{c}}{2} &= \tau_i\frac{\lambda_i\lambda_i^* - \tau_i\lambda_i\bar{\mathfrak{g}}(A^*_{\mathcal{L}}\zeta_i,\xi_i)}{\gamma} + \frac{\lambda_i\lambda_i^* - \tau_i\lambda_i^*\bar{\mathfrak{g}}(A_{\mathcal{L}}\zeta_i,\xi_i)}{\gamma^*} \\ &\quad - 2(ds + d^*s^*)(\zeta_i,\xi_i).\end{aligned} \tag{41}$$

So, we get the assertion. □

Corollary 3.8. *Let* $(\mathcal{M},\mathfrak{g},\nabla)$ *be a screen locally conformal lightlike real hypersurface of an almost para-hyperhermitian statistical manifold* $(\bar{\mathcal{M}},\bar{\mathfrak{g}},\bar{\nabla},\mathfrak{X} = (\mathfrak{X}_i)_{i=1,2,3})$ *of constant holomorphic sectional curvature* $\bar{c}$. *If the vector fields* ζ_i, ζ_i $i = 1,2,3$ *are eigenvalues of the shape operators* $A_{\mathcal{L}}, A^*_{\mathcal{L}}$, *such that*

$$\begin{aligned} A_{\mathcal{L}}\xi_i &= \lambda_i\xi_i, \qquad A^*_{\mathcal{L}}\xi_i = \lambda_i^*\xi_i \\ A_{\mathcal{L}}\zeta_i &= \alpha_i\zeta_i, \qquad A^*_{\mathcal{L}}\zeta_i = \alpha_i^*\zeta_i \end{aligned} \tag{42}$$

then we have

$$2(ds + d^*s^*)(\xi_i,\zeta_j) = 0. \tag{43}$$

Proof. Putting $F = \xi_i$ and $E = \zeta_j$ into relation (32) and using (20), we get the assertion. □

Example 3.9. Let consider $\bar{\mathcal{M}} = \mathbb{R}^8_4$ equipped with para-hyperhermitian structure $\mathfrak{X} = (\mathfrak{X}_1,\mathfrak{X}_2,\mathfrak{X}_3)$ as follows

$$\begin{aligned} \mathfrak{X}_1(y_1,y_2,y_3,y,y_5,y_6,y_7,y_8) &= (-y_3,y_4,-y_1,y_2,-y_7,y_8,-y_5,y_6) \\ \mathfrak{X}_2(y_1,y_2,y_3,y_4,y_5,y_6,y_7,y_8) &= (y_4,y_3,y_2,y_1,y_8,y_7,y_6,y_5) \\ \mathfrak{X}_3(y_1,y_2,y_3,y_4,y_5,y_6,y_7,y_8) &= (-y_2,y_1,-y_4,y_3,-y_6,y_5,-y_8,y_7). \end{aligned}$$

with Cartesian coordinate $(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8)$ with the metric $\bar{\mathfrak{g}}$

$$\bar{\mathfrak{g}} = -dy_1^2 - dy_2^2 - dy_3^2 - dy_4^2 + dy_5^2 + dy_6^2 + dy_7^2 + dy_8^2.$$

By taking $\frac{\partial}{\partial y_i} = w_i$, we define statistical connections $\bar{\nabla}, \bar{\nabla}^*$ on $\bar{\mathcal{M}}$ as follows

$$\bar{\nabla}_{w_i} w_i = w_i = -\bar{\nabla}^*_{w_i} w_i, \qquad i = 1, .., 8$$

and other components are zero. Then, $\bar{\mathcal{M}}$ is an almost para-hyperhermitian statistical manifold.

Let $\mathcal{M}$ be a hypersurface of $(\mathbb{R}^8_4, \bar{\mathfrak{g}}, \mathfrak{X} = (\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3))$ such that

$$\begin{aligned} y_1 &= t_1 + cos\alpha t_5, & y_2 &= t_4 \\ y_3 &= -t_2, & y_4 &= t_3 + t_7 \\ y_5 &= cos\alpha t_1 - sin\alpha t_4 + t_5, & y_6 &= sin\alpha t_1 + cos\alpha t_4 \\ y_7 &= -cos\alpha t_2 + sin\alpha t_3 + t_6, & y_8 &= sin\alpha t_2 + cos\alpha t_3 \end{aligned}$$

where $\alpha \in \mathbb{R} - \{\pi + k\pi, k \in \mathbb{Z}\}$. Then, $T\mathcal{M}$ is spanned by

$$\begin{aligned} E_1 &= \partial y_1 + cos\alpha\partial y_5 + sin\alpha\partial y_6, & E_2 &= -\partial y_3 - cos\alpha\partial y_7 + sin\alpha y_8, \\ E_3 &= \partial y_4 + sin\alpha\partial y_7 + cos\alpha\partial y_8, & E_4 &= \partial y_2 - sin\alpha\partial y_5 + cos\alpha\partial y_6, \\ E_5 &= cos\alpha\partial y_1 + \partial y_5, & E_6 &= \partial y_7, \\ E_7 &= \partial y_4 \end{aligned}$$

and we can see that $\mathfrak{X}_1 E_1 = E_2, \mathfrak{X}_2 E_1 = E_2$ and $\mathfrak{X}_3 E_1 = E_3$. Considering $E' = cot\alpha\partial y_2 + csc\alpha\partial y_6$, we get $\mathcal{L} = cot\alpha\partial y_2 - \frac{1}{2}(\partial y_1 + cos\alpha\partial y_5) + (csc\alpha - \frac{1}{2}sin\alpha)\partial y_6$. Thus, we have

$$\begin{aligned} \xi_1 &= cot\alpha\partial y_4 + \frac{1}{2}\partial y_3 + \frac{1}{2}cos\alpha\partial y_7 + (csc\alpha - \frac{1}{2}sin\alpha)\partial y_8, \\ \xi_2 &= cot\alpha\partial y_3 - \frac{1}{2}\partial y_4 - \frac{1}{2}cos\alpha\partial y_8 + (csc\alpha - \frac{1}{2}sin\alpha)\partial y_7, \\ \xi_3 &= -cot\alpha\partial y_1 - \frac{1}{2}\partial y_2 - \frac{1}{2}cos\alpha\partial y_6 - (csc\alpha - \frac{1}{2}sin\alpha)\partial y_5 \end{aligned}$$

$$\begin{aligned} &\zeta_1 = -\partial y_3 - cos\alpha\partial y_7 + sin\alpha\partial y_8, \\ &\zeta_2 = \partial y_4 + cos\alpha\partial y_8 + sin\alpha\partial y_7, \qquad \zeta_3 = \partial y_2 + cos\alpha\partial y_6 - sin\alpha\partial y_5. \end{aligned}$$

Using the Gauss formulas for $\mathcal{M}$ and $\mathfrak{s}(T\mathcal{M})$, we obtain

$$\begin{aligned}
\bar{\nabla}_{E_1}E_1 &= -\partial y_1 + cos^2\alpha\partial y_5 + sin^2\alpha\partial y_6,\\
\bar{\nabla}_{E_2}E_2 &= \partial y_3 + cos^2\alpha\partial y_7 + sin^2\alpha\partial y_8\\
\bar{\nabla}_{E_3}E_3 &= \partial y_4 + sin^2\alpha\partial y_7 + cos^2\partial y_8,\\
\bar{\nabla}_{E_4}E_4 &= \partial y_2 + sin^2\alpha\partial y_5 + cos^2\alpha\partial y_6,\\
\bar{\nabla}_{E_5}E_5 &= cos^2\alpha\partial y_1 + \partial y_5,\\
\bar{\nabla}_{E_6}E_6 &= \partial y_7,\\
\bar{\nabla}_{E_7}E_7 &= \partial y_4,\\
\bar{\nabla}_{\xi_1}\xi_1 &= cot^2\alpha\partial y_4 + \frac{1}{4}\partial y_3 + \frac{1}{4}cos^2\alpha\partial y_7 + (csc\alpha - \frac{1}{2}sin\alpha)\partial y_8\\
\bar{\nabla}_{\xi_1}\zeta_1 &= \frac{1}{2}cos^2\alpha\partial y_7 + sin\alpha(csc\alpha - \frac{1}{2}sin\alpha)\partial y_8.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\sigma(E_1,E_1) = cos^3\alpha + sin^3\alpha, &\qquad \sigma(E_2,E_2) = 0,\\
\sigma(E_3,E_3) = 0, &\qquad \sigma(E_4,E_4) = sin^2\alpha cos\alpha + cos^2\alpha sin\alpha,\\
\sigma(E_5,E_5) = -cos^2\alpha + cos\alpha, &\qquad \sigma(E_6,E_6) = 0,\\
\sigma(E_7,E_7) = 0. &
\end{aligned}$$

We compute

$$\begin{aligned}
\nabla_{\xi_1}\xi_1 &= cot^2\alpha\partial_4 + \frac{1}{4}\partial y_3 + \frac{1}{4}cos^2\alpha\partial y_7 + (csc\alpha - \frac{1}{2}sin\alpha)\partial y_8 \quad (44)\\
\nabla_{\xi_1}\zeta_1 &= \frac{1}{2}cos^2\alpha\partial y_7 + sin\alpha(csc\alpha - \frac{1}{2}sin\alpha)\partial y_8\\
\rho(\xi_1,\xi_1) &= \rho(\xi_1,\zeta_1) = 0.
\end{aligned}$$

Thus, $\mathcal{M}$ is a real lightlike hypersurface of $(\mathbb{R}^8_4, \bar{\mathfrak{g}}, \mathfrak{X} = (\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3), \bar{\nabla})$.

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