Journal of Mathematical Extension
Vol. 17, No. 11, (2023) (6)1-33
URL: https://doi.org/10.30495/JME.2023.2873
ISSN: 1735-8299
Original Research Paper

Investigation of the Solution for the k-dimensional Device of Differential Inclusion of Laplacein Fraction with Sequential Derivatives and Boundary Conditions of Integral and Derivative

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Abstract. In this paper, we intend to investigate the solutions of the fractional differential inclusion system with successive derivatives with the Laplace operator and according to the derivative and integral boundary conditions. In this regard, we use the fixed point and end point theorems. Finally, we will show this device's effectiveness by providing some practical examples.

AMS Subject Classification: MSC 39A13; 39B72; 30C45
Keywords and Phrases: p-Laplacian, end point, Caputo fractional derivatives, k-dimensional differential inclusion, Riemann-Liouville fractional derivative

Received: October 2023; Accepted: December 2023
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1 Introduction

The calculation of differential calculus and fractional integrals is the same as generalizing the correct orders to the desired rankings. With the development and the breadth of fractional differential (FD) calculations, fraction equations (FDE), and inclusions (FDI) in modeling engineering, physical and medical disciplines have many applications, such as use in mechanics, heat, chemistry, genetics and so on [1, 5, 11, 12, 21, 23, 34, 35]. There are papers of interest called FDEs with different boundary conditions [2, 3, 9, 10, 13, 14, 16, 17, 18, 19, 20, 24, 25, 29, 30, 33]. Among these papers, we can name the FDEs with the p-Laplacian operator and integral boundary conditions [4, 6, 7, 8, 15, 22, 27].

Han *et al.* in [17] investigated the following problem with derivative boundary conditions on cones using the p-Laplacien operator,

$$\begin{cases} D^{\zeta}\left(\phi_p(D_0^{\beta}\tilde{p}(n))\right) = \lambda f(\tilde{p}(n)), & n, \lambda \in \mathfrak{J} := [0,1], \\ \tilde{p}(0) = 0,\ \tilde{p}'(0) = 0,\ \tilde{p}'(1) = 0, \\ \phi_p\left(D^{\beta}\tilde{p}(0)\right) = \phi_p\left(D^{\beta}\tilde{p}(0)\right)' = 0, \end{cases}$$

where $1 < \zeta < 2$, $2 < \beta < 3$. They obtain new results for the FDE with the above boundary conditions. Using the Leggett-Williams fixed point theorem, Günendi and Yaslan examined the existence of positive solutions to the following problem

$$\begin{cases} -D^{\zeta-2}\left(\tilde{p}''(n)\right) + f\left(\tilde{p}(n)\right) = 0, & n \in \mathfrak{J}, \\ \tilde{p}''(0) = \tilde{p}'''(0) = \tilde{p}^{\zeta-2}(1) = 0,\ \tilde{p}'''(1) = 0, \\ \beta_1\tilde{p}(0) - \alpha_1\tilde{p}'(0) = \displaystyle\sum_{t=1}^{h-2} a_t \int_0^{\nu_t} \tilde{p}(s)\,\mathrm{d}s, \\ \beta_2\tilde{p}(1) + \alpha_2\tilde{p}'(1) = \displaystyle\sum_{t=1}^{h-2} b_t \int_0^{\nu_t} \tilde{p}(s)\,\mathrm{d}s, \end{cases}$$

where $k-1 < \nu < k$,$k > 3$, $\alpha_i, \beta_i > 0$, $i = 1, 2$, $a_t, b_t > 0$ are considered as fixed [15]. In 2021, Aydogan *et al.* studied the k-dimensional hybrid system of FDIs

$$\begin{cases} \eta_{1_1}\left({}^cD_0^{\zeta}+\eta_{2_1}{}^cD_0^{\zeta-1}\right)\left[\frac{\tilde{p}_1(j)}{g_1(j,\tilde{p}_1(j),{}^RI^{\beta}\tilde{p}_1(j))}\right]\in\widehat{B}_1\left(j,\tilde{p}'_{1,2,\dots,k}(j)\right),\\ \eta_{1_2}\left({}^cD_0^{\zeta}+\eta_{2_2}{}^cD_0^{\zeta-1}\right)\left[\frac{\tilde{p}_2(j)}{g_2(j,\tilde{p}_2(j),{}^RI^{\beta}\tilde{p}_2(j))}\right]\in\widehat{B}_2\left(j,\tilde{p}'_{1,2,\dots,k}(j)\right),\\ \qquad\dots\\ \eta_{1_k}\left({}^cD_0^{\zeta}+\eta_{2_k}{}^cD_0^{\zeta-1}\right)\left[\frac{\tilde{p}_k(j)}{g_k(j,\tilde{p}_k(j),{}^RI^{\beta}\tilde{p}_k(j))}\right]\in\widehat{B}_k\left(j,\tilde{p}'_{1,2,\dots,k}(j)\right), \end{cases}$$

where

$$\widehat{B}_{\varkappa}(j,\tilde{p}'_{1,2,\dots,k}(j))=B_k\left(j,\tilde{p}_1(j),\dots,\tilde{p}_k(j),\tilde{p}'_1(j),\dots,\tilde{p}'_k(j)\right),$$

here $\varkappa=1,2,\dots,k$, under three-point hybrid boundary conditions

$$\begin{cases} \left[\frac{\tilde{p}_i(j)}{g_i(j,\tilde{p}_i(j),{}^RI^{\beta}\tilde{p}_i(j))}\right]\Big|_{j=0}=0,\\ {}^cD_0^1\left[\frac{\tilde{p}_i(j)}{g_i(j,\tilde{p}_i(j),{}^RI^{\beta}\tilde{p}_i(j))}\right]\Big|_{j=0}+{}^cD_0^2\left[\frac{\tilde{p}_i(j)}{g_i(j,\tilde{p}_i(j),{}^RI^{\beta}\tilde{p}_i(j))}\right]\Big|_{j=0}=0,\\ \left[\frac{\tilde{p}_i(j)}{g_i(j,\tilde{p}_i(j),{}^RI^{\beta}\tilde{p}_i(j))}\right]\Big|_{j=1}+{}^RI^{\xi}\left[\frac{\tilde{p}_i(j)}{g_i(j,\tilde{p}_i(j),{}^RI^{\beta}\tilde{p}_i(j))}\right]\Big|_{j=\epsilon}=0, \end{cases}$$

where $1\le i\le k$, $j\in\mathfrak{J}$, $\zeta\in(2,3]$, $\beta\in(0,1)$, $\eta_{1_1},\dots,\eta_{1_k}$, $\eta_{2_1},\dots,\eta_{2_k}$, $\beta,\xi>0$, $g_i\in C(\mathfrak{J}\times\mathbb{R}\times\mathbb{R})$ with $g\neq0$ and $B\in C(\mathfrak{J}\times\mathbb{R}^{2k},\mathcal{P}(\mathbb{R}))$ is a multifunction [3].

We state essential definitions and lemmas in Section 2. Using the idea of the above problems, in Section 3, we intend to investigate the existence of a solution for the FDI system:

$$\begin{cases} {}^cD_0^{\zeta_1}\Big[\Phi_a\left({}^cD_0^{\sigma_1}u_1(j)\right)\Big]+g_1\left(j,u_1(j)\right)\in\widehat{B}_1\left(j,\hat{v}_{1,\dots,k}(j)\right),\\ {}^cD_0^{\zeta_2}\Big[\Phi_a\left({}^cD_0^{\sigma_2}u_2(j)\right)\Big]+g_2\left(j,u_2(j)\right)\in\widehat{B}_2\left(j,\hat{v}_{1,\dots,k}(j)\right),\\ \qquad\dots\\ {}^cD_0^{\zeta_k}\Big[\Phi_a\left({}^cD_0^{\sigma_k}u_k(j)\right)\Big]+g_k\left(j,u_k(j)\right)\in\widehat{B}_k\left(j,\hat{v}_{1,\dots,k}(j)\right), \end{cases}\tag{1}$$

where

$$\widehat{B}_{\varkappa}(j,\hat{v}_{1,\dots,k}(j))=B_{\varkappa}\left(j,v_1(j),\dots,v_k(j),v'_1(j),\dots,v'_k(j)\right),$$

$\varkappa = 1, 2, \ldots, k$, under a combination of integral and fractional derivative boundary conditions

$$\begin{cases} \Phi_a \left({}^c D_0^{\sigma_i} u_i(1)\right) = 0, \quad \Phi_a({}^c D_0^{\sigma_i} u_i(\eta)) = D^{1/2} b, \\ u_i''(0) = {}^R I^{\sigma_i} b, \quad u_i'(1) = {}^R I^{\sigma_i} b + D^{1/2} b, \quad u_i(\epsilon) = \int_0^\eta s \, \mathrm{d}s, \end{cases} \tag{2}$$

for $j, s \in \mathfrak{J}$, where $2 < \sigma_i < 3$, $1 < \zeta_i < 2$, $\Phi_a(j) = |j|^{a-2} j$, $\epsilon, b, \eta \in (0, \infty)$, ${}^c D_0^\zeta$ and ${}^c D_0^\sigma$ are the fractional derivative Caputo sense and ${}^R I^{\sigma_i}$ is Riemann-Liouville (R-L) integral of order σ, $B_\varkappa \in C(\mathfrak{J} \times \mathbb{R}^{2k}, \mathcal{P}(\mathbb{R}))$ is a multifunction and $g_\varkappa : \mathfrak{J} \times (0, \infty) \to (0, \infty)$ is continuous for each $1 \leq \varkappa \leq k$. A few examples are illustrated which guarantee the validity of our outcomes in Section 4. Finally, with the conclusion, we introduce the views of the future works.

2 Preliminaries and Notations

At the beginning of the work, we will state the definition of integrals, derivatives with fractional order, and will state the fixed point theorems that we use [28, 32].

Throughout this article, we consider the following assumptions:

(S1) $g(j, u(j)) : \mathfrak{J} \times (0, \infty) \to (0, \infty)$ is continuous;

(S2) $a + b = ab$ and $\Phi_b(s)$ is inverse $\Phi_a(s)$;

(S3) (S3) $\epsilon, b, \eta \in (0, \infty)$ and $D^{1/2}$ is the R-L fractional derivative.

Assuming that $\zeta > 0$, $\zeta \in (m-1, m)$ and $m = [\zeta] + 1$. The R-L integral for $p : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is defined by

$${}^R I^\zeta p(s) = \int_0^s \frac{(s-t)^{\zeta - 1}}{\Gamma(\zeta)} p(t) \, \mathrm{d}t,$$

whenever the the integral exists [28, 32]. If $p \in BC^{(m)}(\mathbb{R}_{\geq 0})$, the fractional Caputo derivative is defined by

$${}^c D^\zeta p(s) = \int_0^s \frac{(s-\varrho)^{m-\zeta-1}}{\Gamma(n-\zeta)} p^{(m)}(\varrho) \, \mathrm{d}\varrho,$$

provided that the integral is finite-valued [28, 32]. Also, smooth enough for the function $p : \mathbb{R}_{\geq 0} \to \mathbb{R}$, the sequential fractional derivative is expressed by

$$D^{\zeta} p(s) = \left(D^{\zeta_1} D^{\zeta_2} \dots D^{\zeta_m} \right) p(s),$$

for a multi-index $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_m)$ [26]. First, we note that the sequential derivative operator D^{ζ}, can be versions such as R-L, Caputo, or any other copy of the fractional derivative operator. In this article, we use Caputo's ordinal derivatives of different orders. Caputo's ordinal fractional derivative for $m-1 < \zeta < m$ where $p : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is a smooth function is expressed by

$${}^{c}D^{\zeta} p(s) = D^{-(m-\zeta)} \left(\tfrac{\mathrm{d}}{\mathrm{d}s} \right)^{m} p(s),$$

where $D^{-(m-\zeta)} p(s) = {}^{R}I^{(m-\zeta)} p(s)$ is the fractional R-L integral of order $k - \zeta$ [28]. It is definitely proved that the general solution for the homogeneous FDE ${}^{c}D^{\zeta}_{0^+} p(s) = 0$ is given by

$${}^{R}I^{\zeta} \left({}^{c}D^{\zeta} p(s) \right) = p(s) + \sum_{n=0}^{m-1} \tilde{d}_n s^{v} = p(s) + \tilde{d}_0 + \tilde{d}_1 t + \tilde{d}_2 t^2 + \cdots + \tilde{d}_{m-1} s^{m-1},$$

where $\tilde{d}_0, \dots, \tilde{d}_{m-1} \in \mathbb{R}$ with $m = [\zeta] + 1$ [26]. Consider $(\mathcal{W}, d_{\mathcal{W}})$ is a metric space where $d_{\mathcal{W}}$ is the meter of this space. Pompeiu-Hausdorff metric can be expressed as follows ([11]):

$$\mathrm{P.H}_{d_{\mathcal{W}}} : \mathcal{P}_{c.l.s}(\mathcal{W}) \times \mathcal{P}_{c.l.s}(\mathcal{W}) \to \mathbb{R}^* = \mathbb{R} \cup \{\infty\},$$

in the way that

$$\mathrm{P.H}_{d_{\mathcal{W}}}(R_1, R_2) = \max \left\{ \sup_{r_1 \in R_1} d_{\mathcal{W}}(r_1, R_2), \sup_{r_2 \in R_2} d_{\mathcal{W}}(R_1, r_2) \right\},$$

where

$$d_{\mathcal{W}}(R_1, r_2) = \inf_{r_1 \in R_1} d_{\mathcal{W}}(r_1, r_2), \qquad d_{\mathcal{W}}(r_1, R_2) = \inf_{r_2 \in R_2} d_{\mathcal{W}}(r_1, r_2).$$

Assuming that $(\mathcal{W}, \|\cdot\|_{\mathcal{W}})$ be a normed space. For convenience, we can use symbols. $\mathbb{P}_{c.l.s}(\mathcal{W})$, $\mathbb{P}(\mathcal{W})$, $\mathbb{P}_{c.m.p}(\mathcal{W})$, $\mathbb{P}_{b.n.d}(\mathcal{W})$, and $\mathbb{P}_{c.v.x}(\mathcal{W})$ for the sets of all closed, all subsets, all compact, all bounded and all convex subsets of the space $\mathcal{W}$, respectively. An element like $p^* \in \mathcal{W}$ is called a fixed point for mapping $\mathcal{S} : \mathcal{W} \to \mathbb{P}(\mathcal{W})$ with the value of the given set whenever $p^* \in \mathcal{S}(p^*)$ [11]. Using the symbol $FIX(\mathcal{S})$ to express the point [11]. A multifunction $\mathcal{S} : \mathcal{W} \to \mathbb{P}_{c.l.s}(\mathcal{W})$ is Lipschitzian with a positive constant such as $\hat{\theta}$ exists such that we have

$$\mathrm{P.H}_{d_{\mathcal{W}}}\big(\mathcal{S}(\acute{h}), \mathcal{S}(\acute{\acute{h}})\big) \leq \hat{\theta} d_{\mathcal{W}}(\acute{h}, \acute{\acute{h}}), \qquad \forall\, \acute{h}, \acute{\acute{h}} \in \mathcal{W}.$$

If $\hat{\theta}$ exists for a Lipschitz mapping such that $\hat{\theta} \in (0,1)$ is, we say there is a contraction [11]. In the following, $\mathcal{T}$ is said to be completely continuous if $\mathcal{T}(Q)$ is relatively compact for each $Q \in \mathbb{P}_{b.n.d}(\mathcal{W})$, with $\mathcal{T} : \mathfrak{J} \to \mathbb{P}_{c.l.s}(\mathbb{R})$ is called measurable if

$$s \longmapsto d_{\mathcal{W}}(\upsilon, \mathcal{T}(s)) = \inf \Big\{ |\upsilon - x| : x \in \mathcal{T}(s) \Big\},$$

is measurable for any $\upsilon \in \mathbb{R}$ [1, 11]). At the end, $\mathcal{T}$ is an upper semi-continuous if for $w^* \in \mathcal{W}$, the set $\mathcal{T}(w^*)$ belongs to $\mathbb{P}_{c.l.s}(\mathcal{W})$ and also, for each open set $\mathcal{Y}$ of $\mathcal{W}$ containing $\mathcal{T}(w^*)$, there is a neighborhood $\mathcal{Q}_0^*$ of w^* provided that $\mathcal{T}(\mathcal{Q}_0^*) \subseteq \mathcal{Y}$ [11]. We denote the graph of the multifunction $\mathcal{T} : \mathcal{W} \to \mathbb{P}_{c.l.s}(\mathcal{Y})$ by

$$\mathrm{Graph}\,(\mathcal{T}) = \Big\{ (\mathfrak{w}, \mathfrak{y}) \in \mathcal{W} \times \mathcal{Y} \,:\, \mathfrak{y} \in \mathcal{T}(\mathfrak{w}) \Big\}.$$

The Graph$\,(\mathcal{T})$ is closed whenever there are two arbitrary sequences $\{p_m\}_{m\geq 1}$, $\{x_m\}_{m\geq 1}$ in $\mathcal{W}$, $\mathcal{X}$ respectively, such as with conditions $m \to \infty$, we have $x_0 \in \mathcal{T}(p_0)$ [1, 11]. Therefore, the results that if the multifunction $\mathcal{T} : \mathcal{W} \to \mathbb{P}_{c.l.s}(\mathcal{X})$ It has a upper semi-continuous property, then Graph$(\mathcal{T})$ is a closed subset of $\mathcal{W} \times \mathcal{X}$ [11]. Suppose that $\mathcal{T}$ is a property of graph continuity, closed and complete, in this case $\mathcal{S}$ is upper semi-continuous [11]. Also, $\mathcal{T}$ has convex values if $\mathcal{T}(E) \in \mathbb{P}_{c.v.x}(\mathcal{W})$ for all $E \subset \mathcal{W}$. In addition to, a set of choices $\mathcal{T}$ at a point such as $p \in C_{\mathbb{R}}(\mathfrak{J})$ for each $s \in \mathfrak{J}$ is represented by ([1, 11]),

$$(\mathcal{S}.\mathcal{E}.\mathcal{L})_{\mathcal{T},p} := \Big\{ \hat{\upsilon} \in \mathcal{L}^1_{\mathbb{R}}(\mathfrak{J}) \,:\, \hat{\upsilon}(s) \in \mathcal{T}(s, p(s)) \Big\}.$$

Consider $\mathcal{T}$ is an arbitrary multifunction, so for $p \in C_{\mathcal{W}}(\mathfrak{J})$, we have $(\mathcal{S}.\mathcal{E}.\mathcal{L})_{\mathcal{T},p} \neq \emptyset$ whenever $\dim(\mathcal{W}) < \infty$ [31]. We state that $\mathcal{T} : \mathfrak{J} \times$

$\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is expressed Carathéodory if $t \mapsto \mathcal{T}(s,p)$ is measurable for each $p \in \mathbb{R}$ and $p \mapsto \mathcal{T}(s,p)$ is an upper semi-continuous for almost all $p \in \mathfrak{J}$ [1, 11]. A Carathéodory multifunction $\mathcal{T} : \mathfrak{J} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is called $\mathcal{L}^1$-Carathéodory whenever for each $\varphi > 0$ there is $\vartheta_\varphi \in \mathcal{L}^1_{\mathbb{R}^+}(\mathfrak{J})$ with

$$\|\mathcal{T}(s,p)\| = \sup_{s\in\mathfrak{J}} \Big\{ |w| \, : \, w \in \mathcal{T}(s,p) \Big\} \leq \vartheta_\varphi(s),$$

for all $|p| \leq \varphi$ and for almost any $t \in \mathfrak{J}$ [1, 11].

We need to use the case. following key famous theorems.

Theorem 2.1 ([23])**.** *Consider $\mathcal{W}$ be a separable Banach space, $\mathcal{D} : \mathfrak{J} \times \mathcal{W} \to \mathbb{P}_{c.m.p,c.v.x}(\mathcal{W})$ an $\mathcal{L}^1$-Carathéodory set-valued map and $\mu : \mathcal{L}^1_{\mathcal{W}}(\mathfrak{J}) \to C_{\mathcal{W}}(\mathfrak{J})$ a linear continuous map. Then the map*

$$\begin{cases} \mu \circ (\mathcal{S}.\mathcal{E}.\mathcal{L})_{\mathcal{D}} : C_{\mathcal{W}}(\mathfrak{J}) \to \mathbb{P}_{c.m.p,c.v.x}(C_{\mathcal{W}}(\mathfrak{J})), \\ p \mapsto \left(\mu \circ (\mathcal{S}.\mathcal{E}.\mathcal{L})_{\mathcal{D}}\right)(p) = \mu((\mathcal{S}.\mathcal{E}.\mathcal{L})_{\mathcal{D},p}), \end{cases}$$

is an operator in $C_{\mathcal{W}}(\mathfrak{J}) \times C_{\mathcal{W}}(\mathfrak{J})$ and has the closed graph property.

Theorem 2.2 ([12])**.** *Consider that $\mathcal{F}$ is an open subset of a closed convex subset $\mathcal{Q}$ of Banach space $\mathcal{W}$, with $0 \in \mathcal{F}$ and $H : \overline{\mathcal{F}} \to \mathbb{P}_{c.m.p,c.v.x}(\mathcal{Q})$ is a upper semi-continuous compact map, where $\mathbb{P}_{c.m.p,c.v.x}(\mathcal{Q})$ represents the family of compact, nonempty, and convex subsets of $\mathcal{Q}$. Therefore either H has a fixed point in $\overline{\mathcal{F}}$ or there exist $u \in \partial\mathcal{F}$ and $\theta \in (0,1)$ such that $u \in \theta H(u)$.*

3 Main Results

First, we state the key Lemma 3.1.

Lemma 3.1. *Let $\tilde{p} \in \mathcal{Q}$. Then $u(j)$ is a solution for the* FDE*,*

$${}^cD_0^\zeta \left[\Phi_a \left({}^cD_0^\sigma u(j)\right)\right] + g(j, u(j)) = \tilde{p}(j), \tag{3}$$

fort $j \in \mathfrak{J}$, $\sigma \in (2,3]$, and $\zeta \in (1,2]$, with a combination of integral and fractional derivative boundary conditions

$$\begin{cases} \Phi_a \left({}^cD_0^\sigma u(1)\right) = 0, \quad \Phi_a({}^cD_0^\sigma(\eta)) = D^{1/2}b, \\ u''(0) = {}^RI^\sigma b, \quad u'(1) = {}^RI^\sigma b + D^{1/2}b, \quad u(\epsilon) = \displaystyle\int_0^\eta t\,\mathrm{d}t, \end{cases} \tag{4}$$

iff $u(j)$ is a solution for the integral equation

$$\begin{aligned} u(j) &= \int_0^j \frac{(j-\varrho)^{\sigma-1}}{\Gamma(\sigma)}\tilde{p}(\varrho)\,\mathrm{d}\varrho + (\epsilon - j)\int_0^1 \frac{(1-\varrho)^{\sigma-2}}{\Gamma(\sigma-1)}\tilde{p}(\varrho)\,\mathrm{d}\varrho \\ &\quad - \int_0^\epsilon \frac{(\epsilon-\varrho)^{\sigma-1}}{\Gamma(\sigma)}\tilde{p}(\varrho)\,\mathrm{d}\varrho + \frac{\eta^2}{2} \\ &\quad + \frac{b(j-\epsilon)}{2(j\pi)^{1/2}\Gamma(\sigma+1)}\Big[2\Gamma(\sigma+1) + j^\sigma (j\pi)^{1/2}(j+\epsilon)\Big], \end{aligned} \tag{5}$$

where

$$\tilde{p}(j) = \int_0^1 T(j,\imath)\, g(\imath,(u(\imath)))\,\mathrm{d}\imath + \tfrac{1-j}{(1-\eta)\Gamma(\zeta)}\Big[\int_0^\eta (\eta-\imath)^{\zeta-1} g(\imath,u(\imath))\,\mathrm{d}\imath + b\Big],$$

and for $t, j \in \mathfrak{J}$,

$$T(j,\varrho) = \begin{cases} \frac{-\left[(1-\eta)(j-\varrho)^{\zeta-1} + (j-\eta)(1-\varrho)^{\zeta-1}\right]}{(1-\eta)\Gamma(\zeta)}, & \varrho < j, \\ \frac{(j-\eta)(1-\varrho)^{\zeta-1}}{(1-\eta)\Gamma(\zeta)}, & \varrho > j. \end{cases} \tag{6}$$

Proof. It is observed

$${}^{R}I^{\zeta}\left({}^{c}D_0^{\zeta}\Big[\Phi_a\left({}^{c}D_0^{\sigma}u(j)\right)\Big]\right) = -{}^{R}I^{\zeta} g(j,u(j)),$$

and so, $\Phi_a({}^{c}D_0^{\sigma}u(j)) = -I^{\zeta}g(j,u(j)) + c_0 + c_1 j$. Since the $\Phi_a({}^{c}D_0^{\sigma}u(1)) = 0$, we conclude

$$-\int_0^1 \tfrac{(1-\varrho)^{\zeta-1}}{\Gamma(\zeta)} g(\varrho,u(\varrho))\,\mathrm{d}\varrho + c_0 + c_1 = 0.$$

The first and second conditions imply that

$$c_0 + c_1 = \int_0^1 \tfrac{(1-\varrho)^{\zeta-1}}{\Gamma(\zeta)} g(\varrho,u(\varrho))\,\mathrm{d}\varrho,$$

$$\Phi_a\left({}^{c}D_0^{\sigma}u(\eta)\right) = D^{1/2}b = -\int_0^\eta \tfrac{(\eta-t)^{\zeta-1}}{\Gamma(\zeta)} g(\varrho,u(\varrho))\,\mathrm{d}\varrho + c_0 + c_1\eta.$$

Now, we can get the value of $D^{1/2}b$ according to the conditions of the problem

$$D^{1/2}b = \tfrac{1}{\Gamma(n-\sigma)}\left(\tfrac{\mathrm{d}}{\mathrm{d}j}\right)^n \int_0^j (j-s)^{n-\sigma-1} g(t,u(t))\,\mathrm{d}t,$$

which we have for $n=1$, $D^{1/2}b = b(j\pi)^{-1/2}$. Thus,

$$-\int_0^\eta \tfrac{(\eta-\varrho)^{\zeta-1}}{\Gamma(\zeta)} g(\varrho, u(\varrho))\,\mathrm{d}\varrho + c_0 + c_1\eta = b(j\pi)^{-1/2}.$$

Hence,

$$\begin{aligned} c_1 &= \int_0^1 \frac{(1-\imath)^{\zeta-1}}{\Gamma(\zeta)} g(\imath, u(\imath))\,\mathrm{d}\imath - c_0, \\ c_0 &= \frac{1}{1-\eta}\Bigg[\int_0^\eta \frac{(\eta-\imath)^{\zeta-1}}{\Gamma(\zeta)} g(\imath, u(\imath))\,\mathrm{d}\imath \\ &\quad - \eta\int_0^1 \frac{(1-\imath)^{\zeta-1}}{\Gamma(\zeta)} g(\imath, u(\imath))\,\mathrm{d}\imath + b(j\pi)^{-1/2}\Bigg]. \end{aligned}$$

Thus

$$\begin{aligned} c_1 = \frac{1}{1-\eta}\Bigg[&-\int_0^\eta \frac{(\eta-\imath)^{\zeta-1}}{\Gamma(\zeta)} g(\imath, u(\imath))\,\mathrm{d}\imath \\ &+ \int_0^1 \frac{(1-\imath)^{\zeta-1}}{\Gamma(\zeta)} g(\imath, u(\imath))\,\mathrm{d}\imath - b(j\pi)^{-1/2}\Bigg], \end{aligned}$$

and so, we get

$$\begin{aligned} \Phi_a\left({}^cD_0^\sigma u(\eta)\right) &= -\int_0^\eta \frac{(\eta-\imath)^{\zeta-1}}{\Gamma(\zeta)} g(\imath, u(\imath))\,\mathrm{d}\imath + c_0 + c_1\eta \\ &= \int_0^\eta \frac{(\eta-\imath)^{\zeta-1}}{\Gamma(\zeta)} g(\imath, u(\imath))\,\mathrm{d}\imath \\ &\quad + \frac{1}{1-\eta}\Bigg[\int_0^\eta \frac{(\eta-\imath)^{\sigma-1}}{\Gamma(\zeta)} g(\imath, u(\imath))\,\mathrm{d}\imath \\ &\quad - \eta\int_0^1 \frac{(1-\imath)^{\zeta-1}}{\Gamma(\zeta)} g(\imath, u(\imath))\,\mathrm{d}\imath + b(\eta\pi)^{-1/2}\Bigg] \\ &\quad + \frac{j}{1-\eta}\Bigg[-\int_0^\eta \frac{(\eta-\imath)^{\zeta-1}}{\Gamma(\zeta)} g(\imath, u(\imath))\,\mathrm{d}\imath \\ &\quad + \int_0^1 \frac{(1-\imath)^{\zeta-1}}{\Gamma(\zeta)} g(\imath, u(\imath))\,\mathrm{d}\imath - b(j\pi)^{-1/2}\Bigg] \\ &= \int_0^1 T(j,\imath) g(\imath, u(\imath))\,\mathrm{d}\imath \end{aligned}$$

$$+\frac{(1-j)}{(1-\eta)\Gamma(\zeta)}\bigg[\int_0^\eta (\eta-t)^{\zeta-1}\, g(\imath,u(\imath))\,\mathrm{d}\imath + b\bigg].$$

The inverse of the Laplacian operator implies that

$$\begin{aligned}\Phi_b\left(\Phi_a{}^c D_0^\sigma u(j)\right) = \Phi_b\bigg(&\int_0^1 T(j,\imath) g(\imath,u(\imath))\,\mathrm{d}\imath \\ &+ \tfrac{(1-j)}{(1-\eta)\Gamma(\zeta)}\bigg[\int_0^\eta (\eta-\imath)^{\zeta-1} g(\imath,u(\imath))\,\mathrm{d}\imath + b\bigg]\bigg).\end{aligned}$$

We can now calculate $u(j)$. ${}^cD_0^\sigma u(j) = \tilde{p}(j)$, and so

$$I^\sigma\ \left({}^cD_0^\sigma u(j)\right) = \int_0^j \frac{(j-\varrho)^{\sigma-1}}{\Gamma(\sigma)}\tilde{p}(\varrho)\,\mathrm{d}\varrho + d_0 + d_1 j + d_2 j^2.$$

Given that ${}^cD_0^\sigma b = \frac{1}{\Gamma(\sigma+1)} b j^\sigma$ and $D^{1/2} b = b(j\pi)^{-1/2}$, we obtain

$$\begin{aligned} d_0 &= \frac{\eta^2}{2} + \epsilon\int_0^1 \frac{(1-\varrho)^{\sigma-2}}{\Gamma(\sigma-1)}\tilde{p}(\varrho)\,\mathrm{d}\varrho \\ &\quad - \int_0^\epsilon \frac{(\epsilon-\varrho)^{\sigma-1}}{\Gamma(\sigma)}\tilde{p}(\varrho)\,\mathrm{d}\varrho - b\epsilon(j\pi)^{-1/2} - \frac{b\epsilon^2 j^2}{2\Gamma(\sigma+1)}, \\ d_1 &= b(j\pi)^{-1/2} - \int_0^1 \frac{(1-\varrho)^{\sigma-2}}{\Gamma(\sigma-1)}\tilde{p}(\varrho)\,\mathrm{d}\varrho, \qquad d_2 = \frac{bj^2}{2\Gamma(\sigma+1)}. \end{aligned}$$

Therefore $u(j)$, in relation (5), should be obtained. □

At this point, we can examine the next k-dimensional system with derivative and integral boundary conditions (1)-(2). At the beginning, we need to defined the space $\mathcal{W}_i = \{v_i(t) : v_i'(t) \in C(\mathfrak{J})\}$, $i = 1, 2, \dots, k$, endowed with the norm

$$\|v\|_{\mathcal{W}_i} = \sup_{\varrho\in\mathfrak{J}} |v_i(\varrho)| + \sup_{\varrho\in\mathfrak{J}} |v_i'(\varrho)|.$$

In this case, the product space $(\mathcal{W}, \|\cdot\|)$, $\mathcal{W} = W_1 \times W_2 \times \cdots \times W_k$ endowed with the norm

$$\|(v_1, v_2, \dots, v_k)\| = \sum_{i=1}^k \|v\|_{\mathcal{W}_i}, \tag{7}$$

is a Banach space. More on that, we define the set $\mathcal{W}_{B_{i,v}}$ as follows:

$$\mathcal{W}_{B_{i,v}} = \Big\{ p \in L^1(\mathfrak{J}) \,:\, p(s) \in \widehat{B}_i(s, \hat{v}_{1,2,\ldots,k}(s)),\, \forall\, v = (v_1, \ldots, v_k) \in \mathcal{W} \Big\},$$

for almost all $s \in \mathfrak{J}$ and $1 \leq i \leq k$, where

$$\widehat{B}_i(s, \hat{v}_{1,2,\ldots,k}(s)) = B_i\left(s, v_1(s), \ldots, v_k(s), v_1'(s), \ldots, v_k'(s)\right). \tag{8}$$

We say that $(v_1, v_2, \ldots, v_k)$ is a solution for the system (1)-(2), whenever there exists functions $\{p_1, p_2, \ldots, p_k\} \in L^1(\mathfrak{J})$ with

$$p_i(\varrho) \in \mathcal{W}_{B_{i,v}}\left(\varrho, v_1(\varrho), v_2(\varrho), \ldots, v_i(\varrho), v_1'(\varrho), v_2'(\varrho), \ldots, v_i'(\varrho)\right), \tag{9}$$

for each i, almost all $\varrho \in \mathfrak{J}$ and

$$\begin{aligned} p_i(j) = &\int_0^j \frac{(j-\varrho)^{\sigma_i - 1}}{\Gamma(\sigma_i)} v_i(\varrho)\,\mathrm{d}\varrho + (\epsilon - j)\int_0^1 \frac{(1-\varrho)^{\sigma_i - 2}}{\Gamma(\sigma_i - 1)} v_i(\varrho)\,\mathrm{d}\varrho \\ &- \int_0^\epsilon \frac{(\epsilon-\varrho)^{\sigma_i - 1}}{\Gamma(\sigma_i)} v_i(\varrho)\,\mathrm{d}\varrho + \frac{\eta^2}{2} \\ &+ \frac{b(j-\epsilon)}{2(j\pi)^{1/2}\Gamma(\sigma_i+1)}\Big[2\Gamma(\sigma_i+1) + j^{\sigma_i}(j\pi)^{1/2}(j+\epsilon)\Big], \end{aligned} \tag{10}$$

for $1 < i < k$, where

$$\begin{aligned} v_i(j) = &\int_0^1 T_i(j,\varrho) g(\varrho, p_i(\varrho))\,\mathrm{d}\varrho \\ &+ \frac{1-j}{(1-\eta)\Gamma(\zeta_i)}\bigg[\int_0^\eta (\eta-\varrho)^{\zeta_i - 1} g(\varrho, p_i(\varrho))\,\mathrm{d}\varrho + b\bigg], \end{aligned}$$

and for $\varrho, j \in \mathfrak{J}$,

$$T_i(j,\varrho) = \begin{cases} \frac{-\left[(1-\eta)(j-\varrho)^{\zeta_i - 1} + (j-\eta)(1-\varrho)^{\sigma_i - 1}\right]}{(1-\eta)\Gamma(\zeta_i)}, & \varrho < j, \\ \frac{(j-\eta)(1-\varrho)^{\zeta_i - 1}}{(1-\eta)\Gamma(\zeta_i)}, & t > j, \end{cases} \tag{11}$$

$2 < \sigma_i < 3$, $1 < \zeta_i < 2$, $\Phi_a(j) = |j|^{a-2}\, j$. We are careful that, ${}^cD^1_{0+} = \frac{\mathrm{d}}{\mathrm{d}s}$ and ${}^cD^2_{0+} = \frac{\mathrm{d}^2}{\mathrm{d}s^2}$.

Theorem 3.2. *Consider Carathéodory multifunctions*

$$B_1, \dots, B_k : \mathfrak{J} \times \mathbb{R}^{3k} \to \mathbb{P}_{c.m.p,c.v.x}(\mathbb{R}),$$

continuous, bounded, nondecreasing map $\tau : [0, \infty) \to (0, \infty)$ *and continuous functions* $b_1, \dots, b_k : \mathfrak{J} \to (0, \infty)$ *with*

$$\begin{aligned} \left\| \widehat{B}_i(s, \hat{v}_{1,2,\dots,k}(s)) \right\| &= \sup \left\{ |y| \,:\, y \in \widehat{B}_i(s, \hat{v}_{1,2,\dots,k}(s)) \right\} \\ &\leq b_i(\varrho)\, \tau(\|v_1, v_2, \dots, v_k\|), \end{aligned} \tag{12}$$

for all $1 \leq i \leq k$, $(v_1, \dots, v_k) \in \mathcal{W}$ *and almost all* $s \in \mathfrak{J}$. *Let there exist constants* E_i *such that* $E_i \leq \Omega_{i_1} + \Omega_{i_2}$, *where*

$$\begin{aligned} \Omega_{i_1} &= \int_0^1 \left| \left[(1-\varrho)^{\sigma_i - 1} + (\epsilon - 1)(\sigma_i - 1)(1-\varrho)^{\sigma_i - 2} \right] \frac{y_i(\varrho)}{\Gamma(\sigma_i)} \right| \mathrm{d}\varrho \\ &\quad + \int_0^\epsilon \frac{(\epsilon - \varrho)^{\sigma_i - 1}}{\Gamma(\sigma_i)} |y_i(\varrho)| \, \mathrm{d}\varrho + \frac{\eta^2}{2} \\ &\quad + \left| \frac{b(1-\epsilon)}{2\pi^{1/2}\Gamma(\sigma_i + 1)} \left[2b\Gamma(\sigma_i + 1) + \pi^{1/2}(1-\epsilon) \right] \right|, \\ \Omega_{i_2} &= b\left(\frac{1 + 2\epsilon}{2\pi^{1/2}} \right) + \frac{b((\sigma_i + 2) - 2\epsilon^2)}{2\Gamma(\sigma_i + 1)}, \end{aligned} \tag{13}$$

and $\|b_i\| = \sup_{s \in \mathfrak{J}} |b_i(s)|$ *with* $i = 1, \dots, k$, $y_i(s) \in B_i$. *Therefore, the* k*-dimensional* FDI *system* (1) *under boundary conditions* (2) *has at least one solution.*

Proof. We express the operator $L : \mathcal{W} \to 2^{\mathcal{W}}$ by

$$L(v_1, \dots, v_k) = \big(L_1(v_1, \dots, v_k), L_2(v_1, \dots, v_k), \dots, L_k(v_1, \dots, v_k) \big),$$

where

$$L_i(v_1,\ldots,v_k) = \left\{ \begin{array}{l} y \in \mathcal{W}_i \,\Big|\, \exists p \in \mathcal{W}_{B_{i,(v_1,\ldots,v_k)}} \;:\; y(j) = \int_0^j \frac{(j-\varrho)^{\sigma_i-1}}{\Gamma(\sigma_i)} v_i(\varrho)\,\mathrm{d}\varrho \\ \quad +(\epsilon-\varrho)\int_0^1 \frac{(1-\varrho)^{\sigma_i-2}}{\Gamma(\sigma_i-1)} v_i(\varrho)\,\mathrm{d}\varrho \\ \quad -\int_0^\epsilon \frac{(\epsilon-\varrho)^{\sigma_i-1}}{\Gamma(\sigma_i)} v_i(\varrho)\,\mathrm{d}\varrho + \frac{\eta^2}{2} \\ \quad +\frac{b(j-\epsilon)}{2(j\pi)^{1/2}\Gamma(\sigma_i+1)}\left[2\Gamma(\sigma_i+1)+j^{\sigma_i}(j\pi)^{1/2}(j+\epsilon)\right] \end{array} \right\},$$

and we will show that L has at least one fixed point. First, we prove the expression $L(v_1, v_2, \ldots, v_k)$ is convex for all $(v_1, v_2, \ldots, v_k) \in \mathcal{W}$. Consider $(y_1, \ldots, y_k), (y_{j_1}, \ldots, y_{j_k}) \in L(v_1, v_2, \ldots, v_k)$. Choose $p_i, p_{j_i} \in \mathcal{W}_{B_{i,(v_1,v_2,\ldots,v_k)}}$ with

$$\begin{aligned} y_i(j) = &\int_0^j \frac{(j-\varrho)^{\sigma_i-1}}{\Gamma(\sigma_i)} v_i(\varrho)\,\mathrm{d}\varrho + (\epsilon-j)\int_0^1 \frac{(1-\varrho)^{\sigma_i-2}}{\Gamma(\sigma_i-1)} v_i(\varrho)\,\mathrm{d}\varrho \\ &-\int_0^\epsilon \frac{(\epsilon-\varrho)^{\sigma_i-1}}{\Gamma(\sigma_i)} v_i(\varrho)\,\mathrm{d}\varrho + \frac{\eta^2}{2} \\ &+\frac{b(j-\epsilon)}{2(j\pi)^{1/2}\Gamma(\sigma_i+1)}\left[2\Gamma(\sigma_i+1)+j^{\sigma_i}(j\pi)^{1/2}(j+\epsilon)\right], \end{aligned} \tag{14}$$

and

$$\begin{aligned} y_{j_i}(j) = &\int_0^j \frac{(j-\varrho)^{\sigma_i-1}}{\Gamma(\sigma_i)} v_{j_i}(\varrho)\,\mathrm{d}\varrho + (\epsilon-j)\int_0^1 \frac{(1-\varrho)^{\sigma_i-2}}{\Gamma(\sigma_i-1)} v_{j_i}(\varrho)\,\mathrm{d}\varrho \\ &-\int_0^\epsilon \frac{(\epsilon-t)^{\sigma_i-1}}{\Gamma(\sigma_i)} v_{j_i}(\varrho)\,\mathrm{d}\varrho + \frac{\eta^2}{2} \\ &+\frac{b(j-\epsilon)}{2(j\pi)^{1/2}\Gamma(\sigma_i+1)}\left[2\Gamma(\sigma_i+1)+j^{\sigma_i}(j\pi)^{1/2}(j+\epsilon)\right], \end{aligned} \tag{15}$$

for $i = 1, 2, \dots, k$. Consider $\hbar \in \mathfrak{J}$. Then,

$$\begin{aligned}\Big[\hbar y_i + (1-\hbar) y_{j_i}\Big](j) &= \int_0^j \frac{(j-\varrho)^{\sigma_i - 1}}{\Gamma(\sigma_i)} \left[\hbar y_i(\varrho) + (1-\hbar) y_{j_i}(\varrho)\right] \mathrm{d}\varrho \\ &+ (\epsilon - j) \int_0^1 \frac{(1-\varrho)^{\sigma_i - 2}}{\Gamma(\sigma_i - 1)} \left[\hbar y_i(\varrho) + (1-h) y_{j_i}(\varrho)\right] \mathrm{d}\varrho \\ &- \int_0^\epsilon \frac{(\epsilon-\varrho)^{\sigma_i - 1}}{\Gamma(\alpha_i)} \left[\hbar y_i(\varrho) + (1-\hbar) y_{j_i}(\varrho)\right] \mathrm{d}\varrho + \frac{\eta^2}{2} \\ &+ \frac{b(j-\epsilon)}{2(j\pi)^{1/2}\Gamma(\sigma_i+1)} \Big[2\Gamma(\sigma_i+1) + j^{\sigma_i}(j\pi)^{1/2}(j+\epsilon)\Big]. \qquad (16)\end{aligned}$$

Thanks to convex valued B_i,

$$\left[\hbar y_i + (1-\hbar) y_{j_i}\right](\varrho) \in L_i(v_1, \dots, v_k), \qquad \forall\, 1 \leq i \leq k.$$

Thus,

$$\begin{aligned}&\hbar(y_1, \dots, y_k) + (1-\hbar)(y_{j_1}, \dots, y_{j_k}) \\ &\qquad = (\hbar y_1 + (1-\hbar) y_{j_1}, \dots, \hbar y_k + (1-\hbar) y_{j_k}) \in L(v_1, \dots, v_k).\end{aligned}$$

Let $\rho > 0$,

$$A_\rho = \Big\{(v_1, \dots, v_k) \in \mathcal{W} \,:\, \|(v_1, \dots, v_k)\| \leq \rho\Big\},$$

$(v_1, \dots, v_k) \in A_\rho$ and $(v_1, \dots, v_k) \in L(v_1, \dots, v_k)$. Choose

$$(v_1, \dots, v_k) \in \mathcal{W}_{B_{1,(v_1,\dots,v_k)}} \times \cdots \times \mathcal{W}_{B_{k,(v_1,\dots,v_k)}},$$

such that

$$\begin{aligned}y_i(j) &= \int_0^j \frac{(j-\varrho)^{\sigma_i - 1}}{\Gamma(\sigma_i)} v_i(\varrho)\, \mathrm{d}\varrho + (\epsilon - j) \int_0^1 \frac{(1-\varrho)^{\sigma_i - 2}}{\Gamma(\sigma_i - 1)} v_i(\varrho)\, \mathrm{d}t \\ &- \int_0^\epsilon \frac{(\epsilon-\varrho)^{\sigma_i - 1}}{\Gamma(\sigma_i)} v_i(\varrho)\, \mathrm{d}\varrho + \frac{\eta^2}{2} \\ &+ \frac{b(j-\epsilon)}{2(j\pi)^{1/2}\Gamma(\sigma_i+1)} \Big[2\Gamma(\sigma_i+1) + j^{\sigma_i}(j\pi)^{1/2}(j+\epsilon)\Big], \qquad (17)\end{aligned}$$

for $1 \leq i \leq k$. Hence,

$$
\begin{aligned}
y_i'(j) = & \int_0^j \frac{(j-t)^{\sigma_i-2}}{\Gamma(\sigma_i-1)} v_i(\varrho)\,\mathrm{d}\varrho - \int_0^1 \frac{(1-\varrho)^{\sigma_i-2}}{\Gamma(\sigma_i-1)} v_i(\varrho)\,\mathrm{d}\varrho \\
& + \frac{1}{\left(2(j\pi)^{1/2}\Gamma(\sigma_i+1)\right)^2}\Bigg[2b\Gamma(\sigma_i+1) + \frac{b\pi}{2(j\pi)^{1/2}}(j^{\sigma_i+2}-\epsilon^2 j^2) \\
& + \left((\sigma_i+2)j^{\sigma_i+1} - 2\epsilon^2 j\left(b(j\pi)^{1/2}\right)\right]\left[2(j\pi)^{1/2}\Gamma(\sigma_i+1)\right] \\
& - \frac{1}{\left(2(j\pi)^{1/2}\Gamma(\sigma_i+1)\right)^2}\left(\frac{\pi\Gamma(\sigma_i+1)}{(j\pi)^{1/2}}\right)\Bigg[(j-\epsilon)(2b\Gamma(\sigma_i+1)) \\
& + b(j\pi)^{1/2}(j^{\sigma_i+2}-\epsilon^2 j^2)\Bigg] \\
= & \int_0^j \frac{(j-\varrho)^{\alpha_i-2}}{\Gamma(\sigma_i-1)} v_i(\varrho)\,\mathrm{d}t - \int_0^1 \frac{(1-\varrho)^{\sigma_i-2}}{\Gamma(\sigma_i-1)} v_i(\varrho)\,\mathrm{d}\varrho \\
& + \frac{b}{2(j\pi)^{1/2}} + \frac{b\epsilon\pi}{(j\pi)^{3/2}} + \frac{b}{2\Gamma(\sigma_i+1)}\left((\sigma_i+2)j^{\sigma_i+1} - 2\epsilon^2 j\right). \qquad (18)
\end{aligned}
$$

So,

$$
\begin{aligned}
|y_i(j)| \leq & \int_0^j \frac{(j-\varrho)^{\sigma_i-1}}{\Gamma(\sigma_i)}|v_i(\varrho)|\,\mathrm{d}\varrho + (\epsilon - j)\int_0^1 \frac{(1-\varrho)^{\sigma_i-2}}{\Gamma(\sigma_i-1)}|v_i(\varrho)|\,\mathrm{d}\varrho \\
& - \int_0^\epsilon \frac{(\epsilon-\varrho)^{\sigma_i-1}}{\Gamma(\sigma_i)}|v_i(\varrho)|\,\mathrm{d}\varrho + \frac{\eta^2}{2} \\
& + \left|\frac{b(j-\epsilon)}{2(j\pi)^{1/2}\Gamma(\sigma_i+1)}\right|\left[2\Gamma(\sigma_i+1) + j^{\sigma_i}(j\pi)^{1/2}(j+\epsilon)\right] \\
\leq & \int_0^1 \left|\frac{(1-\varrho)^{\sigma_i-1} + (\epsilon-1)(\sigma_i-1)(1-\varrho)^{\sigma_i-2}}{\Gamma(\sigma_i)} v_i(\varrho)\right|\mathrm{d}\varrho \\
& + \int_0^\epsilon \left|\frac{(\epsilon-\varrho)^{\sigma_i-1}}{\Gamma(\sigma_i)} v_i(\varrho)\right|\mathrm{d}\varrho + \frac{\eta^2}{2} \\
& + \left|\frac{b(1-\epsilon)}{2(\pi)^{1/2}\Gamma(\sigma_i+1)}\right|\left[2b\Gamma(\sigma_i+1) + (\pi)^{1/2}(1-\epsilon)\right] \\
\leq & \|b_i\|\tau\left(\|v_1, v_2, \ldots, v_i\|\right)\Omega_{i_1}, \qquad (19)
\end{aligned}
$$

$$
\begin{aligned}
|y_i'(j)| &= \bigg| \int_0^j \frac{(j-\varrho)^{\sigma_i-2}}{\Gamma(\sigma_i-1)} v_i(\varrho)\,\mathrm{d}\varrho - \int_0^1 \frac{(1-\varrho)^{\sigma_i-2}}{\Gamma(\sigma_i-1)} v_i(\varrho)\,\mathrm{d}\varrho \\
&\quad + \frac{b}{2(j\pi)^{1/2}} + \frac{b\epsilon\pi}{(j\pi)^{1/2}} + \frac{b}{2\Gamma(\sigma_i+1)} \left[(\sigma_i+2)j^{\sigma_i+1} - 2\epsilon^2 j\right] \bigg| \\
&\le \int_0^j \frac{(j-\varrho)^{\sigma_i-2}}{\Gamma(\sigma_i-1)} |v_i(\varrho)|\,\mathrm{d}\varrho - \int_0^1 \frac{(1-\varrho)^{\sigma_i-2}}{\Gamma(\sigma_i-1)} |v_i(\varrho)|\,\mathrm{d}\varrho \\
&\quad + \frac{b}{2(j\pi)^{1/2}} + \frac{b\epsilon\pi}{(j\pi)^{3/2}} + \frac{b}{2\Gamma(\sigma_i+1)} \left((\sigma_i+2)j^{\sigma_i+1} - 2\epsilon^2 j\right) \\
&\le b\left(\frac{1+2\epsilon}{2(\pi)^{1/2}}\right) + \frac{b((\sigma_i+2)-2\epsilon^2)}{2\Gamma(\sigma_i+1)} \\
&\le \|b_i\| \tau(\|v_1, v_2, \dots, v_i\|)\Omega_{i_2},
\end{aligned} \tag{20}
$$

for $1 \le i \le k$. Thus, $\|y_i\|_i \le (\Omega_{i_1} + \Omega_{i_2})\|b_i\|_{\mathcal{L}^1}$ and

$$
|y_1, \dots, y_k| = \sum_{i=1}^{k} \|y_i\| \le \sum_{i=1}^{k} (\Omega_{i_1} + \Omega_{i_2})\|b_i\|_{\mathcal{L}^1}.
$$

Indeed, L maps bounded sets of $\mathcal{W}$ into bounded sets. Let $(p_1, \dots, p_k) \in \mathcal{A}_\rho$, $j_1, j_2 \in \mathfrak{J}$ with $j_1 \le j_2$ and $(y_1, \dots, y_k) \in L(p_1, \dots, p_k)$. Then, $\forall\, 1 \le i \le k$, we have

$$
\begin{aligned}
|p_i(j_2) - p_i(j_1)| \le \bigg| & \int_0^{j_2} \frac{(j_2-\varrho)^{\sigma_i-1}}{\Gamma(\sigma_i)} y_i(\varrho)\,\mathrm{d}\varrho \\
& + (\epsilon - j_2) \int_0^1 \frac{(1-\varrho)^{\sigma_i-2}}{\Gamma(\sigma_i-1)} y_i(\varrho)\,\mathrm{d}\varrho \\
& - \int_0^{\epsilon} \frac{(\epsilon-\varrho)^{\sigma_i-1}}{\Gamma(\sigma_i)} y_i(\varrho)\,\mathrm{d}\varrho + \frac{\eta^2}{2} \\
& + \frac{b(j_2-\epsilon)}{2(j_2\pi)^{1/2}\Gamma(\sigma_i+1)} \Big[2\Gamma(\sigma_i+1) \\
& + j_2^{\sigma_i}(j_2\pi)^{1/2}(j_2+\epsilon)\Big] - \bigg[\int_0^{j_1} \frac{(j_1-\varrho)^{\sigma_i-1}}{\Gamma(\sigma_i)} y_i(\varrho)\,\mathrm{d}\varrho \\
& + (\epsilon - j_1) \int_0^1 \frac{(1-\varrho)^{\sigma_i-2}}{\Gamma(\sigma_i-1)} y_i(\varrho)\,\mathrm{d}\varrho \\
& - \int_0^{\epsilon} \frac{(\epsilon-\varrho)^{\sigma_i-1}}{\Gamma(\sigma_i)} y_i(\varrho)\,\mathrm{d}\varrho + \frac{\eta^2}{2}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{b(j_1-\epsilon)}{2(j_1\pi)^{1/2}\Gamma(\alpha_i+1)}\Big[2\Gamma(\sigma_i+1) \\
&+ j_1^{\sigma_i}(j_1\pi)^{1/2}(j_1+\epsilon)\Big]\Big]\Big| \\
&\le \tau(\|v_1,v_2,\ldots,v_i\|)\,\Omega_{i_1}\|b_i\|_{\mathcal{L}^1},
\end{aligned}
$$

and

$$
\begin{aligned}
|p_i'(j_2)-p_i'(j_1)| \le \Big| &\int_0^{j_2}\frac{(j_2-\varrho)^{\sigma_i-2}}{\Gamma(\sigma_i-1)}y_i'(\varrho)\,\mathrm{d}\varrho - \int_0^1\frac{(1-\varrho)^{\sigma_i-2}}{\Gamma(\sigma_i-1)}y_i'(\varrho)\,\mathrm{d}\varrho \\
&+ \frac{b}{2(j_2\pi)^{1/2}} + \frac{b\epsilon\pi}{(j_2\pi)^{3/2}} \\
&+ \frac{b}{2\Gamma(\sigma+1)}\big((\sigma_i+2)j^{\sigma_i+1}-2\epsilon^2 j\big) \\
&- \Big[\int_0^{j_1}\frac{(j_1-\varrho)^{\sigma_i-2}}{\Gamma(\sigma_i-1)}y_i'(\varrho)\,\mathrm{d}\varrho - \int_0^1\frac{(1-\varrho)^{\sigma_i-2}}{\Gamma(\sigma_i-1)}y_i'(\varrho)\,\mathrm{d}\varrho \\
&+ \frac{b}{2(j_1\pi)^{1/2}} + \frac{b\epsilon\pi}{(j_1\pi)^{3/2}} \\
&+ \frac{b}{2\Gamma(\sigma_i+1)}\big((\sigma_i+2){j_1}^{\sigma_i+1}-2\epsilon^2 j_1\big)\Big]\Big| \\
\le\ & \tau(\|v_1,v_2,\ldots,v_i\|)\,\Omega_{i_2}\|b_i'\|_{\mathcal{L}^1},
\end{aligned}
$$

Indeed,

$$
\begin{aligned}
&\lim_{j_2\to j_1}|p_1(j_2)-p_1(j_1)|,\ldots,|p_k(j_2)-p_k(j_1)| = \mathbf{0}, \\
&\lim_{j_2\to j_1}|p_1'(j_2)-p_1'(j_1)|,\ldots,|p_k'(j_2)-p_k'(j_1)| = \mathbf{0}.
\end{aligned}
$$

According to Theorem Arzelà-Ascoli for every bounded subset $\mathcal{A}_\rho$ of $\mathcal{W}$, $\mathcal{T}(\mathcal{A}_\rho)$ is relatively compact, i.e L is completely continuous. We will prove further L has a closed graph. Suppose $(p_1^n,\ldots,p_k^n)\in\mathcal{W}$ and $(y_1^n,\ldots,y_k^n)\in L(p_1^0,\ldots,p_k^0)$ with $(p_1^n,\ldots,p_k^n)\to(p_1^0,\ldots,p_k^0)$ and $(y_1^n,\ldots,y_k^n)\to(y_1^0,\ldots,y_k^0)$. We will prove $(y_1^0,\ldots,y_k^0)\in L(p_1^0,\ldots,p_k^0)$.

For each $n \in \mathbb{N}$, choose $(u_1^n, \dots, u_k^n) \in \mathcal{W}_{B_{1,v}} \times \cdots \times \mathcal{W}_{B_{k,v}}$ such that

$$\begin{aligned} y_i^n(j) = & \int_0^j \frac{(j-t)^{\sigma_i - 1}}{\Gamma(\sigma_i)} v_i^n(t)\,\mathrm{d}t + (\epsilon - j) \int_0^1 \frac{(1-t)^{\sigma_i - 2}}{\Gamma(\sigma_i - 1)} v_i^n(t)\,\mathrm{d}t \\ & - \int_0^\epsilon \frac{(\epsilon - t)^{\sigma_i - 1}}{\Gamma(\sigma_i)} v_i^n(t)\,\mathrm{d}t + \frac{\eta^2}{2} \\ & + \frac{b(j-\epsilon)}{2(j\pi)^{1/2}\Gamma(\sigma_i+1)} \left[2\Gamma(\sigma_i+1) + j^{\sigma_i}(j\pi)^{1/2}(j+\epsilon)\right]. \end{aligned} \tag{21}$$

Define the continuous linear operator $\kappa_i : \mathcal{L}^1(\mathfrak{J}) \to \mathcal{W}_i$ by

$$\begin{aligned} (\kappa_i v)(j) = & \int_0^j \frac{(j-\varrho)^{\sigma_i - 1}}{\Gamma(\sigma_i)} v(\varrho)\,\mathrm{d}\varrho + (\epsilon - j) \int_0^1 \frac{(1-\varrho)^{\sigma_i - 2}}{\Gamma(\sigma_i - 1)} v(\varrho)\,\mathrm{d}\varrho \\ & - \int_0^\epsilon \frac{(\epsilon - \varrho)^{\sigma_i - 1}}{\Gamma(\sigma_i)} v(\varrho)\,\mathrm{d}\varrho + \frac{\eta^2}{2} \\ & + \frac{b(j-\epsilon)}{2(j\pi)^{1/2}\Gamma(\sigma_i+1)} \left[2\Gamma(\sigma_i+1) + j^{\sigma_i}(j\pi)^{1/2}(j+\epsilon)\right]. \end{aligned} \tag{22}$$

Theorem 2.1 implies that $\kappa_i \circ \mathcal{W}_{B_{i,v}}$ has a closed graph. Because $y_i^n \in \kappa_i(\mathcal{W}_{B_{i,(p_1,\dots,p_k)}})$, for each n, $1 \le i \le k$, and $(p_1^n, \dots, p_k^n) \to (p_1^0, \dots, p_k^0)$, there exists $u_i^0 \in \mathcal{W}_{B_{i,(n_1,\dots,n_k)}}$ such that

$$\begin{aligned} y_1^0(j) = & \int_0^j \frac{(j-\varrho)^{\sigma_i - 1}}{\Gamma(\sigma_i)} v_i(\varrho)\,\mathrm{d}\varrho + (\epsilon - j) \int_0^1 \frac{(1-\varrho)^{\sigma_i - 2}}{\Gamma(\sigma_i - 1)} v_i(\varrho)\,\mathrm{d}t \\ & - \int_0^\epsilon \frac{(\epsilon - \varrho)^{\sigma_i - 1}}{\Gamma(\sigma_i)} v_i(\varrho)\,\mathrm{d}\varrho + \frac{\eta^2}{2} \\ & + \frac{b(j-\epsilon)}{2(j\pi)^{1/2}\Gamma(\sigma_i+1)} \left[2\Gamma(\sigma_i+1) + j^{\sigma_i}(j\pi)^{1/2}(j+\epsilon)\right]. \end{aligned} \tag{23}$$

Hence, $y_i^0 \in L(p_1^0, \dots, p_k^0)$. This means that L_i has a closed graph $\forall\, 1 \le i \le k$ and so L has a closed graph. Now, we consider the number $\lambda \in (0,1)$ so that $(p_1, \dots, p_n) \in \lambda L(p_1, \dots, p_n)$. Then there exists

$(p_1, \ldots, p_n) \in \mathcal{W}_{B_{1,(p_1,\ldots,p_k)}} \times \cdots \times \mathcal{W}_{B_{k,(p_1,\ldots,p_k)}}$, with

$$\begin{aligned} p_i(j) = \lambda \int_0^j \frac{(j-\varrho)^{\sigma_i-1}}{\Gamma(\sigma_i)} v_i(\varrho)\,\mathrm{d}\varrho + \lambda(\epsilon - j) \int_0^1 \frac{(1-j)^{\sigma_i-2}}{\Gamma(\sigma_i-1)} v_i(\varrho)\,\mathrm{d}\varrho \\ - \lambda \int_0^\epsilon \frac{(\epsilon-\varrho)^{\sigma_i-1}}{\Gamma(\sigma_i)} v_i(\varrho)\,\mathrm{d}\varrho + \lambda \frac{\eta^2}{2} \\ + \frac{\lambda b(j-\epsilon)}{2(j\pi)^{1/2}\Gamma(\sigma_i+1)} \left[2\Gamma(\sigma_i+1) + j^{\sigma_i}(j\pi)^{1/2}(j+\epsilon) \right], \end{aligned} \tag{24}$$

for $1 \leq i \leq k$. Since $\frac{1}{\Omega_1^i + \Omega_2^i} \|p_i\| \, \|b_i\| \leq 1$, $\|p_i\|_i \leq E_i$ for all $i = 1, 2, \ldots, k$. We put the set T like this,

$$\mathrm{T} = \left\{ (p_1, \ldots, p_k) \in \mathcal{W} \,:\, \|(p_1, \ldots, p_k)\| \leq 1 + \sum_{i=1}^k E_i \right\}.$$

So, there are no $(p_1, \ldots, p_k) \in \partial\mathrm{T}$ and $\lambda \in (0,1)$ such that $(q_1, \ldots, q_k) \in \lambda\, L(p_1, \ldots, p_k)$. Also, the operator $L : \overline{\mathrm{T}} \to \mathbb{P}_{cmp,cvx}(\overline{\mathrm{T}})$ is upper semi-continuous because it is completely continuous and has closed graph. By using definition of L, there is no $(p_1, \ldots, p_k) \in \partial\mathrm{T}$ with $(l_1, \ldots, l_k) \in \lambda L(p_1, \ldots, p_k)$ for some $\lambda \in (0, 1)$. Thus, Theorem 2.2 confirm that L has a fixed point in $\overline{T}$ which is a solution of the k-dimensional FDI system (1)-(2). $\square$

4 Application with Illustrative Examples

In the following, we state a system for our results and show that it has a solution according to Theorem 3.2. In the first example, all parts of Theorem 3.2 were examined along with its proof.

Example 4.1. Consider a two-dimensional sequential FDI system

$$\begin{cases} {}^{c}D_{0}^{11/6}\left[\Phi_a\left({}^{c}D_{0}^{\sigma_1}u_1(j)\right)\right] + \frac{\sqrt{21}\,j\left(3.5+\tan^{-1}(u_1(j))\right)}{15(\sqrt{7}+e^j)} \\ \quad \in \left[0,\ \frac{1.5\,e^j(j+\ln 18)}{14(|j|+2)}\left(\frac{v_1'(j)|\cos v_2'(j)|(\sin v_1(j)-\sin v_2(j))}{2+v_1'(j)|\cos v_2'(j)|}\right)\right], \\ {}^{c}D_{0}^{4/3}\left[\Phi_a\left({}^{c}D_{0}^{12/5}u_2(j)\right)\right] - \frac{\sqrt[3]{2}\,j(2.75+\sin(u_2(j)))}{10(2+|j|)} \\ \quad \in \left[0,\ \frac{\sqrt{5}\,(j+\ln 4)}{12(j^2+6)}\left(\frac{v_1'(j)\exp(|v_2'(j)|)\left(\tan^{-1}v_1(j)-\tan^{-1}v_2(j)\right)}{3\left(v_1'(j)\exp(|v_2'(j)|)+7\right)}\right)\right], \end{cases} \tag{25}$$

for $j \in \mathfrak{J} = [0,1]$ with three different cases

$$\sigma_1 = \left\{\tfrac{27}{10},\ \tfrac{14}{5},\ \tfrac{29}{10}\right\} \subset (2,3),$$

under integral and fractional derivative boundary conditions

$$\begin{cases} \Phi_a\left({}^{c}D_{0}^{\sigma_1}u_1(1)\right) = \Phi_a\left({}^{c}D_{0}^{12/5}u_2(1)\right) = 0, \\ \Phi_a\left({}^{c}D_{0}^{\sigma_1}\left(\tfrac{3}{7}\right)\right) = \Phi_a\left({}^{c}D_{0}^{12/5}\left(\tfrac{3}{7}\right)\right) = D^{1/2}\sqrt{14}, \\ u_1''(0) = {}^{R}I^{\sigma_1}\sqrt{14}, \quad u_1'(1) = {}^{R}I^{\sigma_1}\sqrt{14} + D^{1/2}\sqrt{14}, \\ u_2''(0) = {}^{R}I^{12/5}\sqrt{14}, \quad u_2'(1) = {}^{R}I^{12/5}\sqrt{14} + D^{1/2}\sqrt{14}, \\ u_1\left(\tfrac{\ln 21}{9}\right) = \int_0^{3/7} t\,\mathrm{d}t, \qquad u_2\left(\tfrac{\ln 21}{9}\right) = \int_0^{3/7} t\,\mathrm{d}t. \end{cases} \tag{26}$$

Clearly, $\zeta_1 = \frac{11}{6} \in (1,2)$, $\zeta_2 = \frac{4}{3} \in (1,2)$, $\sigma_2 = \frac{12}{5} \in (2,3)$, $b = \sqrt{14}$, $\epsilon = \frac{\ln 21}{9}$, $\eta = \frac{3}{7}$, $\nu = \frac{3}{7}$. We define continuous functions

$$g_1(j, u_1(j)) = \frac{\sqrt{21}\,j\left(3.5+\tan^{-1}(u_1(j))\right)}{15(\sqrt{7}+e^j)},$$
$$g_2(j, u_2(j)) = -\frac{\sqrt[3]{2}\,j(2.75+\sin(u_2(j)))}{10(2+|j|)},$$

and set-valued map

$$B_1\left(j, v_1(j), v_2(j), v_1'(j), v_2'(j)\right) = \left[0,\ \frac{1.5\,e^j(j+\ln 18)}{14(|j|+2)}\left(\frac{v_1'(j)|\cos v_2'(j)|(\sin v_1(j)-\sin v_2(j))}{2+v_1'(j)|\cos v_2'(j)|}\right)\right],$$
$$B_2\left(j, v_1(j), v_2(j), v_1'(j), v_2'(j)\right) = \left[0,\ \frac{\sqrt{5}\,(j+\ln 4)}{12(j^2+6)}\left(\frac{v_1'(j)\exp(|v_2'(j)|)\left(\tan^{-1}v_1(j)-\tan^{-1}v_2(j)\right)}{3\left(v_1'(j)\exp(|v_2'(j)|)+7\right)}\right)\right].$$

Thus, we have

$$\begin{aligned}
&\Big|B_1\left(j, v_1(j), v_2(j), v_1'(j), v_2'(j)\right)\Big| \\
&\quad= \left|\tfrac{1.5\,e^j\,(j+\ln 18)}{14(|j|+2)}\left(\tfrac{v_1'(j)|\cos v_2'(j)|(\sin v_1(j)-\sin v_2(j))}{2+v_1'(j)|\cos v_2'(j)|}\right)\right| \\
&\quad= \left|\tfrac{1.5\,e^j\,(j+\ln 18)}{14(|j|+2)}\right|\,\left|\tfrac{v_1'(j)|\cos v_2'(j)|(\sin v_1(j)-\sin v_2(j))}{2+v_1'(j)|\cos v_2'(j)|}\right| \\
&\quad\le \tfrac{1.5\,e^j\,(j+\ln 18)}{28}\,\big|\sin v_1(j)-\sin v_2(j)\big| \\
&\quad\le \tfrac{1.5\,e^j\,(j+\ln 18)}{28}\,\big|v_1(j)-v_2(j)\big| \\
&\quad\le \tfrac{1.5\,e^j\,(j+\ln 18)}{28}\,(|v_1(j)|+|v_2(j)|),
\end{aligned}$$

and

$$\begin{aligned}
&\Big|B_2\left(j, v_1(j), v_2(j), v_1'(j), v_2'(j)\right)\Big| \\
&\quad= \left|\tfrac{\sqrt{5}\,(j+\ln 4)}{12(j^2+6)}\left(\tfrac{v_1'(j)\exp(|v_2'(j)|)\left(\tan^{-1} v_1(j)-\tan^{-1} v_2(j)\right)}{3\left(v_1'(j)\exp(|v_2'(j)|)+7\right)}\right)\right| \\
&\quad= \left|\tfrac{\sqrt{5}\,(j+\ln 4)}{12(j^2+6)}\right|\,\left|\tfrac{v_1'(j)\exp(|v_2'(j)|)\left(\tan^{-1} v_1(j)-\tan^{-1} v_2(j)\right)}{3\left(v_1'(j)\exp(|v_2'(j)|)+7\right)}\right| \\
&\quad\le \tfrac{\sqrt{5}\,(j+\ln 4)}{216}\,\big|\tan^{-1} v_1(j)-\tan^{-1} v_2(j)\big| \\
&\quad\le \tfrac{\sqrt{5}\,(j+\ln 4)}{216}\,\big|v_1(j)-v_2(j)\big| \\
&\quad\le \tfrac{\sqrt{5}\,(j+\ln 4)}{216}\,(|v_1(j)|+|v_2(j)|).
\end{aligned}$$

Hence,

$$\begin{aligned}
\Big\|B_1\left(j, v_1(j), v_2(j), v_1'(j), v_2'(j)\right)\Big\| &\le b_1(j)\,\tau(j), \\
\Big\|B_2\left(j, v_1(j), v_2(j), v_1'(j), v_2'(j)\right)\Big\| &\le b_2(j)\,\tau(j),
\end{aligned}$$

where

$$b_1(j) = \tfrac{1.5\,e^j\,(j+\ln 18)}{28}, \qquad b_2(j) = \tfrac{\sqrt{5}\,(j+\ln 4)}{216},$$

and according to Eq. (7), $\tau(j) = j$. Hence, by using the given data and relations (13), we obtain

$$\begin{aligned}
\Omega_{1_1} &= \int_0^1 \left| \left[(1-\varrho)^{\sigma_1 - 1} + (\sigma_1 - 1)\left(\tfrac{\ln 21}{9} - 1\right)(1-\varrho)^{\sigma_1 - 2} \right] \tfrac{u_i(\epsilon)}{\Gamma(\sigma_1)} \right| \mathrm{d}\varrho \\
&\quad + \int_0^{\ln 21/9} \frac{\left(\frac{\ln 21}{9} - \varrho\right)^{\sigma_1 - 1}}{\Gamma(\sigma_1)} |u_i(\epsilon)| \, \mathrm{d}\varrho + \tfrac{9}{98} \\
&\quad + \left| \frac{\sqrt{14}\left(1 - \frac{\ln 21}{9}\right)}{2\pi^{1/2}\Gamma(\sigma_1 + 1)} \left[2\sqrt{14}\Gamma(\sigma_1 + 1) + \pi^{1/2}\left(1 - \tfrac{\ln 21}{9}\right) \right] \right| \\
&\simeq \begin{cases} 5.7165, & \sigma_1 = \frac{27}{10}, \\ 5.6707, & \sigma_1 = \frac{14}{5}, \\ 5.6293, & \sigma_1 = \frac{29}{10}, \end{cases} \\
\Omega_{1_2} &= \sqrt{14}\left(\frac{1 + \frac{2\ln 21}{9}}{2\pi^{1/2}} \right) + \frac{\sqrt{14}\left(\sigma_1 + 2 - 2\left(\frac{\ln 21}{9}\right)^2\right)}{2\Gamma(\sigma_1 + 1)} \\
&\simeq \begin{cases} 3.7752, & \sigma_1 = \frac{27}{10}, \\ 3.5914, & \sigma_1 = \frac{14}{5}, \\ 3.4186, & \sigma_1 = \frac{29}{10}, \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\Omega_{2_1} &= \int_0^1 \left| \left[(1-\varrho)^{7/5} + \tfrac{7}{5}\left(\tfrac{\ln 21}{9} - 1\right)(1-\varrho)^{2/5} \right] \frac{u_i(\epsilon)}{\Gamma\left(\frac{12}{5}\right)} \right| \mathrm{d}\varrho \\
&\quad + \int_0^{\ln 21/9} \frac{\left(\frac{\ln 21}{9} - \varrho\right)^{7/5}}{\Gamma(12/5)} |u_i(\epsilon)| \, \mathrm{d}\varrho + \tfrac{9}{98} \\
&\quad + \left| \frac{\sqrt{14}\left(1 - \frac{\ln 21}{9}\right)}{2\pi^{1/2}\Gamma\left(\frac{17}{5}\right)} \left[2\sqrt{14}\Gamma\left(\tfrac{17}{5}\right) + \pi^{1/2}\left(1 - \tfrac{\ln 21}{9}\right) \right] \right| \simeq 5.8821, \\
\Omega_{2_2} &= \sqrt{14}\left(\frac{1 + \frac{2\ln 21}{9}}{2\pi^{1/2}} \right) + \frac{\sqrt{14}\left(\frac{22}{5}\right) - 2\left(\frac{\ln 21}{9}\right)^2}{2\Gamma\left(\frac{17}{5}\right)} \simeq 4.3871,
\end{aligned}$$

for three cases $\sigma_1 = \frac{27}{10}, \frac{14}{5}, \frac{29}{10}$. One can see the calculated data in Table 1 for different cases of the order σ_1 on $j \in \mathcal{J}$. It can be seen that as the order of derivative σ_1 increases towards the number three, parameter Ω_{1_1} decreases and parameter Ω_{1_2} increases. The curves drawn in Figs. 1a and 1b show these changes well. Note that the two parameters Ω_{2_1} and Ω_{2_2}, because those do not depend on the order of the derivative σ_1,

Table 1: Numerical values of Ω_{1_1}, Ω_{1_1} and Ω_{2_1}, Ω_{2_2} with three cases of σ_1 for system (25) in Example 4.1.

j	Ω_{1_1}	Ω_{1_2}	Ineq. (27)	Ω_{1_1}	Ω_{1_2}	Ineq. (28)
			$\boldsymbol{\sigma_1 = \frac{27}{10}}$			
0.00	5.5161	3.7752	0.5056	5.5956	4.3872	0.0222
0.10	5.5690	3.7752	0.5059	5.6638	4.3872	0.0222
0.20	5.6122	3.7752	0.5062	5.7216	4.3872	0.0223
0.30	5.6465	3.7752	0.5064	5.7693	4.3872	0.0223
0.40	5.6727	3.7752	0.5066	5.8076	4.3872	0.0223
0.50	5.6918	3.7752	0.5067	5.8371	4.3872	0.0223
0.60	5.7046	3.7752	0.5068	5.8585	4.3872	0.0223
0.70	5.7123	3.7752	0.5068	5.8725	4.3872	0.0223
0.80	5.7160	3.7752	0.5068	5.8802	4.3872	0.0223
0.90	5.7169	3.7752	0.5068	5.8827	4.3872	0.0223
1.00	5.7166	3.7752	0.5068	5.8821	4.3872	0.0223
			$\boldsymbol{\sigma_1 = \frac{14}{5}}$			
0.00	5.4940	3.5914	0.5042	5.5956	4.3872	0.0222
0.10	5.5421	3.5914	0.5045	5.6638	4.3872	0.0222
0.20	5.5810	3.5914	0.5048	5.7216	4.3872	0.0223
0.30	5.6114	3.5914	0.5050	5.7693	4.3872	0.0223
0.40	5.6343	3.5914	0.5051	5.8076	4.3872	0.0223
0.50	5.6506	3.5914	0.5052	5.8371	4.3872	0.0223
0.60	5.6614	3.5914	0.5053	5.8585	4.3872	0.0223
0.70	5.6676	3.5914	0.5053	5.8725	4.3872	0.0223
0.80	5.6704	3.5914	0.5054	5.8802	4.3872	0.0223
0.90	5.6710	3.5914	0.5054	5.8827	4.3872	0.0223
1.00	5.6708	3.5914	0.5054	5.8821	4.3872	0.0223
			$\boldsymbol{\sigma_1 = \frac{29}{10}}$			
0.00	5.4739	3.4187	0.5028	5.5956	4.3872	0.0222
0.10	5.5175	3.4187	0.5031	5.6638	4.3872	0.0222
0.20	5.5523	3.4187	0.5034	5.7216	4.3872	0.0223
0.30	5.5791	3.4187	0.5036	5.7693	4.3872	0.0223
0.40	5.5990	3.4187	0.5037	5.8076	4.3872	0.0223
0.50	5.6129	3.4187	0.5038	5.8371	4.3872	0.0223
0.60	5.6219	3.4187	0.5039	5.8585	4.3872	0.0223
0.70	5.6269	3.4187	0.5039	5.8725	4.3872	0.0223
0.80	5.6291	3.4187	0.5039	5.8802	4.3872	0.0223
0.90	5.6295	3.4187	0.5039	5.8827	4.3872	0.0223
1.00	5.6293	3.4187	0.5039	5.8821	4.3872	0.0223

remain constant. Furthermore, by definition of functions b_i, g_i, $i = 1, 2$, we got

$$g_1^* = \sup_{j \in \mathfrak{J}} g_1(j, 0) = \frac{3.5\sqrt{21}}{15(\sqrt{7}+1)}, \qquad g_2^* = \sup_{j \in \mathfrak{J}} g_2(j, 0) = -\frac{2.75\sqrt[3]{2}}{20},$$

$$\|b_1\| = \sup_{j\in\mathfrak{J}} |b_1(j)| = \tfrac{1.5e(1+\ln 18)}{28} < 1,$$

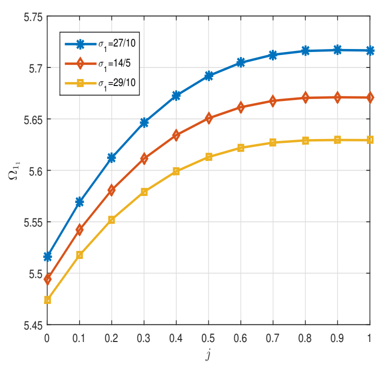
$$\|b_2\| = \sup_{j\in\mathfrak{J}} |b_2(j)| = \tfrac{\sqrt{5}(1+\ln 4)}{216} < 1.$$

Thus, by choosing $E_1 \leq 8.4917,\ 8.2621,\ 8.0479$ for $\sigma_1 = \frac{27}{10},\ \frac{14}{5},\ \frac{29}{10}$ respectively, and $E_2 \leq 9.2692$, we have

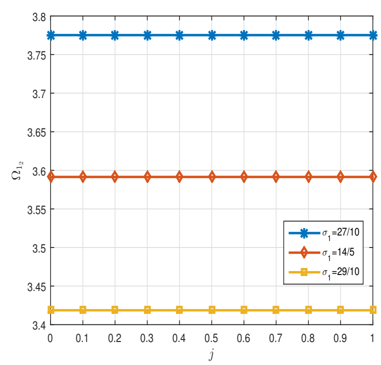
$$\tfrac{1}{\Omega_{i1}+\Omega_{i2}} \|p_i\| \, \|b_i\| \simeq \left\{ \begin{array}{ll} 0.5067, & \sigma_1 = \frac{27}{10}, \\ 0.5054, & \sigma_1 = \frac{14}{5}, \\ 0.5039, & \sigma_1 = \frac{29}{10}, \end{array} \right\} \leq 1, \tag{27}$$

$$\tfrac{1}{\Omega_{21}+\Omega_{22}} \|p_i\| \, \|b_i\| \simeq 0.0223 \leq 1. \tag{28}$$

We plot the numerical results of inequality (27) in Fig. 2 for three cases of derivative order σ_1, which nicely shows that as σ_1 increases towards three, the inequality decreases, but is still smaller than one. By using



(a) Ω_{1_1}



(b) Ω_{1_2}

Figure 1: 2D plots of Ω_{1_1} and Ω_{1_2} with three cases of σ_1 when $j \in \mathcal{J}$ for system (25) in Example 4.1.

the theorem 3.2, the 2-dimensional FDI system (25) with the condition (26) has at least one solution.

In the next example, we show that our results are correct with different cases of the order σ_2 in System (1)-(2).

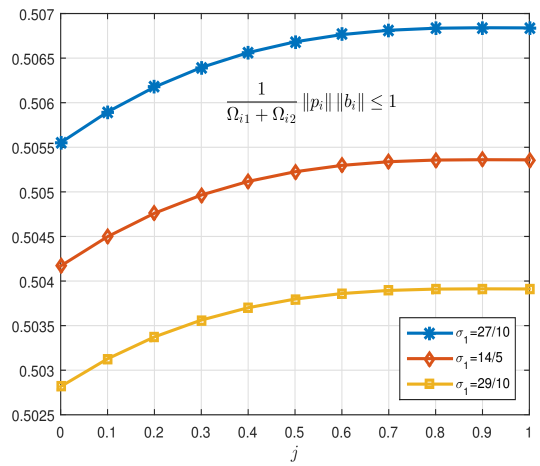


Figure 2: 2D plots of Ineq. (27) with three cases of σ_1 when $j \in \mathcal{J}$ for system (25) in Example 4.1.

Example 4.2. We consider the same two-dimensional sequential FDI system (25) in Example 4.1 as form

$$\begin{cases} {}^{c}D_{0}^{11/6}\left[\Phi_a\left({}^{c}D_{0}^{8/3}u_1(j)\right)\right] + \frac{\sqrt{21}\,j\big(3.5+\tan^{-1}(u_1(j))\big)}{15(\sqrt{7}+e^{j})} \\ \quad \in \left[0,\ \frac{1.5\,e^{j}(j+\ln 18)}{14(|j|+2)}\left(\frac{v_1'(j)|\cos v_2'(j)|(\sin v_1(j)-\sin v_2(j))}{2+v_1'(j)|\cos v_2'(j)|}\right)\right], \\ {}^{c}D_{0}^{4/3}\left[\Phi_a\left({}^{c}D_{0}^{\sigma_2}u_2(j)\right)\right] - \frac{\sqrt[3]{2}\,j(2.75+\sin(u_2(j)))}{10(2+|j|)} \\ \quad \in \left[0,\ \frac{\sqrt{5}\,(j+\ln 4)}{12(j^2+6)}\left(\frac{v_1'(j)\exp(|v_2'(j)|)\big(\tan^{-1}v_1(j)-\tan^{-1}v_2(j)\big)}{3\big(v_1'(j)\exp(|v_2'(j)|)+7\big)}\right)\right], \end{cases} \tag{29}$$

for $j \in \mathfrak{J} = [0,1]$ with three different cases

$$\sigma_2 = \left\{\tfrac{11}{5},\ \tfrac{12}{5},\ \tfrac{13}{5}\right\} \subset (2,3),$$

under integral and fractional derivative boundary conditions

$$\begin{cases} \Phi_a\left({}^cD_0^{8/3}u_1(1)\right) = \Phi_a\left({}^cD_0^{12/5}u_2(1)\right) = 0, \\ \Phi_a\left({}^cD_0^{8/3}\left(\frac{3}{7}\right)\right) = \Phi_a\left({}^cD_0^{\sigma_2}\left(\frac{3}{7}\right)\right) = D^{1/2}\sqrt{14}, \\ u_1''(0) = {}^RI^{8/3}\sqrt{14}, \quad u_1'(1) = {}^RI^{8/3}\sqrt{14} + D^{1/2}\sqrt{14}, \\ u_2''(0) = {}^RI^{\sigma_2}\sqrt{14}, \quad u_2'(1) = {}^RI^{\sigma_2}\sqrt{14} + D^{1/2}\sqrt{14}, \\ u_1\left(\frac{\ln 21}{9}\right) = \int_0^{3/7} t\,\mathrm{d}t, \quad u_2\left(\frac{\ln 21}{9}\right) = \int_0^{3/7} t\,\mathrm{d}t. \end{cases} \tag{30}$$

Clearly, $\zeta_1 = \frac{11}{6} \in (1,2)$, $\zeta_2 = \frac{4}{3} \in (1,2)$, $\sigma_1 = \frac{8}{3} \in (2,3)$, $b = \sqrt{14}$, $\epsilon = \frac{\ln 21}{9}$, $\eta = \frac{3}{7}$, $\nu = \frac{3}{7}$. Also, we consider the functions g_i and set-valued maps $B_i\left(j, v_1(j), v_2(j), v_1'(j), v_2'(j)\right)$, $i = 1,2$ in Example 4.1. We have show that

$$\left|B_1\left(j, v_1(j), v_2(j), v_1'(j), v_2'(j)\right)\right| \leq \frac{1.5\,e^j\,(j+\ln 18)}{28}\left(|v_1(j)| + |v_2(j)|\right),$$
$$\left|B_2\left(j, v_1(j), v_2(j), v_1'(j), v_2'(j)\right)\right| \leq \frac{\sqrt{5}\,(j+\ln 4)}{216}\left(|v_1(j)| + |v_2(j)|\right),$$

ans so

$$b_1(j) = \frac{1.5\,e^j\,(j+\ln 18)}{28}, \qquad b_2(j) = \frac{\sqrt{5}\,(j+\ln 4)}{216},$$

and according to Eq. (7), $\tau(j) = j$. Now, by using the given data and relations (13), we have $\Omega_{1_1} \simeq 5.7328$, $\Omega_{1_2} \simeq 3.8388$ and

$$\Omega_{2_1} \simeq \begin{cases} 6.0173, & \sigma_2 = \frac{11}{5}, \\ 5.8821, & \sigma_2 = \frac{12}{5}, \\ 5.7669, & \sigma_2 = \frac{13}{5}, \end{cases} \qquad \Omega_{2_2} \simeq \begin{cases} 4.8345, & \sigma_2 = \frac{11}{5}, \\ 4.3871, & \sigma_2 = \frac{12}{5}, \\ 3.9696, & \sigma_2 = \frac{13}{5}. \end{cases}$$

One can see the data in Table 1 for different cases of the order σ_2 on $j \in \mathcal{J}$. It can be seen that as the order of derivative σ_2 increases towards the number three, parameters Ω_{1_1} and Ω_{1_2} decrease. The curves drawn in Figs. 3a and 3b show these changes well. The values b_i, g_i, $i = 1,2$, were calculated earlier in Example 4.1. Thus, by choosing $E_1 \leq 8.5717$, and $E_2 \leq 9.8519$, 9.2692, 8.7365 for $\sigma_2 = \frac{11}{5}$, $\frac{12}{5}$, $\frac{13}{5}$ respectively, we have

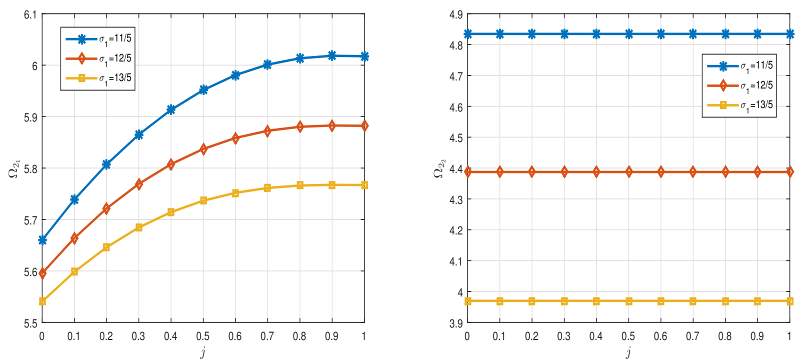
$$\frac{1}{\Omega_{i1} + \Omega_{i2}}\,\|p_i\|\,\|b_i\| \simeq 0.5073 \leq 1, \tag{31}$$

Table 2: Numerical values of Ω_{1_1}, Ω_{1_1} and Ω_{2_1}, Ω_{2_2} with three cases of σ_2 for system (29) in Example 4.2.

j	Ω_{1_1}	Ω_{1_2}	Ineq. (27)	Ω_{1_1}	Ω_{1_2}	Ineq. (28)
			$\boldsymbol{\sigma_2 = \frac{11}{5}}$			
0.00	5.5240	3.8389	0.5060	5.6600	4.8346	0.0223
0.10	5.5785	3.8389	0.5064	5.7388	4.8346	0.0224
0.20	5.6232	3.8389	0.5067	5.8071	4.8346	0.0224
0.30	5.6588	3.8389	0.5069	5.8651	4.8346	0.0224
0.40	5.6862	3.8389	0.5070	5.9132	4.8346	0.0224
0.50	5.7063	3.8389	0.5072	5.9516	4.8346	0.0224
0.60	5.7199	3.8389	0.5073	5.9808	4.8346	0.0224
0.70	5.7282	3.8389	0.5073	6.0012	4.8346	0.0224
0.80	5.7322	3.8389	0.5073	6.0134	4.8346	0.0224
0.90	5.7332	3.8389	0.5073	6.0182	4.8346	0.0224
1.00	5.7328	3.8389	0.5073	6.0174	4.8346	0.0224
			$\boldsymbol{\sigma_2 = \frac{12}{5}}$			
0.00	5.5240	3.8389	0.5060	5.5956	4.3872	0.0222
0.10	5.5785	3.8389	0.5064	5.6638	4.3872	0.0222
0.20	5.6232	3.8389	0.5067	5.7216	4.3872	0.0223
0.30	5.6588	3.8389	0.5069	5.7693	4.3872	0.0223
0.40	5.6862	3.8389	0.5070	5.8076	4.3872	0.0223
0.50	5.7063	3.8389	0.5072	5.8371	4.3872	0.0223
0.60	5.7199	3.8389	0.5073	5.8585	4.3872	0.0223
0.70	5.7282	3.8389	0.5073	5.8725	4.3872	0.0223
0.80	5.7322	3.8389	0.5073	5.8802	4.3872	0.0223
0.90	5.7332	3.8389	0.5073	5.8827	4.3872	0.0223
1.00	5.7328	3.8389	0.5073	5.8821	4.3872	0.0223
			$\boldsymbol{\sigma_2 = \frac{13}{5}}$			
0.00	5.5240	3.8389	0.5060	5.5404	3.9697	0.0221
0.10	5.5785	3.8389	0.5064	5.5983	3.9697	0.0221
0.20	5.6232	3.8389	0.5067	5.6461	3.9697	0.0221
0.30	5.6588	3.8389	0.5069	5.6845	3.9697	0.0221
0.40	5.6862	3.8389	0.5070	5.7144	3.9697	0.0222
0.50	5.7063	3.8389	0.5072	5.7366	3.9697	0.0222
0.60	5.7199	3.8389	0.5073	5.7519	3.9697	0.0222
0.70	5.7282	3.8389	0.5073	5.7614	3.9697	0.0222
0.80	5.7322	3.8389	0.5073	5.7661	3.9697	0.0222
0.90	5.7332	3.8389	0.5073	5.7673	3.9697	0.0222
1.00	5.7328	3.8389	0.5073	5.7669	3.9697	0.0222

$$\frac{1}{\Omega_{21}+\Omega_{22}} \|p_i\| \|b_i\| \simeq \left\{ \begin{array}{ll} 0.0224, & \sigma_2 = \frac{11}{5}, \\ 0.0223, & \sigma_2 = \frac{12}{5}, \\ 0.0222, & \sigma_2 = \frac{13}{5}. \end{array} \right\} \leq 1. \tag{32}$$

We plot the numerical results of inequality (31) in Fig. 4 for three cases of derivative order σ_2, which nicely shows that as σ_2 increases towards three, the inequality decreases, but is still smaller than one.



(a) Ω_{2_1} **(b)** Ω_{2_2}

Figure 3: 2D plots of Ω_{2_1} and Ω_{2_2} with three cases of σ_2 when $j \in \mathcal{J}$ for system (29) in Example 4.2.

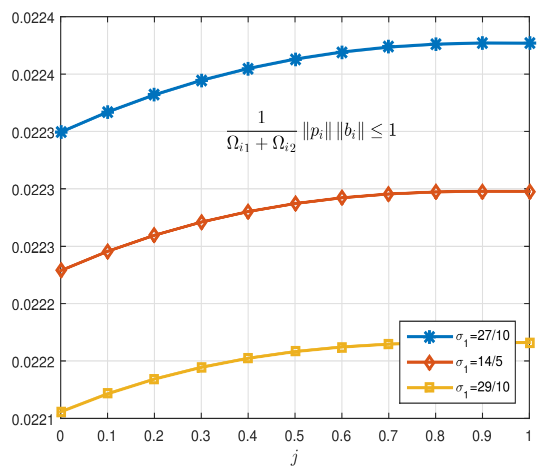


Figure 4: 2D plots of Ineq. (28) with three cases of σ_2 when $j \in \mathcal{J}$ for system (29) in Example 4.2.

Therefore, all conditions of theorem 3.2 hold. hence, the 2-dimensional FDI system (29)-(30) has at least one solution.

5 Conclusion

In today's world, solving fractional differential devices and checking the solution for these devices has become very important. We can check the successive derivatives of fractions with different integral conditions. In this article, we first found a solution for fractional differential inclusion. Then we checked that this solution could be a solution for the fractional differential inclusion device. In the end, we gave some illustrative examples to clarify our case.

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