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Hybrid Numbers with Hybrid Leonardo Number Coefficients

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Abstract. In this study we define a new generalization of the hybrid Leonardo sequence consisting of the hybrid numbers with hybrid Leonardo numbers coefficients. We investigate some algebraic properties of this new sequence and also the generating function, exponential generating function, and the Binet formula related to this type of sequence. In addition, some identities are provided, such as Catalan´s and Cassini's identities, and sums are related.

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1 Introduction

In [1], Catarino and Borges devoted to the Leonardo sequence and adopt the expression Le_n to denote the n-th Leonardo number and conse-

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quently, the Leonardo sequence is denoted by $\{Le_n\}_{n=0}^{\infty}$. This sequence is defined by the following recurrence relation

$$Le_n = Le_{n-1} + Le_{n-2} + 1,\ n \geq 2,$$

with initial conditions $Le_0 = Le_1 = 1$. The first thirty Leonardo numbers are

$$1, 1, 3, 5, 9, 15, 25, 41, 67, 109, 177, 287, 465, 753, 1219, 1973, 3193, 5167, 8361, 13529, 21891, 35421, 57323, 92745, 150069, 242815, 392885, 635701, 1028587, 1664289.$$

In [8], Ozdemir defined the set of hybrid numbers which contains complex, dual, and hyperbolic numbers as

$$\begin{aligned}\mathbb{K} =&\{a + bi + c\epsilon + dh : a, b, c, d \in \mathbb{R}, i^2 = -1,\\ &\epsilon^2 = 0, h^2 = 1, ih = -hi = \epsilon + i\}.\end{aligned}$$

This number system is a generalization of complex $(i^2 = -1)$, hyperbolic $(h^2 = 1)$ and dual number $(\epsilon^2 = 0)$ systems. Here, i is a complex unit, ϵ is the dual unit and h is the hyperbolic unit. We call these units hybrid units. In the last few years, researchers from many different fields have taken this number system and used it in various fields of applied sciences. For some applications of hybrid numbers, we refer the reader to [8]. There is no doubt that this number system will be studied by other applied science researchers in the near future.

According to [8] two hybrid numbers are equal if all their components are equal, one by one. The sum of two hybrid numbers is defined by summing their components. Addition operation in the hybrid numbers is both commutative and associative. Zero is the null element. With respect to the addition operation, the symmetric element of k is $-k$, which is defined as having all the components of k changed in their signs. This implies that, $(\mathbb{K}, +)$ is an Abelian group. The conjugate of a hybrid number $k = a + bi + c\epsilon + dh$ is defined by $\overline{k} = a - bi - c\epsilon - dh$. From the definition of hybrid numbers, the multiplication table of the hybrid units is given by the following table:

Table 1 shows that the multiplication of hybrid numbers is not commutative.

Table 1: The Multiplication Table for Hybrid Units

$\bullet$	1	i	ϵ	h
1	1	i	ϵ	h
i	i	-1	$1-h$	$\epsilon+i$
ϵ	ϵ	$h+1$	0	$-\epsilon$
h	h	$-(\epsilon+i)$	ϵ	1

The hybrid Leonardo numbers were introduced by [10] and defined by

$$HLe_n = Le_n + Le_{n+1}i + Le_{n+2}\epsilon + Le_{n+3}h \ ,$$

for all $n \geq 0$. The authors gave some algebraic properties of the hybrid Leonardo numbers such as recurrence relation, generating function, Binet's formula, sum formulas, Catalan's identity, and Cassini's identity.

In this perspective, new extensions of these sequences of numbers continue to be studied. We can cite here recent papers [2, 3, 4, 5, 6, 7], where generalizations of Leonardo numbers are considered.

An important article to be cited is [9], where the author introduced the hybrid numbers with Fibonacci and Lucas hybrid number coefficients. Moreover, it is established the Binet formulas, generating functions, and exponential generating functions for these numbers.

Motivated by the generalization and results given in [9] and the direct connections that the Fibonacci and Leonardo sequences have, in this article we will introduce the hybrid numbers with Leonardo hybrid number coefficients. Also, we will establish some algebraic properties such as recurrence relation, generating function, Binet's formula, sum formulas, and Cassini's identity.

Note that in this article we chose to multiply imaginary units to the right of the coefficients.

This article is organized as follows. In Section 2, we introduce the hybrid numbers with hybrid Leonardo numbers coefficients and provide some algebraic properties of these numbers. Section 3 is devoted to establishing the Binet formula and the generating function of the hybrid numbers with hybrid Leonardo numbers coefficients. As a consequence, we derive the exponential generating function. Finally, in Section 4 and 5, we provide some identities, such as Cassini's identity, and sums involving this new sequence of hybrid numbers.

2 Hybrid Numbers with Hybrid Leonardo Number Coefficients

In this section, we define hybrid numbers with Leonardo hybrid number coefficients and provide some properties of this new sequence of numbers. We start considering the following definition,

Definition 2.1. For integers $n \geq 0$, the hybrid numbers with Leonardo hybrid number coefficients are defined recursively by

$$\mathbb{L}e_n = HLe_n + HLe_{n+1}i + HLe_{n+2}\epsilon + HLe_{n+3}h, \tag{1}$$

where HLe_n is the $n-$th hybrid Leonardo number, $\mathbf{i}, \epsilon$ and h are hybrid units.

According to [10], the recurrence relation of the hybrid Leonardo sequence is given by

$$HLe_n = HLe_{n-1} + HLe_{n-2} + A \tag{2}$$

for $n \geq 2$, where $A = 1 + i + \epsilon + h$ and the initial conditions are $HLe_0 = 1 + i + 3\epsilon + 5h$ and $HLe_1 = 1 + 3i + 5\epsilon + 9h$.

We have the next auxiliary result.

Lemma 2.2. *Consider* $A = 1 + i + \epsilon + h$ *then* $A^2 = 2A + 1$.

Proof. We have that $A = 1 + i + \epsilon + h$, then

$$\begin{aligned} A^2 &= A \cdot A \\ &= (1 + i + \epsilon + h)(1 + i + \epsilon + h) \\ &= A + Ai + A\epsilon + Ah \\ &= 3 + 2i + 2\epsilon + 2h \\ &= 2A + 1 \ , \end{aligned}$$

which completes the proof. □

Expressions (1) and (2) give us the recurrence relation for the hybrid numbers with Leonardo hybrid number coefficients,

Proposition 2.3. *For $n \geq 0$ the hybrid sequence $\{\mathbb{L}e_n\}_{n\geq 0}$ satisfies the recurrence relation*

$$\mathbb{L}e_n = \mathbb{L}e_{n-1} + \mathbb{L}e_{n-2} + 2A + 1$$

for $n \geq 2$, where $A = 1 + i + \epsilon + h$ and the initial conditions are $\mathbb{L}e_0 = 28 + 2i + 6\epsilon + 10h$ and $\mathbb{L}e_1 = 55 + 6i + 10\epsilon + 18h$.

Now, by considering that it is verified the recurrence relation [Equation 2.2 in [10]]

$$HLe_{n+1} = 2HLe_n - HLe_{n-2},$$

we obtain other recurrence relation for the hybrid numbers with Leonardo hybrid number coefficients, given in the next result.

Proposition 2.4. *For $n \geq 2$ the hybrid sequence $\{\mathbb{L}e_n\}_{n\geq 0}$ satisfies the recurrence relation*

$$\mathbb{L}e_{n+1} = 2\mathbb{L}e_n - \mathbb{L}e_{n-2},$$

with the initial conditions are $\mathbb{L}e_0 = 28 + 2i + 6\epsilon + 10h$ and $\mathbb{L}e_1 = 55 + 6i + 10\epsilon + 18h$.

Theorem 2.3 in [10] gives us the closed relation between the n-th hybrid Leonardo number and the n-th hybrid Fibonacci number, namely,

$$HLe_n = 2HF_{n+1} - A, \tag{3}$$

for any integer $n \geq 0$, where $A = 1 + i + \epsilon + h$ and HF_n is the n-th hybrid Fibonacci number. Then, by replacing the (3) in Expression (1) we obtain

$$\begin{aligned}
\mathbb{L}e_n &= (2HF_{n+1} - A) + (2HF_{n+2} - A)i + (2HF_{n+3} - A)\epsilon \\
&\quad + (2HF_{n+4} - A)h \\
&= 2(HF_{n+1} + HF_{n+2}i + HF_{n+3}\epsilon + HF_{n+4}h) - 2A - 1, \\
&= 2(\mathbb{F}_{n+1} - A) - 1,
\end{aligned}$$

where $A = 1 + i + \epsilon + h$.

Under the previous discussion, the next result is given.

Proposition 2.5. *For any integer $n \geq 0$*

$$\mathbb{L}e_n = 2(\mathbb{F}_{n+1} - A) - 1, \tag{4}$$

where $A = 1 + i + \epsilon + h$ and $\mathbb{F}_n$ is the n-th hybrid number with hybrid Fibonacci number coefficients.

3 The Binet Formula and Generating Function

In this section, we establish the Binet formula for the hybrid numbers with hybrid Leonardo coefficients by considering the Binet formula [Theorem 2.4, [10]] given by

$$\mathbb{F}_n = \frac{(\underline{\alpha})^2\alpha^n - (\underline{\beta})^2\beta^n}{\alpha - \beta} \tag{5}$$

where $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation of the Fibonacci sequence, $\underline{\alpha} = 1 + i\alpha + \epsilon\alpha^2 + h\alpha^3$, and $\underline{\beta} = 1 + i\beta + \epsilon\beta^2 + h\beta^3$.

Thus we have the next result.

Theorem 3.1. *For $n \geq 0$, the Binet formula for hybrid numbers with hybrid Leonardo numbers coefficients is given by,*

$$\mathbb{L}e_n = 2\left(\frac{(\underline{\alpha})^2\alpha^{n+1} - (\underline{\beta})^2\beta^{n+1}}{\alpha - \beta}\right) - 2A - 1, \tag{6}$$

where $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation of the Fibonacci sequence, $\underline{\alpha} = 1 + i\alpha + \epsilon\alpha^2 h + \alpha^3$, $\underline{\beta} = 1 + i\beta + \epsilon\beta^2 + h\beta^3$, and $A = 1 + i + \epsilon + h$.

Proof. The proof follows by replacing Expression (5) in (4). □

As a consequence, we can provide the exponential generating function for the generating function of the hybrid numbers with hybrid Leonardo coefficients by considering Expression (6).

Proposition 3.2. *For $n \geq 0$, the exponential generating function for hybrid numbers with hybrid Leonardo number coefficients is given by,*

$$\sum_{n=0}^{\infty} \mathbb{L}e_n \frac{t^n}{n!} = 2\left(\frac{(\underline{\alpha})^2\alpha e^{\alpha t} - (\underline{\beta})^2\beta e^{\beta t}}{\alpha - \beta}\right) - (2A+1)e^t$$

where $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation of the Fibonacci sequence, $\underline{\alpha} = 1 + i\alpha + \epsilon\alpha^2 h + \alpha^3$, $\underline{\beta} = 1 + i\beta + \epsilon\beta^2 + h\beta^3$, and $A = 1 + i + \epsilon + h$.

Proof. Let $\sum_{n=0}^{\infty} \mathbb{L}e_n \frac{t^n}{n!}$ be the exponential generating function of the hybrid numbers with hybrid Leonardo number coefficients. By using Expression (6) we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \mathbb{L}e_n \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \left(2 \left(\frac{(\underline{\alpha})^2 \alpha^{n+1} - (\underline{\beta})^2 \beta^{n+1}}{\alpha - \beta} \right) - (2A+1) \right) \frac{t^n}{n!} \\ &= 2 \left(\frac{\alpha (\underline{\alpha})^2}{\alpha - \beta} \right) \sum_{n=0}^{\infty} \frac{\alpha^n t^n}{n!} - 2 \left(\frac{\beta (\underline{\beta})^2}{\alpha - \beta} \right) \sum_{n=0}^{\infty} \frac{\beta^n t^n}{n!} \\ &- (2A+1) \sum_{n=0}^{\infty} \frac{t^n}{n!} \\ &= 2 \left(\frac{(\underline{\alpha})^2 \alpha e^{\alpha t} - (\underline{\beta})^2 \beta e^{\beta t}}{\alpha - \beta} \right) - (2A+1) e^t,\end{aligned}$$

which completes the proof. □

In what follows, we establish the generating function of the hybrid numbers with hybrid Leonardo coefficients by considering the generating function [Theorem 2.5, [10]] given by

$$\sum_{n=0}^{\infty} \mathbb{F}_n t^n = \frac{11 + 7t + 2i + 2(1+t)\epsilon + (4+2t)h}{1 - t - t^2}. \tag{7}$$

Hence, we have the next result.

Theorem 3.3. *For $n \geq 0$, the generating function for hybrid numbers with hybrid Leonardo number coefficients is given by,*

$$\sum_{n=0}^{\infty} \mathbb{L}e_n t^n = \frac{22 + 14t + 4i + 4(1+t)\epsilon + (8+4t)h)}{t(1 - t - t^2)} - \frac{(2A+1)}{1-t}$$

where $A = 1 + i + \epsilon + h$.

Proof. Let $\sum_{n=0}^{\infty} \mathbb{L}e_n t^n$ be the generating function of the hybrid numbers with hybrid Leonardo number coefficients. Expression (4) gives us

$$t \sum_{n=0}^{\infty} \mathbb{L}e_n t^n = 2 \sum_{n=0}^{\infty} \mathbb{F}_{n+1} t^{n+1} - (2A+1) \sum_{n=0}^{\infty} t^{n+1}. \tag{8}$$

Then, by replacing (7) in (8) we obtain,

$$\begin{aligned} t\sum_{n=0}^{\infty}\mathbb{L}e_n t^n &= \frac{22+14t+4i+4(1+t)\epsilon+(8+4t)h)}{t(1-t-t^2)} - B\sum_{n=0}^{\infty}t^{n+1} \\ &= \frac{22+14t+4i+4(1+t)\epsilon+(8+4t)h)}{(1-t-t^2)} - \frac{(2A+1)t}{1-t} \end{aligned}$$

where $A = 1+i+\epsilon+h$. □

4 Catalan and Cassini Identities

In this section, we provide the Catalan and Cassini identities related to the hybrid numbers with hybrid Leonardo coefficients by considering that the multiplication is not commutative in the hybrid sets.

Theorem 4.1 (First Cassini)**.** *For positive integer n, the following identity is verified*

$$\begin{aligned} (\mathbb{L}e_n)^2 - \mathbb{L}e_{n-1}\mathbb{L}e_{n+1} &= 4(-1)^n(1-34i+12\epsilon-6h) \\ &\quad - 2\mathbb{L}e_{n-2}A + (2A+1)\mathbb{L}e_{n-1} - 2\mathbb{L}e_n + \mathbb{L}e_{n+1}, \end{aligned}$$

where $A = 1+i+\epsilon+h$.

Proof. Denote $B_n = \left(\frac{(\underline{\alpha})^2\alpha^{n+1} - (\underline{\beta})^2\beta^{n+1}}{\alpha-\beta} - A\right) = \mathbb{F}_{n+1} - A$. From the Binet formula (6) we obtain

$$\begin{aligned} (\mathbb{L}e_n)^2 - \mathbb{L}e_{n-1}\mathbb{L}e_{n+1} &= (2B_n-1)(2B_n-1) - (2B_{n-1}-1)(2B_{n+1}-1) \\ &= 4(B_n)^2 - 4B_n + 1 \\ &- (4B_{n-1}B_{n+1} - 2(B_{n-1}+B_{n+1}) + 1) \\ &= 4(B_n)^2 - 4B_n - 4B_{n-1}B_{n+1} + 2(B_{n-1}+B_{n+1}) \\ &= 4\big((B_n)^2 - B_{n-1}B_{n+1}\big) - 4B_n + 2B_{n-1} + 2B_{n+1}. \end{aligned}$$

Thus,

$$\begin{aligned}
4\big((B_n)^2 - B_{n-1}B_{n+1}\big) &= 4\big((\mathbb{F}_{n+1} - A)(\mathbb{F}_{n+1} - A) \\
&- (\mathbb{F}_n - A)(\mathbb{F}_{n+2} - A)\big) \\
&= 4\big((\mathbb{F}_{n+1})^2 - \mathbb{F}_{n+1}A - A\mathbb{F}_{n+1} + A^2\big) \\
&- 4\big(\mathbb{F}_n\mathbb{F}_{n+2} - \mathbb{F}_nA - A\mathbb{F}_{n+2} + A^2\big) \\
&= 4\big((\mathbb{F}_{n+1})^2 - \mathbb{F}_n\mathbb{F}_{n+2}\big) \\
&- 4(\mathbb{F}_{n+1} - \mathbb{F}_n)A - 4A(\mathbb{F}_{n+1} - \mathbb{F}_{n+2}) \\
&= 4\big((\mathbb{F}_{n+1})^2 - \mathbb{F}_n\mathbb{F}_{n+2}\big) - 4\mathbb{F}_{n-1}A + 4A\mathbb{F}_n.
\end{aligned}$$

Since it is valid Expression (4), $\mathbb{F}_{n+1} = \frac{\mathbb{L}e_n+2A+1}{2}$, where $A = 1+i+\epsilon+h$, then

$$\begin{aligned}
-4B_n + 2B_{n-1} + 2B_{n+1} &= -4\mathbb{F}_{n+1} + 2\mathbb{F}_n + 2\mathbb{F}_{n+2} \\
&= -2\mathbb{L}e_n + \mathbb{L}e_{n-1} + \mathbb{L}e_{n+1},
\end{aligned}$$

and also

$$-4\mathbb{F}_{n-1}A + 4A\mathbb{F}_n = -2\mathbb{L}e_{n-2}A + 2A\mathbb{L}e_{n-1}.$$

Moreover, since it is verified the identity $\mathbb{F}_{n-1}\mathbb{F}_{n+1} - (\mathbb{F}_n)^2 = (-1)^n(1 - 34i + 12\epsilon - 6h)$ by [Theorem 3.1, [9]], then we get

$$\begin{aligned}
(\mathbb{L}e_n)^2 - \mathbb{L}e_{n-1}\mathbb{L}e_{n+1} &= 4(-1)^n(1 - 34i + 12\epsilon - 6h) \\
&- 2\mathbb{L}e_{n-2}A + (2A+1)\mathbb{L}e_{n-1} - 2\mathbb{L}e_n + \mathbb{L}e_{n+1}.
\end{aligned}$$

This completes the proof. □

A similar result can be obtained by considering that the multiplication is not commutative in the hybrid sets.

Theorem 4.2 (Second Cassini)**.** *For positive integer n, then the following identity is verified*

$$\begin{aligned}
(\mathbb{L}e_n)^2 - \mathbb{L}e_{n+1}\mathbb{L}e_{n-1} &= 4(-1)^n(1 - 26i + 28\epsilon - 14h) \\
&+ \mathbb{L}e_{n-1}(-2A+1) + 2A\mathbb{L}e_{n-2} - 2\mathbb{L}e_n + \mathbb{L}e_{n+1},
\end{aligned}$$

where $A = 1 + i + \epsilon + h$.

Proof. Denote $B_n = \left(\frac{(\underline{\alpha})^2 \alpha^{n+1} - (\underline{\beta})^2 \beta^{n+1}}{\alpha - \beta} - A \right) = \mathbb{F}_{n+1} - A$. From the Binet formula (6) we obtain

$$\begin{aligned}(\mathbb{L}e_n)^2 - \mathbb{L}e_{n+1}\mathbb{L}e_{n-1} &= (2B_n - 1)(2B_n - 1) - (2B_{n+1} - 1)(2B_{n-1} - 1) \\ &= 4\big((B_n)^2 - B_{n+1}B_{n-1}\big) - 4B_n + 2(B_{n+1} + B_{n-1}).\end{aligned}$$

Thus,

$$4\big((B_n)^2 - B_{n+1}B_{n-1}\big) = 4\big((\mathbb{F}_{n+1})^2 - \mathbb{F}_{n+2}\mathbb{F}_n\big) - 4\mathbb{F}_n A + 4A\mathbb{F}_{n-1}.$$

Furthermore, since it is valid Expression (4), $\mathbb{F}_{n+1} = \frac{\mathbb{L}e_n + 2A + 1}{2}$, where $A = 1 + i + \epsilon + h$, then

$$\begin{aligned}-4B_n + 2B_{n-1} + 2B_{n+1} &= -4\mathbb{F}_{n+1} + 2\mathbb{F}_n + 2\mathbb{F}_{n+2} \\ &= -2\mathbb{L}e_n + \mathbb{L}e_{n-1} + \mathbb{L}e_{n+1},\end{aligned}$$

and also

$$-4\mathbb{F}_n A + 4A\mathbb{F}_{n-1} = -2\mathbb{L}e_{n-1}A + 2A\mathbb{L}e_{n-2}.$$

Moreover, since it is verified the identity $\mathbb{F}_{n+1}\mathbb{F}_{n-1} - (\mathbb{F}_n)^2 = (-1)^n(1 - 26i + 28\epsilon - 14h)$ by [Theorem 3.2, [9]], then

$$\begin{aligned}(\mathbb{L}e_n)^2 - \mathbb{L}e_{n-1}\mathbb{L}e_{n+1} &= 4(-1)^n(1 - 26i + 28\epsilon - 14h) \\ &\quad + \mathbb{L}e_{n-1}(-2A + 1) + 2A\mathbb{L}e_{n-2} - 2\mathbb{L}e_n + \mathbb{L}e_{n+1},\end{aligned}$$

which completes the proof. □

By using $\alpha\beta = -1$, $(\alpha - \beta)^2 = 5$, $\frac{1}{\beta} = \frac{-\alpha}{2}$ and $\frac{1}{\alpha} = \frac{-\beta}{2}$, a straightforward calculus permits us to obtain the Catalan identities for $\mathbb{F}_n$ in terms of α and β.

Proposition 4.3. *For positive integers n and r, the following identity is verified*

$$(\mathbb{F}_n)^2 - \mathbb{F}_{n+r}\mathbb{F}_{n-r} = \frac{(-1)^{n-r}}{2^r\sqrt{5}} F_r\big((\underline{\alpha}\underline{\beta})^2(\alpha)^r + (\underline{\beta}\underline{\alpha})^2(\beta)^r\big),$$

where F_r is r-th Fibonacci number, $\alpha = \dfrac{1+\sqrt{5}}{2}$, $\beta = \dfrac{1-\sqrt{5}}{2}$, $\underline{\alpha} = 1 + i\alpha + \epsilon\alpha^2 h + \alpha^3$, and $\underline{\beta} = 1 + i\beta + \epsilon\beta^2 + h\beta^3$.

Proposition 4.4. *For positive integers n and r, the following identity is verified*

$$(\mathbb{F}_n)^2 - \mathbb{F}_{n-r}\mathbb{F}_{n+r} = \frac{(-1)^{n-r}}{2^r\sqrt{5}}F_r((\underline{\alpha}\underline{\beta})^2(\beta)^r + (\underline{\beta}\underline{\alpha})^2(\alpha)^r),$$

where F_r is r-th Fibonacci number, $\alpha = \dfrac{1+\sqrt{5}}{2}$, $\beta = \dfrac{1-\sqrt{5}}{2}$, $\underline{\alpha} = 1 + i\alpha + \epsilon\alpha^2 h + \alpha^3$, *and* $\underline{\beta} = 1 + i\beta + \epsilon\beta^2 + h\beta^3$.

In addition, since $\mathbb{L}e_n = 2(\mathbb{F}_{n+1} - A) - 1$, in a similar way to what was done for Cassini identities, we obtain Catalan identities for $\mathbb{L}e_n$.

Theorem 4.5 (First Catalan)**.** *For positive integers n and r, holds:*

$$(\mathbb{L}e_n)^2 - \mathbb{L}e_{n+r}\mathbb{L}e_{n-r} = \frac{(-1)^{n+1-r}}{2^{r-2}\sqrt{5}}F_r((\underline{\alpha}\underline{\beta})^2(\alpha)^r + (\underline{\beta}\underline{\alpha})^2(\beta)^r) + (2A+1)(\mathbb{L}e_{n+r} + \mathbb{L}e_{n-r}) - (4A+2)(\mathbb{L}e_n),$$

where F_r is r-th Fibonacci number, $\alpha = \dfrac{1+\sqrt{5}}{2}$, $\beta = \dfrac{1-\sqrt{5}}{2}$, $\underline{\alpha} = 1 + i\alpha + \epsilon\alpha^2 h + \alpha^3$, $\underline{\beta} = 1 + i\beta + \epsilon\beta^2 + h\beta^3$, *and* $A = 1 + i + \epsilon + h$.

Theorem 4.6 (Second Catalan)**.** *For positive integers n and r, holds:*

$$(\mathbb{L}e_n)^2 - \mathbb{L}e_{n-r}\mathbb{L}e_{n+r} = \frac{(-1)^{n+1-r}}{2^{r-2}\sqrt{5}}F_r((\underline{\alpha}\underline{\beta})^2(\beta)^r + (\underline{\beta}\underline{\alpha})^2(\alpha)^r) + (\mathbb{L}e_{n+r} + \mathbb{L}e_{n-r})(2A+1) - (\mathbb{L}e_n)(4A+2),$$

where F_r is r-th Fibonacci number, $\alpha = \dfrac{1+\sqrt{5}}{2}$, $\beta = \dfrac{1-\sqrt{5}}{2}$, $\underline{\alpha} = 1 + i\alpha + \epsilon\alpha^2 h + \alpha^3$, $\underline{\beta} = 1 + i\beta + \epsilon\beta^2 + h\beta^3$, *and* $A = 1 + i + \epsilon + h$.

5 Some Summation Formulas

In this section, we present some results concerning sum and the alternating sum of terms of the hybrid Leonardo sequence with hybrid Leonardo sequence by using some results of the hybrid Leonardo sequence.

We recall from [10] some auxiliary results for the hybrid Leonardo sequence.

Lemma 5.1. *[10, Proposition 2.5] For $n \geq 0$,*

1. $\displaystyle\sum_{j=0}^{n} HLe_j = HLe_{n+2} - (n+2)A - (2i + 4\epsilon + 8h)$,

2. $\displaystyle\sum_{j=0}^{n} HLe_{2j} = HLe_{2n+1} - nA - (2i + 2\epsilon + 4h)$,

3. $\displaystyle\sum_{j=0}^{n} HLe_{2j+1} = HLe_{2n+2} - (n+2)A - (2\epsilon + 4h)$.

A consequence of the Lemma 5.1 is that:

Proposition 5.2. *For $n \geq 0$,*

$$\sum_{j=0}^{n} (-1)^j HLe_j = HLe_{2n+1} - HLe_{2n+2} + 2A - 2i; \tag{9}$$

if the last term is negative;

$$\sum_{j=0}^{n} (-1)^j HLe_j = HLe_{2n+3} - HLe_{2n+2} + A - 2i; \tag{10}$$

if the last term is positive.

Proof. First consider that the last term is negative, so

$$\begin{aligned}
\sum_{k=0}^{2n+1} (-1)^j HLe_j &= HLe_0 - HLe_1 + \cdots + HLe_{2n} - HLe_{2n+1} \\
&= (HLe_0 + \cdots + HLe_{2n}) - (HLe_1 + \cdots + HLe_{2n+1}) \\
&= \sum_{k=0}^{n} HLe_{2k} - \sum_{k=0}^{n} HLe_{2k+1} \\
&= HLe_{2n+1} - nA - (2i + 2\epsilon + 4h) \\
&\quad -\big(HLe_{2n+2} - (n+2)A - (2\epsilon + 4h)\big) \\
&= HLe_{2n+1} - HLe_{2n+2} + 2A - 2i\ .
\end{aligned}$$

We use the Lemma 5.1, items (2) and (3), and Equation (9) is verified. Similarly, we can prove Equation (10) . □

Now, sums of terms of sequence $\{\mathbb{L}e_n\}_{n\geq 0}$.

Proposition 5.3. *For $n \geq 0$,*

$$\sum_{j=0}^{n} \mathbb{L}e_j = HLe_{n+2}A - n(2A+1) - 3A - (32 + 4i + 21h) \ , \tag{11}$$

$$\sum_{j=0}^{n} \mathbb{L}e_{2j} = HLe_{2n+1}A - n(2A+1) - 5A - (70 + 12i + 9h) \tag{12}$$

$$\sum_{j=0}^{n} \mathbb{L}e_{2j+1} = HLe_{2n+2}A - n(2A+1) - 11A - (116 + 28i + 13h) \tag{13}$$

Proof. As the sum of two hybrid numbers is defined by summing their components, and $\mathbb{L}e_n = HLe_n + HLe_{n+1}i + HLe_{n+2}\epsilon + HLe_{n+3}h$. We have that

$$\begin{aligned}\sum_{j=0}^{n} \mathbb{L}e_j &= \sum_{j=0}^{n} (HLe_j + HLe_{j+1}i + HLe_{j+2}\epsilon + HLe_{j+3}h) \\ &= \left(\sum_{j=0}^{n} HLe_j\right) + \left(\sum_{j=0}^{n} HLe_{j+1}\right) i + \left(\sum_{j=0}^{n} HLe_{j+2}\right) \epsilon \\ &+ \left(\sum_{j=0}^{n} HLe_{j+3}\right) h \ .\end{aligned}$$

So, following Lemma 5.1

$$\begin{aligned}\sum_{j=0}^{n} HLe_j &= HLe_{n+2} - (n+2)A - (2i + 4\epsilon + 8h) \\ &= HLe_{n+2} - nA - (2 + 4i + 6\epsilon + 10h) \ .\end{aligned} \tag{14}$$

$$\begin{aligned}\left(\sum_{j=0}^{n} HLe_{j+1}\right) i &= \left(\sum_{j=0}^{n} HLe_j - HLe_0\right) i \\ &= (HLe_{n+2} - (n+2)A - (2i + 4\epsilon + 8h) - HLe_0)\, i \\ &= (HLe_{n+2})\, i - nAi - (2 - 10i - 15\epsilon + 9h) \ .\end{aligned} \tag{15}$$

$$\begin{aligned}\left(\sum_{j=0}^{n} HLe_{j+2}\right)\epsilon &= \left(\sum_{j=0}^{n} HLe_{j} - HLe_{0} - HLe_{1}\right)\epsilon \\ &= \left(HLe_{n+2} - (n+2)A - (2i+4\epsilon+8h)\right)\epsilon \\ &- \left(HLe_{0} + HLe_{1}\right)\epsilon \\ &= \left(HLe_{n+2}\right)\epsilon - nA\epsilon - (8+22\epsilon-2h)\ . \qquad (16)\end{aligned}$$

$$\begin{aligned}\left(\sum_{j=0}^{n} HLe_{j+3}\right)h &= \left(\sum_{j=0}^{n} HLe_{j} - HLe_{0} - HLe_{1} - HLe_{2}\right)h \\ &= \left(HLe_{n+2} - (n+2)A - (2i+4\epsilon+8h)\right)h \\ &- \left(HLe_{0} + HLe_{1} + HLe_{2}\right)h \\ &= \left(HLe_{n+2}\right)h - nAh - (23+13i-10\epsilon+7h)\ . \qquad (17)\end{aligned}$$

From the Equations (14), (15), (16), and (17) we have that

$$\sum_{j=0}^{n} \mathbb{L}e_{j} = \left(HLe_{n+2}\right)A - nA\cdot A - (35+7i+3\epsilon+24h).$$

According to Lemma 2.2, Equation (11) is verified. Similarly, we can prove Equations (12) and (13). □

As a direct consequence of the Proposition 5.3 we have:

Proposition 5.4. *For $n \geq 0$,*

$$\sum_{j=0}^{n} (-1)^{j}\mathbb{L}e_{j} = (HLe_{2n+1} - HLe_{2n+2} + 6)A + 46 + 16i + 4h;$$

if the last term is negative, and

$$\sum_{j=0}^{n} (-1)^{j}\mathbb{L}e_{j} = (HLe_{2n+3} - HLe_{2n+2} + 6)A - (2A+1) + 46 + 16i + 4h;$$

if the last term is positive.

6 Conclusion

In this paper, we presented a new generalization of the hybrid Leonardo sequence, the hybrid numbers with hybrid Leonardo number coefficients. Moreover, the algebraic properties of this sequence are studied and also the generating function and several identities are provided. In addition, some properties for these sequences were established.

It seems to us that all results given here are new in the literature and this new sequence of numbers introduced is a subject that can still be studied in several aspects such as combinatorial, analytical, and matrix perspectives.

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