Journal of Mathematical Extension
Vol. 18, No. 6, (2024) (2)1-25
URL: https://doi.org/10.30495/JME.2024.2910
ISSN: 1735-8299
Original Research Paper

An Inverse Free Disposal Hull Model for the Inputs/Outputs Estimation

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Abstract. Inverse data envelopment analysis (InvDEA) has been widely used for estimating the expected input (output) variation level while maintaining constant their relative efficiency score. However, the InvDEA may underestimate(overestimate) the input (output) variation in a non-convex setting. To solve this problem, we develop the InvDEA notion to the Free Disposal Hull (FDH) technology which relies only on the free disposability assumption and removes the convexity. The Inverse FDH model sets an observed unit as the target unit for learning and following to estimate the input/output changes. Estimating the input(output) variation for FDH technology requires solving linear/nonlinear mixed integer programs which are computationally expensive. In this paper, we provide the enumeration method for estimating the input(output) variations without solving any mathematical model. Furthermore, the Inverse convex DEA and non-convex FDH models are extended in the situation in which the input prices are available. Finally, an application of inverse DEA using real data has been presented.

Keywords and Phrases: Data envelopment analysis (DEA), Efficiency, Inverse DEA, Free disposal hull (FDH)

Received: October 2023; Accepted: May 2024
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1 Introduction

The main objective of techniqual efficiency evaluation is to compare the performance of the decision making units (DMUs) against the best peer units. There are two frameworks for this task, called parametric and non-parametric. In the non-parametric framework, the performance of the DMU is assessed by estimating a best practice frontier from a data sample. Then the techniqual inefficiency of the DMU under evaluation is measured as the distance from unit to the frontier. Let an input vector be given, the production technology yields the output attainable from that input bundle. A subset of the boundary of the production technology is called efficient frontier. Following the seminal work on production theory, any production technology and its estimator, must satisfy a certain set of basic postulates, Färe[8] . Two basic postulates are convexity and free disposability in input and output factors. In particular, free disposability says that if an input-output bundle is feasible (producible), then the consumption of more inputs and/or producing less outputs are also possible. In case of producing only one type of output, free disposability in inputs and outputs means that the corresponding production frontier must be monotonically nondecreasing, whereas convexity of the technology is translated into a concave production frontier. The production technology estimators are classified into two main classes: convex and non-convex technologies. Two of the widely used estimators of the true production technology are convex Data envelopment analysis (DEA) and non-convex Free disposal Hull (FDH) (see Charnes et al. [6], Banker et al. [3], Deprins et al. [8]). DEA is a linear programming-based technique for evaluating the performance of the decision-making units (DMUs) with multiple inputs and outputs. This technique was expanded and developed by scholars to include a wide variety of applications, see Charnes et al. [6], Seiford, and Thrall [32]. It is well-known that a variety of reasons may generate non-convexities in technology (see Kerstens and Woestyne [24]). This class of models, presented by Deprins et al.[8], evaluate the DMUs relying only on the free disposability by removing the convexity assumption (Cherchye et al., [7]). FDH models have been studied by several scholars, including Tulkens [36], Kerstens and Eeckaut [23], Podinovski [30], Leleu [25], Briec and Kerstens [4], Soleimani-damaneh and Reshadi [34], Soleimani-damaneh and Mostafaee [35], Kerstens and de Woestyne [24].

Inverse DEA (InvDEA) was formally introduced in a seminal research paper by Wei et al.[37], to estimate the maximum increase of outputs in response to the given input increments while preserving the efficiency score, though its idea was appeared in Zhang and Cui [39]. In Zhang and Cui[39], the input increments of a DMU are measured for its given output increments under the CCR efficiency-fixed constraints. Wei et al. [37] utilized multiple-objective programming techniques to estimate the desired output levels.

To the best of our knowledge, all of the existing works study the variations of inputs and outputs under the convex DEA models. However, in the non-convex setting, the inverse DEA may underestimate/overestimate the variation of input/output change while maintainingthe the efficiency score with respect to the current technology. To overcome this issue, we develop the InvDEA model to the non-convex FDH model which is simple to interpret and easy to use. The inverse FDH model designates an observed unit as the target unit to learn and follow in response to input/output changes. Furthermore, we use the inverse FDH model to estimate the required input costs to produce a pre-specified output level when the input prices are available. Moreover, estimating the input/output variations required solving a linear/nonlinear mixed integer programming problem, which is computationally expensive. We propose the enumeration method to solve the inverse FDH models without solving any mathematical model.

The rest of the paper is organized as follows. Section 2 includes literature review on InvDEA. Section 3 is devoted to some preliminaries. Section 4 develops the InvDEA models to the non-convex FDH technology. An illustrative example is given in Section 4. Finally, Section 5 concludes the paper.

2 Literature Review

Wei et al. [37] proposed the InvDEA model at first to estimate the output variation subject to an increase in inputs under preservation of the efficiency score. They applied multi-objective programming (MOP) to estimate output levels ([9]). Yan et al. [38] and Jahanshahloo et al.

[18, 19] converted multi-objective programming into single-objective linear programming using decision priorities. In addition, other InvDEA models were proposed by some scholars, see Li and Cui [27]. Hadi-Vencheh and Foroughi [15] introduced a model to increase some input levels while decreasing others. Hadi-Vencheh et al. [16] introduced another InvDEA model to calculate the inputs under given increased outputs and preserve the efficiency score. They used multiple-objective programming tools for input-estimating under increasing outputs and preserving the efficiency score. Lertworasirikul et al [26] Proposed an InvDEA model assuming variable returns to scale regarding the concurrent increases of some outputs and decreases of other outputs. Jahanshahloo et al. [20] used Russell's enhanced non-radial model to investigate InvDEA ([31]). Jahanshahloo et al. [21] introduced a periodic weak Pareto solution for MOLP to solve InvDEA problems. Zhang and Cui [39] classified different InvDEA models under twelve different scenarios. Li and Cui [28] discusse a new type of InvDEA model with considering returns to scale and elasticity of DMU. Unlike the original InvDEA model, the proposed model allows the efficiency score being changed.

Many studies have been done about the applications of InvDEA in real-world problems: Ghyasi [14] employed InvDEA to estimate cost efficiency when price data are available. Eyni, et al [10] used the InvDEA to analyze the sensitivity of DMUs with undesirable input and output. Amin et al. [1] used the InvDEA in Enterprise merger. Hassanzadeh et al. [17] assessed the sustainability of countries using InvDEA. Kamyab et al. [22] proposed a two-stage InvDEA model for the merger of DMUs based on linear programming models. They provided some applications of the proposed model for the mergers of universities, insurance companies, and commercial banks. Ghobadi [12] presents an application of InvDEA for choosing a suitable strategy for spreading educational departments in a university. Amin and Al-Muharrami [2] used InvDEA to merge units with negative data. Sohrabi et al. [33] proposed the InvDEA models in the presence of ratio data. They presented the inputs/output estimation process based on ratio-based DEA models. Ghiyasi [14] developed the InvDEA model based on preserving the cost and revenue efficiency score. Ghobadi [13] applied the InvDEA for the estimation of inputs and outputs. Gattoufi et al. [11] proposed an InvDEA method for merging banks.

In summary, InvDEA is used to solve three types of problems. The first type is resource allocation problems, which determines the minimum required increase of inputs for producing specific outputs while maintaining efficiency and the current technology. The second type is investment analysis, which determines the maximum increase in outputs for specific increase of inputs preserving efficiency score. When the input prices are available, the cost efficiency model evaluates the ability of a DMU to produce the current outputs at minimal cost, see Mostafaee, and Saljooghi [29]. The third type is cost analysis, which determines the expected increase of inputs cost for producing the targeted outputs while preserving the cost efficiency.

3 Preliminaries

3.1 Inverse DEA

Let us consider a set on n DMUs, $\text{DMU}_j : j \in J = \{1, \cdots, n\}$, which produce multiple output $y_{rj}\,(r = 1, \cdots, s)$, through multiple inputs $x_{ij}\,(i = 1, \cdots, m)$. Suppose inputs and outputs for DMU_j, are represented by vectors: $x_j = (x_{1j}, \cdots, x_{mj})^t$ and $y_j = (y_{1j}, \cdots, y_{sj})^t$ respectively. We consider the following input-oriented generalized DEA model in which DMU_o, $o = \{1, \cdots, n\}$, is the unit under evaluation:

$$
\begin{aligned}
\theta_o = \min \theta & \\
\text{s.t.} \ & \sum_{j \in J} \lambda_j x_j \leq \theta x_o, \\
& \sum_{j \in J} \lambda_j y_j \geq y_o, \\
& \lambda \in \Lambda,
\end{aligned} \tag{1}
$$

in which

$$
\Lambda = \Big\{ \lambda | (\lambda_1, \lambda_2, ..., \lambda_n), \delta_1 (\sum_{j \in J} \lambda_j + \delta_2 (1-)^{\delta_3} v) = \delta_1, v \geq 0, \lambda_j \geq 0
$$
$$
, j = 1, 2, ..., n \Big\}.
$$

It can be easily observed that:

if $\delta_1 = 0$, model (1) is the CCR model,

if $\delta_1 = 1$ and $\delta_2 = 0$, model (1) is the BCC model,

if $\delta_1 = \delta_2 = 1$ and $\delta_3 = 0$, model (1) is the CCR-BCC model,

if $\delta_1 = \delta_2 = \delta_3 = 1$, model (1) is the BCC-CCR model.

The question that arises is that: If the efficiency score θ_o remains unchanged, but the outputs increase by Δy_o, where vector $\Delta y_o \geq 0$ and $\Delta y_o \neq 0$ how much should the inputs of the DMU$_o$ increase? The inputs vectors $x_o + \Delta x_o$ must be estimated such that the efficiency score after input-output variations remains unchanged. We set $\alpha_{io} = x_{io} + \Delta x_{io}$, $(i = 1, \cdots, m)$ and $\beta_{ro} = y_{ro} + \Delta y_{ro}$, $(r = 1, \cdots, s)$, and also $\alpha_o = x_o + \Delta x_o$, $\beta_o = y_o + \Delta y_o$. The following MOLP model is applied for estimating α_o, (Wei et.al. [37]).

$$
\begin{aligned}
&\min(\alpha_{1o}, \cdots, \alpha_{mo}) \\
&\text{s.t.} \sum_{j \in J} \lambda_j x_{ij} \leq \theta_o \alpha_{io}, \\
&\qquad \sum_{j \in J} \lambda_j y_j \geq \beta_o, \\
&\qquad \alpha_o \geq x_o, \\
&\qquad \lambda \in \Lambda.
\end{aligned}
\tag{2}
$$

However, according to Wei et.al [37], if $(\hat{\lambda}, \hat{\alpha_o})$ is a weakly Pareto solution to model (2), then the efficiency score remains unchanged. But Hadi-Vencheh. et al. [16] provided a counter-example to show that this assertion is not valid in general. They proved that problem (2) preserves the efficiency index through the following theorem:

Theorem 3.1. *If the optimal solution of model (1) is θ_o and (λ^*, α_o^*) is a Pareto solution (strongly efficient solution) of model (2), when the inputs of DMU$_o$ are increased to $x_o + \Delta x_o$, then the optimal value of model (2) is θ_o.*

The next theorem shows that we can use some weakly Pareto solutions model (2) for input estimation, Hadi-Vencheh et al. [16].

Theorem 3.2. *If the optimal solution of model (1) is θ_o and (λ^*, α_o^*) is a weakly Pareto solution of model (2) such that $\alpha_o^* > \ x_o$, then the optimal value of model (2) is θ_o.*

3.2 The non-convex FDH model

Another important class of efficiency analysis models is that of Free Disposal Hull (FDH) models. These models, first presented by Deprins et al. [8], evaluate the performance of the Decision-Making Units considering the closest inner approximation of the true strongly disposable (but possibly non-convex) technology.

In this subsection, we first briefly describe some characteristic properties of the FDH model. The FDH technology under different RTS assumptions can be represented as follows

$$T^{\mathrm{FDH}_\Delta} = \Big\{(x,y) | \sum_{j\in J} \lambda_j x_j \leq x,\ \sum_{j\in J} \lambda_j y_j \geq 0,\ \lambda_j = \delta\mu_j;\ \forall j \in J,$$
$$\sum_{j\in J} \mu_j = 1,\ \mu \in (\{0,1\})^n,\ \delta \in \Delta\Big\},$$

where Δ depending on the RTS assumption of the reference technology, is

$$\Delta^{\mathrm{VRS}} \equiv \{\delta |\ \delta = 1\}, \qquad \Delta^{\mathrm{CRS}} \equiv \{\delta |\ \delta \geq 0\},$$

$$\Delta^{\mathrm{NIRS}} \equiv \{\delta |\ 0 \leq \delta \leq 1\}, \qquad \Delta^{\mathrm{NDRS}} \equiv \{\delta |\ \delta \geq 1\}.$$

Here, VRS, CRS, NIRS, and NDRS stand for variable, constant, non-increasing, and non-decreasing RTS, respectively. From now on and for simplicity we use notations FDH_v, FDH_c, $\mathrm{FDH}_{\mathrm{NI}}$, $\mathrm{FDH}_{\mathrm{ND}}$, instead of $\mathrm{FDH}_{\mathrm{VRS}}$, $\mathrm{FDH}_{\mathrm{CRS}}$, $\mathrm{FDH}_{\mathrm{NIRS}}$, and $\mathrm{FDH}_{\mathrm{NDRS}}$ respectively.

Considering $\mathrm{DMU}_o = (x_o, y_o), (o \in J)$ as the unit under assessment, the input-oriented and output-oriented FDH radial efficiency measures of DMU_o are obtained by solving the following mixed-integer nonlinear programming problems, respectively:

$$\begin{aligned} \theta_o^{\Delta} = & \min \theta \\ & \text{s.t.} \sum_{j \in J} \lambda_j x_j \leq \theta x_o, \\ & \sum_{j \in J} \lambda_j y_j \geq y_o. \end{aligned} \tag{3}$$

Plus, the constraints:

$$\begin{aligned} & \lambda_j = \delta \mu_j, \ \mu_j \in \{0,1\}, \ \forall j \in J, \\ & \sum_{j \in J} \mu_j = 1, \ \delta \in \Delta, \end{aligned}$$

and

$$\begin{aligned} \varphi_o^{\Delta} = & \max \varphi \\ & \text{s.t.} \sum_{j \in J} \lambda_j x_j \leq x_o, \\ & \sum_{j \in J} \lambda_j y_j \geq \varphi y_o, \\ & \lambda_j = \delta \mu_j, \ \mu_j \in \{0,1\}, \ \forall j \in J, \\ & \sum_{j \in J} \mu_j = 1, \ \delta \in \Delta, \end{aligned} \tag{4}$$

wherein $\Delta \in \{\mathrm{FDH}_v, \mathrm{FDH}_c, \mathrm{FDH}_{\mathrm{NI}}, \mathrm{FDH}_{\mathrm{ND}}\}$.

In contrast to the convex DEA, the FDH model ensures that the efficiency measurements are only effected from observed performances. Additionally, the convex DEA estimator of the production frontier is piece-wise linear and concave, while the FDH estimator of the production frontier is step-wise.

Definition 3.3. $DMU_o = (x_o, y_o)$ is called Δ-efficient if there exists no $(x, y) \in T^{\Delta}$ such that $x \leq x_o$, $y \geq y_o$ and $(x, y) \neq (x_o, y_o)$, where the vector inequalities are understood component wise.

4 Inverse FDH

It is clear that the non-convex FDH technology is not as popular as its Convex counterpart. Inverse DEA has been widely used for estimating the expected input/output variation level while preserving the efficiency score. However, the inverse DEA may underestimate/overestimate the expected input/output levels in the non-convex setting. To solve this problem, we introduce the inverse FDH model, which is easy to apply and interpret. Furthermore, the enumeration method is provided for estimating the input/output variation level while preserving the technical efficiency score, which is the polynomial time from a computational viewpoint. Moreover, the inverse FDH model is extended to the situation in which input prices are available. Also, we aim to estimate the cost/output variation level while preserving the cost efficiency score.

4.1 Preserving the input-oriented FDH efficiency

Assuming that the output levels of DMU_o in a non-convex setting increase, how much more input should the unit consume (produce) so that the efficiency score remains unchanged? This subsection tries to answer this question by developing the inverse FDH model. Suppose the outputs of DMU_o are changed to $\beta_{r_0} = y_{r_0} + \Delta y_{ro},\ (r = 1, \cdots, s)$. We want to estimate the input vector $\alpha_{i0} = x_{io} + \Delta x_{io},\ (i = 1, ..., m)$ such that the efficiency score remains unchanged. By setting $\alpha_o = x_o + \Delta xo$, $\beta_o = y_o + \Delta y_o$, the efficiency of the system with adjusted input-output is denoted by $\hat{\theta}_o^{\Delta}$, and is obtained by solving the following nonlinear mixed-integer programming model:

$$
\begin{aligned}
\theta_o^{\Delta *} = \min \theta & \\
\text{s.t.} \ & \sum_{j \in J} \lambda_j x_j \leq \theta \hat{\alpha}_o, \\
& \sum_{j \in J} \lambda_j y_j \geq \beta_o, \\
& \hat{\alpha}_o \geq x_o, \\
& \lambda_j = \delta \mu_j,\ \mu_j \in \{0,1\},\ \forall j \in J, \\
& \sum_{j \in J} \mu_j = 1,\ \delta \in \Delta.
\end{aligned} \tag{5}
$$

The goal is to find the minimum value of Δx_0 such that the efficiency remains unchanged, that is: $\hat{\theta}_o^{\Delta} = \theta_o^{\Delta}$. The following theorem provides a sufficient condition for preserving the efficiency score and estimating the expected input/output variation level.

Theorem 4.1. *If* $(\hat{\lambda}_o, \hat{\alpha})$ *is a part of weakly Pareto solution of* (6)*, then the optimal objective value of* (5) *is* θ_o^{Δ}.

$$\begin{aligned} &\min\{\alpha_{io}, i = 1, 2, ..., m\} \\ &s.t. \sum_{j\in J} \lambda_j x_{ij} \leq \theta_o^{\Delta} \hat{\alpha}_{io}, \\ &\quad \sum_{j\in J} \lambda_j y_{rj} \geq \beta_{ro}, \\ &\quad \hat{\alpha}_{io} \geq x_{io}, \\ &\quad \lambda_j = \delta\mu_j,\ \mu_j \in \{0,1\},\ \forall j \in J, \\ &\quad \sum_{j\in J} \mu_j = 1,\ \delta \in \Delta, \end{aligned} \tag{6}$$

Proof. Assume $\left(\hat{\lambda}, \hat{\alpha}_o\right)$ is part of a weakly efficient solution to (6). Therefore, this part of the solution plus $\theta = \theta_o^{\Delta}$ satisfies the constraints of (5). The objective value of (6) corresponding to this feasible solution is θ_o^{Δ}. Since the objective function of (5) is in minimizing form, we get $\theta_o^{\Delta *} \leq \hat{\theta}_o^{\Delta}$. Now, we will show that $\theta_o^{\Delta *} = \hat{\theta}_o^{\Delta}$. Assume, by contradiction, that $\theta_o^{\Delta *} < \hat{\theta}_o^{\Delta}$. Therefore, there exists some $t \in (0,1)$ such that $\theta_o^{\Delta *} = t\,\theta_o^{\Delta}$. From (6), we have

$$\sum_{j\in J} \lambda_j x_j \leq \theta_o^{\Delta *} \hat{\alpha}_o = \theta_o^{\Delta}\, t\hat{\alpha}_o,$$

$$\begin{aligned} &\sum_{j\in J} \lambda_j y_j \geq \beta_o, \\ &\lambda_j = \delta\mu_j,\ \mu_j \in \{0,1\},\ \forall j \in J, \\ &\sum_{j\in J} \mu_j = 1,\ \delta \in \Delta. \end{aligned}$$

This means that $(\lambda, t\hat{\alpha})$ is part of a feasible solution to (6), and weakly dominates $(\lambda, \hat{\alpha})$. This clearly contradicts being weakly Pareto solution of $(\lambda, \hat{\alpha}_o)$, and hence, $\theta_o^{\Delta *} < \hat{\theta}_o^{\Delta}$. This completes the proof. □
Model (6) can be considered as the input-oriented version of the inverse FDH (InvFDH) model. This model is a binary mixed integer multi-objective programming problem, whose weakly efficient solutions determine the minimum expected change of x_0 so that the efficiency of the new production point remains unchanged.

The following theorem provides a practical way of calculating the expected input variation:

Theorem 4.2. *Let*

$$\hat{\alpha}_o = \frac{\theta_0^{S_\Delta}}{\theta_o^{\Delta}} x_0, \qquad \beta_o = y_o + \Delta y_o,$$

then the efficiency of the production point $(\hat{\alpha}_o, \beta_o)$ *is* θ_o^{Δ}, *where*

$$\begin{aligned} \theta_o^{S_\Delta} = & \min \theta \\ & s.t. \sum_{j \neq o} \lambda_j x_j \leq \theta x_o, \\ & \qquad \sum_{j \neq o} \lambda_j y_j \geq \beta_o, \\ & \qquad \lambda_j = \delta \mu_j, \ \mu_j \in \{0, 1\}, \ \forall j \neq o, \\ & \qquad \sum_{j \neq o} \mu_j = 1, \ \delta \in \Delta. \end{aligned} \tag{7}$$

Proof. Assume that $(\lambda_j^*, \mu_j^*; j \neq o, \delta^*, \theta_0^{\mathrm{S}_\Delta})$ is an optimal solution to (7), we have:

$$\begin{aligned} & \sum_{j \neq o} \lambda_j^* x_j \leq \theta_0^{\mathrm{S}_\Delta} x_o = \theta_o^{\Delta} \frac{\theta_0^{\mathrm{S}_\Delta}}{\theta_o^{\Delta}} x_0 = \theta_o^{\Delta} \hat{\alpha}_o, \\ & \sum_{j \neq o} \lambda_j^* y_j \geq \beta_o, \\ & \lambda_j^* = \delta^* \mu_j^*, \mu_j \in \{0, 1\} \\ & \sum_{j \in J} \mu_j^* = 1, \delta^* \in \Delta. \end{aligned}$$

By setting , $\lambda_o^* = \mu_o^* = 0$, it follows that $(\lambda^*, \mu^*, \delta^*, \theta = \theta_o^{\Delta})$ is a feasible solution to (5) with the objective value equals to θ_o^{Δ}. As the objective function of (7) is in minimizing form, we have

$$\theta_o^{\Delta *} \leq \theta_o^{\Delta}. \tag{8}$$

We want to show that $\theta_0^{\Delta} = \theta_o^{\Delta *}$. Assume, by contradiction, that $\theta_o^{\Delta *} < \theta_o^{\Delta}$. Thus, $\theta_o^{\Delta *} = (1-\varepsilon)\theta_0^{\Delta}$, for some $\varepsilon > 0$. If $(\bar{\lambda}, \bar{\mu}, \bar{\delta}, \theta_o^{\Delta *})$ is an optimal solution to (5), then we have

$$\begin{aligned}
\sum_{j \in J} \bar{\lambda}_j x_j \quad &\leq \theta_o^{\Delta *} \frac{\theta_0^{\mathrm{S}\Delta}}{\theta_o^{\Delta}} x_0 \\
&= (1-\varepsilon)\, \theta_0^{\Delta} \frac{\theta_0^{\mathrm{S}\Delta}}{\theta_o^{\Delta}} x_0 \\
&= (1-\varepsilon)\, \theta_0^{\mathrm{S}\Delta} x_o,
\end{aligned}$$

$$\begin{aligned}
&\sum_{j \in J} \bar{\lambda}_j y_j \geq (y_0 + \Delta y_0) = \beta_o, \\
&\bar{\lambda}_j = \bar{\delta}\bar{\mu}_j,\ \bar{\mu}_j \in \{0,1\},\ \forall j \in J, \\
&\sum_{j \in J} \mu_j = 1,\ \delta \in \Delta.
\end{aligned}$$

It can be easily shown that $\bar{\lambda}_o = \bar{\mu}_o = 0$. This implies that $(\theta = (1-\varepsilon)\theta_o^{\mathrm{S}\Delta}, \bar{\lambda}_j, \bar{\mu}_j; j \neq o, \bar{\delta})$ is a feasible solution to (7) with objective value of $(1-\varepsilon)\theta_0^{\mathrm{S}\Delta} < \theta_0^{\mathrm{S}\Delta}$. This is a clear contradiction, and hence, $\theta_0^{\Delta} = \theta_o^{\Delta *}$. These complete the proof. $\square$

4.2 Inverse FDH: Output-oriented Model

This subsection is devoted to inverse FDH under the output-oriented model. If among a set of homogeneous DMUs, the input levels of DMU_o increase, how many more outputs should the unit produce such that the output-oriented efficiency score of φ_o^{Δ} remains unchanged? To answer this question, assume that the inputs of unit DMU_o are changed from x_{io} to $\alpha_{io} = x_{io} + \Delta x_{io}, (i = 1, 2, ..., m)$. We need to estimate the input vector $\beta_{r_o} = y_{r_o} + \Delta y_{ro},\ (r = 1, 2, ..., s)$. By setting $\alpha_o = x_o + \Delta x_o$,

$\beta_o = y_o + \Delta y_o$, the output-oriented efficiency score of this production point is denoted by $\breve{\varphi}_o^{\Delta}$. is obtained via solving the following mixed-integer nonlinear programming problem:

$$\begin{aligned}
\breve{\varphi}_o^{\Delta} = & \max \ \varphi \\
& \text{s.t.} \ \sum_{j \in J} \lambda_j x_j \leq \alpha_o, \\
& \qquad \sum_{j \in J} \lambda_j y_j \geq \varphi \beta_o, \\
& \qquad \lambda_j = \delta \mu_j, \ \mu_j \in \{0,1\}, \ \forall j \in J, \\
& \qquad \sum_{j \in J} \mu_j = 1, \ \delta \in \Delta.
\end{aligned} \tag{9}$$

The goal is to determine the maximum value of Δy_o such that the efficiency remains unchanged, that is, $\breve{\varphi}_o^{\Delta} = \varphi_o^{\Delta}$. The following theorem determines the expected output change after increasing inputs, while the output-oriented efficiency score remains unchanged.

Theorem 4.3. *Let the output-oriented efficiency score of $DMU_o = (x_o, y_o)$ be φ_o^{Δ}. If (λ, β_o) is part of a weakly Pareto solution of (10), then the efficiency score of production point $(\alpha_o, \beta_o = y_o + \Delta y_o)$ remains unchanged.*

$$\begin{aligned}
& \max \ \{y_{ro} + \Delta y_{ro}, r = 1, 2, ..., s\} \\
& s.t. \ \sum_{j \in J} \lambda_j x_j \leq x_o + \Delta x_o, \\
& \qquad \sum_{j \in J} \lambda_j y_j \geq \varphi_o^{\Delta}(y_o + \Delta y_o), \\
& \qquad \beta_o \geq y_o, \\
& \qquad \lambda_j = \delta \mu_j, \ \mu_j \in \{0,1\}, \ \forall j \in J, \\
& \qquad \sum_{j \in J} \mu_j = 1, \ \delta \in \Delta.
\end{aligned} \tag{10}$$

Proof. The proof can be accomplished in a similar way to that of Theorem 4.1. □

Model (10) is known as the output-oriented form of the inverse FDH model, which is a zero-one multi-objective programming problem. The following theorem provides a practical value for calculating β_o:

Theorem 4.4. *If* $\beta_o = \frac{\varphi_o^{S_\Delta}}{\varphi_o^{\Delta}} y_0$, *then the efficiency of the production point* (α_o, β_o) *does not change, where*

$$\begin{aligned}
\varphi_o^{S_\Delta} = &\max\ \varphi \\
&s.t.\ \sum_{j\in J} \lambda_j x_j \leq x_o + \Delta x_o, \\
&\qquad \sum_{j\in J} \lambda_j y_j \geq \varphi y_o, \\
&\qquad \lambda_j = \delta \mu_j,\ \ \mu_j \in \{0,1\},\ \ \forall j \in J, \\
&\qquad \sum_{j\in J} \mu_j = 1,\ \ \delta \in \Delta,
\end{aligned} \tag{11}$$

Proof. Assume that $(\lambda^*, \mu^*, \delta^*, \varphi_o^{\mathrm{S}_\Delta})$ is an optimal solution of (11). We have:

$$\begin{aligned}
&\sum_{j\in J} \lambda_j^* x_j \qquad \leq x_o + \Delta x_o = \alpha_o, \\
&\sum_{j\in J} \lambda_j^* y_j \qquad \geq \varphi_o^{\mathrm{S}_\Delta} y_o = \varphi_o^{\Delta} \frac{\varphi_o^{\mathrm{S}_\Delta}}{\varphi_o^{\Delta}} y_0 = \varphi_o^{\Delta} \beta_o, \\
&\lambda_j^* = \delta^* \mu_j^*, \\
&\sum_{j\in J} \mu_j^* = 1.
\end{aligned}$$

Therefore $(\lambda^*, \mu^*, \varphi_o^{\Delta})$ is a feasible solution to (10) with objective value φ_o^{Δ}. Since the objective function of (10) is in maximization form, we get

$$\breve{\varphi}_o^{\Delta} \geq \varphi_o^{\Delta}. \tag{12}$$

We wish to show that $\breve{\varphi}_o^{\Delta} = \varphi_o^{\Delta}$. Assume, by contradiction, that $\breve{\varphi}_o^{\Delta} > \varphi_o^{\Delta}$. Thus, $\breve{\varphi}_o^{\Delta} = (1+\varepsilon)\varphi_o^{\Delta}$, for some $\varepsilon > 0$. If $(\bar{\lambda}, \bar{\mu}, \bar{\delta}, \breve{\varphi}_o^{\Delta})$ is an optimal solution to (9), then we have:

$$\begin{aligned}
\sum_{j\in J}\bar{\lambda}_j x_j &\leq x_0+\Delta x_0,\\
\sum_{j\in J}\bar{\lambda}_j y_j &\geq \check{\varphi}_o^{\Delta}\frac{\varphi_o^{\mathrm{S}_\Delta}}{\varphi_o^{\Delta}}y_0\\
&= (1+\varepsilon)\varphi_o^{\Delta}\frac{\varphi_o^{\mathrm{S}_\Delta}}{\varphi_o^{\Delta}}y_0=(1+\varepsilon)\varphi_o^{\mathrm{S}_\Delta}y_o,\\
\bar{\lambda}_j &= \bar{\delta}\bar{\mu}_j,\ \bar{\mu}_j\in\{0,1\},\ \forall j\in J,\\
\sum_{j\in J}\bar{\mu}_j &= 1,\ \bar{\delta}\in\Delta.
\end{aligned}$$

Therefore, $(\varphi,\lambda,\mu,\delta)=((1+\varepsilon)\varphi_o^{\mathrm{S}_\Delta},\bar{\lambda},\bar{\mu},\bar{\delta})$ is a feasible solution to (11) with objective value $(1+\varepsilon)\varphi_o^{\mathrm{S}_\Delta}>\varphi_o^{\mathrm{S}_\Delta}$. This is a clear contradiction. Thus,

$$\check{\varphi}_o^{\Delta}=\varphi_o^{\Delta}. \tag{13}$$

This completes the proof. □

4.3 Inverse FDH: Cost efficiency model

Ghiyasi [14] developed the inverse DEA models when price information is available. He tries to estimate the input cost variation under convex technology, while preserving the cost efficiency score. This subsection is devoted to the inverse cost efficiency model under non-convex FDH technology when the input prices are available. He In fact, this subsection aims to estimate the cost/output variation level while preserving the cost efficiency score. Suppose the output level of DMU_o under a non-convex technology is to be increased. The question is how much more input cost must be spent to produce such outputs while preserving the cost efficiency score. Suppose the DMU_o's manager aims to reach an output level $\beta_o=y_o+\Delta y_o;\Delta y_o\geq 0$ when the input cost vector is fully available. The manager wants to estimate the input cost $w\alpha_o=wx_o+w\Delta x_o$ while the cost efficiency score under non-convex FDH technology remains unchanged. The cost efficiency score of production unit (α_o,β_o) is denoted by $\hat{CE}_o^{\Delta}$, and is computed by solving the following nonlinear mixed-integer programming problem:

$$\begin{aligned}
\hat{CE}_o^{\Delta} = \min\ & \frac{w\alpha}{w\alpha_o} \\
\text{s.t.}\ & \sum_{j\in J}\lambda_j x_j \leq \alpha, \\
& \sum_{j\in J}\lambda_j y_j \geq \beta_o, \\
& \lambda_j = \delta\mu_j,\ \mu_j\in\{0,1\},\ \forall j\in J, \\
& \sum_{j\in J}\mu_j = 1,\ \delta\in\Delta,
\end{aligned} \tag{14}$$

Let the input cost vector w is given. The aim is to estimate the input cost value $w\alpha_o$ so that the cost efficiency score remains unchanged. The following theorem proposes a sufficient condition for estimating the expected input cost and the minimum input cost to produce the target output level while the cost efficiency score remains unchanged.

Theorem 4.5. *Let the input cost vector w is given. If $(\tilde{\lambda}, \tilde{\alpha})$ is part of an optimal solution to (15), then the cost efficiency score of production point $(\alpha_o = \frac{\tilde{\alpha}}{CE_o^{\Delta}}, \beta_o)$ is CE_o^{Δ},*

$$\begin{aligned}
\min\ & \sum_{i=1}^{m} w_i\alpha_i, \\
s.t.\ & \sum_{j\in J}\lambda_j x_{ij} \leq \alpha_i, \\
& \sum_{j\in J}\lambda_j y_{rj} \geq \beta_{ro}, \\
& \lambda_j = \delta\mu_j,\ \mu_j\in\{0,1\},\ \forall j\in J, \\
& \sum_{j\in J}\mu_j = 1,\ \delta\in\Delta,
\end{aligned} \tag{15}$$

Proof. Assume $\left(\tilde{\lambda}, \tilde{\alpha}\right)$ is part of an optimal solution to (15). Therefore, this solution satisfies the constraints of (14) with $\alpha_o = \frac{\tilde{\alpha}}{CE_o^{\Delta}}$. The objective value of (14) associated with this feasible solution is

$$\frac{w\tilde{\alpha}}{w\alpha_o} = \frac{w\tilde{\alpha}}{\frac{w\,\tilde{\alpha}}{CE_o^{\Delta}}} = CE_o^{\Delta}.$$

Since the objective function of (14) is in minimizing form, we get $\hat{CE}_o^{\Delta} \leq CE_o^{\Delta}$. Now, we will show that $\hat{CE}_o^{\Delta} = CE_o^{\Delta}$. Assume, by contradiction, that $\hat{CE}_o^{\Delta} = \frac{w\alpha^*}{w\alpha_o} = \frac{w\alpha^*}{\frac{w\,\tilde{\alpha}}{CE_o^{\Delta}}} < CE_o^{\Delta}$, where (λ^*, α^*) is an optimal solution to (14). We get $w\alpha^* < w\,\tilde{\alpha}$, and

$$\begin{aligned}
&\sum_{j\in J} \lambda_j^* y_j \geq \beta_o, \\
&\sum_{j\in J} \lambda_j^* x_j \leq \alpha^*, \\
&\lambda_j^* = \delta^* \mu_j^*,\ \mu_j^* \in \{0,1\},\ \forall j \in J, \\
&\sum_{j\in J} \mu_j^* = 1,\ \delta^* \in \Delta.
\end{aligned}$$

This implies that (λ^*, α^*) is part of a feasible solution to (15) with objective value $w\alpha^* < w\,\tilde{\alpha}$, which is a clear contradiction. Thus, $\hat{CE}_o^{\Delta} = CE_o^{\Delta}$, and the proof is completed. □

5 Illustrative Example

This section contains a numerical example to illustrate the provided theoretical results. We have 12 DMUs which consume three inputs to produce two outputs. The data of this example, listed in Table 1, have been addressed initially by Zhang and Cui [39]:

Table 1: Data of 12 DMUs

DMU	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5	DMU_6
I_1	350	298	422	281	301	360
I_2	39	26	31	16	16	29
I_3	9	8	7	9	6	17
O_1	67	73	75	70	75	83
O_2	751	611	584	665	445	1070
DMU	DMU_7	DMU_8	DMU_9	DMU_{10}	DMU_{11}	DMU_{12}
I_1	540	276	323	444	323	444
I_2	18	33	25	64	25	64
I_3	10	5	5	6	5	6
O_1	73	78	75	74	25	104
O_2	457	590	1074	1072	350	1199

The input-oriented (model (3)) and output-oriented (model (5)) efficiencies under Δ^{VRS} technology have been shown in Tables 4 and 3, respectively. We used the enumeration method presented in Theorems 3 and 4 to compute the efficiency scores.

Table 2: The input-oriented efficiencies under Δ^{VRS} technology

DMU	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5	DMU_6
θ	0.92	1.00	0.81	1.00	1.00	1.00
Refrence	DMU_9	DMU_2	DMU_9	DMU_4	DMU_5	DMU_6
DMU	DMU_7	DMU_8	DMU_9	DMU_{10}	DMU_{11}	DMU_{12}
θ	1.00	1.00	1.00	0.83	1.00	1.00
Refrence	DMU_7	DMU_8	DMU_9	DMU_9	DMU_{11}	DMU_{12}

Table 3: The FDH output-oriented efficiencies

DMU	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5	DMU_6
φ	1.12	1.00	1.00	1.00	1.00	1.00
Refrence	DMU_9	DMU_2	DMU_9	DMU_4	DMU_5	DMU_6
DMU	DMU_7	DMU_8	DMU_9	DMU_{10}	DMU_{11}	DMU_{12}
φ	1.00	1.00	1.00	1.12	3.00	1.00
Refrence	DMU_4	DMU_8	DMU_9	DMU_{12}	DMU_9	DMU_{12}

We increase the value of the first output of DMU_1 from 67 to 68 ($\Delta y_1 = (1, 0)$), the other data remain unchanged.

By using the inverse FDH input-oriented model (Theorem 4.2), the minimum expected change in the inputs of DMU_1 would be $\Delta x_1 = (1.087, 0, 0)$. Notice that the expected increase of inputs under the convex InvDEA are bigger than their corresponding non-convex inverse FDH model. This means that the first input should increase by 1.087 and other inputs remain unchanged. As can be seen, the efficiency scores of all DMUs remain unchanged. Now, assume that the value of the first input of DMU_1 are to be increased from 350 to 351, and other data remain unchanged.

We wish to estimate the expected increase of the outputs of DMU_1. By using the output-oriented form of the inverse FDH model (Theorem 4.4), the second output should be increased to $\Delta = 17.27$ and the first output will remain unchanged. As can be observed, the efficiency of the DMU does not change.

Assume the input price vector is available as $w = (4, 3, 6)$. The cost efficiency measures for all DMUs have been reported in Table 4. Tables 2 and 4 show that the cost efficiency measures are less than or equal to technical efficiencies. Suppose the target outputs for DMU_1 are changed to $y_1 = (69, 754)$. The expected input vector to produce this target output vector is $x_1 = (362.92, 28.09, 5.62)$. The cost efficiency measure is calculated as follows:

$$CE_1 = \frac{1397}{1569.67} = 0.89.$$

Therefore, the cost efficiency score of DMU_1 remains unchanged. This confirms the result provided by Theorem 4.5.

Table 4: The FDH cost efficiencies under VRS technology

DMU	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5	DMU_6
cost efficiency	0.89	1.00	0.68	1.00	0.96	1.00
DMU	DMU_7	DMU_8	DMU_9	DMU_{10}	DMU_{11}	DMU_{12}
cost efficiency	0.54	1.00	1.00	0.70	0.88	1.00

6 Conclusion

InvDEA is used to solve three types of problems. The first type is resource allocation problems, which determine the minimum required increase of inputs for producing specific outputs while maintaining efficiency under the current technology. The second type is investment analysis, which determines the maximum increase in outputs for a specific increase of inputs preserving the efficiency score. When the input prices are available, the cost efficiency model evaluates the ability of a DMU to produce the current outputs at minimal cost, see Mostafaee, and Saljooghi [29]. The third type is cost analysis, which determines the expected increase of input costs for producing the targeted outputs while preserving the cost efficiency score.

The InvDEA models may underestimate (overestimate) the input (output) variation in a non-convex setting because it provides an artificial unit (convex combination of the observed unit) as the target unit to follow and learn from. This artificial unit underestimates (overestimates) the input/output variations. To solve this issue, we developed the InvDEA models to the non-convex FDH models. Unlike the InvDEA models, the inverse FDH model provides an observed DMU as the target unit, which is simple to learn and easy to use.

This is the first attempt to extend the InvDEA concept to the non-convex FDH technology. Developing the inverse FDH model to the non-convex technologies is important from theoretical and practical viewpoints. We proposed the enumeration method to estimate the input/output variation while the efficiency score remains the same. Furthermore, the inverse cost efficiency concept tries to estimate the expected input costs to produce a pre-specified output level when the input prices are fully available. Developing the notion of InvDEA to the DMUs with network structures under non-convex FDH technology can be considered for future research.

Acknowledgments. The authors wish to thank the anonymous referees for their constructive comments, which contributed to the improvement of the present paper. I am especially grateful to the Editor for the valuable comments.

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