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Original Research Paper

Using Lagrange Interpolation for Numerical Solution of Two-Dimensional Fredholm Integral Equations

Gh. R. Farahmand
Hamedan Branch, Islamic Azad University

N. Parandin*
Kermanshah Branch, Islamic Azad University

N. Karamikabir
Hamedan Branch, Islamic Azad University

Abstract. We present a numerical method for solving Fredholm two-dimensional integral equations in this study. Our approach is based on two-dimensional Lagrange interpolation polynomials. The use of interpolation is that instead of the unknown function, we use the Lagrange interpolator polynomial, and then by solving these linear equation system, we obtain the Lagrange coefficients, which are the second components of the support points, approximately. By putting these coefficients in the Lagrange finder function, we get an approximation to the exact answer. A numerical algorithm is described for this purpose, and two cases are solved using this algorithm. Furthermore, a theorem is proved to demonstrate the algorithm's convergence and to obtain an upper bound on the distance between the exact and numerical solutions.

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*Corresponding Author

1 Introduction

The numerical solution of two-dimensional integral equations, including a combination of unknown functions under the symbol of a two-dimensional integral as well as their dervatives, is the main goal of this study. It has been studied by many researchers and is used in many fields, such as mechanical engineering, physics, chemistry, and astronomy, [3, 6, 16, 21, 25]. A recent work on developing and analyzing numerical methods for solving Fredholm integral equations of the second kind is described in [2].

Other numerical methods based on the collocation method can be found in [2, 3, 4, 7, 8, 10, 16, 17], and we mention some of these methods below.

Alipanah and Esmaeili, did the Numerical solution of the two-dimensional Fredholm integral equations using Gaussian radial basis function [2]. Awadzadeh et al., comparison of two-dimensional integral solution The equations were done using the traditional collocation method and radial basis functions [4]. Golbabai and Seifollahi, integral equations of the second type using the radial basis Functional networks, were solved [8]. Mirzaee and Hadadiyan, numerically solved linear Fredholm integral equations via two-dimensional modification of hat functions [16]. Mirzaee and Piroozfar, numerically solved of the linear two-dimensional Fredholm integral equations of the second kind via two-dimensional triangular orthogonal functions [17]. Han and Wang [9] approximated the two-dimensional Fredholm integral equations by the Galerkin iterative method. Avazzadeh and Heydari, presented Chebyshev polynomials for solving two-dimensional linear and nonlinear integral equations of the second kind in [3]. In [25], Tohidi, using the Taylor matrix method, solved the linear two-dimensional Fredholm integral equations with piecewise intervals.

We describe in this study a numerical method based on Chebyshev polynomials for solving the Fredholm integral equation of the second type, which is as follows:

$$w(x,y) = f(x,y) + \beta \int_a^b \int_c^d k(x,y,t,s) w(t,s) ds dt, \quad (x,y) \in \sigma, \tag{1}$$

where $\beta > 0$, $f(x,y)$ and $k(x,y,t,s)$ are given continuous functions defined on $\sigma = [a, b] \times [c, d]$ and $w(x, y)$ is an unknown function on σ, which our goal in this research is to obtain this unknown function.

This project is divided into six sections. Definitions and theorems are presented in Section 2. The recommended method for solving the equation (1) is discussed in Section 3. Convergence analysis is proven in Section 4. In Section 5, the proposed method is applied to numerical examples using Mathematica codes. Finally, a conclusion is given in Section 6.

2 Preliminaries

In this section, we refer to the definitions and theorems that we need in the following sections.

Definition 2.1. (The Lagrange interpolation polynomial)
First, we consider the data points, $\{(t_0, f(t_0)), (t_1, f(t_1)), \cdots, (t_n, f(t_n))\}$ as above. For $i = 0, 1, \cdots, n$, we use the polynomials $L_i(t)$ given by the following formula [20]

$$L_i(t) = \prod_{\substack{j=0 \\ i \neq j}}^{n} \frac{t - t_i}{t_j - t_i},$$

and the interpolation polynomials calculated as

$$P_n(t) = \sum_{i=0}^{n} f(t_i) L_i(t).$$

The Lagrange polynomial $L_i(t)$ corresponding to the node t_j has the property

$$L_i(t_r) = \begin{cases} 1, & i = r, \\ 0, & \textit{otherwise}. \end{cases} \tag{2}$$

Definition 2.2. We difine the two dimensional Lagrange interpolation function about a continuous function $w(x, y), (x, y) \in [a, b] \times [c, d]$ as follows

$$w_{n,m}(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{m} L_{ij}(x, y) w(x_i, y_j), \tag{3}$$

where $L_{ij}(x, y)$ are the Lagrange polynomials.

Definition 2.3. Let $M_{1'} = (A_{1'}, d_{1'})$ and $M_{2'} = (A_{2'}, d_{2'})$ be metric spaces. Let $A_{1'} \times A_{2'}$ be the cartesian product of $A_{1'}$ and $A_{2'}$. The Euclidean metric on $A_{1'} \times A_{2'}$ is defined as:

$$d_2(x,y) = \left((d_{1'}(x_1,y_1))^2 + (d_{2'}(x_2,y_2))^2 \right)^{\frac{1}{2}}$$

where $x = (x_1, x_2)\,,\ y = (y_1, y_2) \in A_{1'} \times A_{2'}$.

Theorem 2.4. *(Existence and Uniqueness Theorem)[23].*
Let $t_0, t_1, \ldots, t_n$ be $n+1$ distinct points in $[a,b]$. There exists a unique polynomial $p_n(t)$ of degree$\leq n$ that interpolates $f(t)$ at the points $\{t_i\}$, such that $p_n(t_i) = f(t_i)$, for $i \leq i \leq n$.

This polynomial clearly has a degree of n or less and has the property that $P_n(t_i) = f(t_i)$ as required, and the Lagrange polynomial, $P_n(t)$, is unique. If there were two such polynomials, $P_n(t)$ and $\widehat{P}_n(t)$, then $P_n(t) - \widehat{P}_n(t)$ would be a polynomial with degree $\leq n$ and $n+1$ zeros. Thus, it would have to be equally zero. So, we need $P_n(t) \equiv \widehat{P}_n(t)$.

Theorem 2.5. *[23].*
Suppose that $t_0, t_1, \cdots, t_n$ are distinct numbers in the interval $[a,b]$ and $f \in C^{n+1}[a,b]$. Then, for each t in $[a,b]$, there exists a number $\zeta(t)$ (generally unknown) between $t_0, t_1, \cdots, t_n$ and hence in (a,b), so that

$$f(t) - p_n(t) = \frac{f^{(n+1)}(\zeta(t))}{(n+1)!}(t-t_0)(t-t_1)\cdots(t-t_n),$$

where $p_n(t)$ is the Lagrange interpolating polynomial.

Remark 2.6. Theorem 2.4 refers to only the polynomial that is made with this number of support points and Theorem 2.5 refers to the amount of polynomial error in point

$$t \neq t_i, \quad i = 0, 1, ..., n \ \textit{and} \ \ t \in (a,b)\,.$$

3 The Main Idea of this Research

In this part, we explain the numerical solution of two-dimensional Fredholm integral equations. To solve Eq.(1), we consider the Lagrange interpolation in the given points $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ and $c = x_0 < x_1 < x_2 < \cdots < x_m = d$, and by substituting (3) into (1), we approximate the two dimensional Fredholm integral equation as follows:

$$\sum_{i=0}^{n}\sum_{j=0}^{m} L_{ij}(x,y)w(x_i,y_j) = \tag{4}$$

$$f(x,y) + \beta \int_a^b \int_c^d k(x,y,t,s) \sum_{i=0}^{n}\sum_{j=0}^{m} L_{ij}(t,s)w(x_i,y_j)dtds.$$

Then taking the collocation points (x_i, y_j) into (4), and exchanging the integral and summation sign, we obtain

$$\sum_{r=0}^{n}\sum_{z=0}^{m} L_{ij}(x_r,y_z)w(x_i,y_j) =$$

$$f(x_r,y_z) + \beta \sum_{r=0}^{n}\sum_{z=0}^{m} k(x_i,y_j t_r,s_z)w(t_r,s_z)B_{rz}(x_i,y_j),$$

where

$$B_{rz}(x_i,y_j) = \int_a^{x_r}\int_c^{y_z} L_{rz}(t_i,s_j)dsdt$$

and $L_{ij}(t,s)$ are the Lagrange polynomials which are defined as follows:

$$L_{ij}(t,s) = L_i(t)L_j(s), \quad 0 \le i \le n, 0 \le j \le m,$$

where

$$L_i(t) = \prod_{\substack{r=0\\ r\neq i}}^{n} \frac{t - t_r}{t_i - t_r},$$

$$L_j(t) = \prod_{\substack{z=0\\ z\neq j}}^{m} \frac{s - s_z}{s_j - s_z},$$

and

$$L_{ij}(t_r,s_z) = \begin{cases} 1, & i = r, j = z, \\ 0, & \textit{otherwise}. \end{cases}$$

Given the known functions $f(x_i, y_j), B_{rz}(x_i, y_j)$ and $k(x_i, y_j, t_r, s_z)$ for each $r = 0, 1, 2, \cdots, n$ and $z = 0, 1, 2, \cdots, m$, we seek for the unknowns in the form of w_i in which the points $\big((x_i, y_j), w(x_i, y_j)\big)$ are the unknowns. A unique polynomial is generated, by the answer to equation Eq.(1) can be approximated as follows:

$$\Big[\sum_{\substack{r=0\\r\neq i}}^{n}\sum_{\substack{z=0\\z\neq j}}^{m} k(x_i, y_j, x_r, y_z)w(x_r, y_z)B_{rz}(x_i, y_j)\Big] + \big[k(x_i, y_j, x_i, y_j)B_{ij}(x_i, y_j) - 1\big] *w(x_i, y_j) = f(x_i, y_j). \tag{5}$$

Eq.(5) produces a linear equation system $[(n+1)(m+1)]\times[(n+1)(m+1)]$ and the same number is unknown $w(x_i, y_j)$. By solving this set of equations and setting unknown values $w(x_i, y_j)$, we can obtain approximate solution of the two-dimensional integral equation (1) as follows:

$$w_{n,m}(x, y) = \sum_{r=0}^{n}\sum_{z=0}^{m} L_{ij}(x, y)w(x_r, y_z),$$
$$i = 0, 1, 2, \cdots, n,\ j = 0, 1, 2, \cdots, m.$$

4 Convergence Analysis

The error estimate for the numerical method proposed in the previous part is obtained in this section.

Theorem 4.1. *Let $(C(\varrho), D)$ be the metric space of all continuous function on $\varrho = [0, 1) \times [0, 1)$ and let D be a bounded meter. Assume that for each $(x, y)$and $(t, s) \in \varrho, |k(x, y, t, s)| \leq h$. We define the error two-dimensional Lagrange interpolation function by $E_{n,m} = |w(x, y) - w_{n,m}(x, y)|$, where $w(x, y)$ and $w_{n,m}(x, y)$ show the exact and approximate solutions, respectively. The solution of the two-dimensional linear Fredholm integral equation of the second kind by using two-dimensional Lagrange interpolation approximation is convergent if for each $\beta > 0$*

$$h = \max_{0\leq x,y,t,s\leq 1} |k(x, y, t, s)| \leq \frac{1}{\beta}.$$

Proof. Suppose that $w_{n,m}(x,y)$ and $w(x,y)$ show the approximate and exact solution of Eq.(1), respectively. Then

$$\begin{aligned}
\left|w\left(x,y\right)-w_{n,m}\left(x,y\right)\right| &= \Big|f(x,y)+\beta\int_0^1\int_0^1 k\left(x,y,t,s\right)w(t,s)dtds- \\
&\qquad f(x,y)-\beta\int_0^1\int_0^1 k\left(x,y,t,s\right)w_{n,m}(t,s)dtds\Big| \\
&\le |\beta|\int_0^1\int_0^1 \left|k\left(x,y,t,s\right)\right|\left|w(t,s)-w_{n,m}(t,s)\right|dtds \\
&\le |\beta|\int_0^1\int_0^1 \max_{0\le x,y,t,s\le 1}\left|k\left(x,y,t,s\right)\right|\left|w(t,s)-w_{n,m}(t,s)\right|dtds \\
&= |\beta|\int_0^1\int_0^1 h\left|w(t,s)-w_{n,m}(t,s)\right|dtds \\
&= |\beta|\,h\int_0^1\int_0^1 \left|w(t,s)-w_{n,m}(t,s)\right|dtds \\
&\le |\beta|\,h\int_0^1\int_0^1 D\left(w(t,s),w_{n,m}(t,s)\right)dtds
\end{aligned}$$

So

$$\left|w(x,y)-w_{n,m}(x,y)\right|\le|\beta|\,hD\left(w(t,s),w_{n,m}(t,s)\right)$$

$$\Rightarrow\ \sup\ D\left(w(x,y),w_{n,m}(x,y)\right)\le|\beta|\,h\sup\ D\left(w(t,s),w_{n,m}(t,s)\right).$$

Suppose, $n=n_l$, $m=m_l$

Then

$$\begin{aligned}
&|\beta|\,h\sup\ D\left(w(t,s),w_{n_l,m_l}(t,s)\right)\le \\
&|\beta|\,h^2\sup\ D\left(w(t,s),w_{n_{l-1},m_{l-1}}(t,s)\right)\le \\
&|\beta|\,h^3\sup\ D\left(w(t,s),w_{n_{l-2},m_{l-2}}(t,s)\right) \\
&\quad\vdots \\
&\le|\beta|\,h^{l+1}\sup\ D\left(w(t,s),w_{n_0,m_0}(t,s)\right).
\end{aligned}$$

Therefore

$$\sup D\left(w(x,y), w_{n_l,m_l}(x,y)\right) \leq |\beta|\, h^{l+1} \sup\ D\left(w(t,s), w_{n_0,m_0}(t,s)\right).$$

Since $|\beta|h < 1$, for each large enough values n and m, we have

$$|\beta|\, h^{l+1} \to 0.$$

Therefore we have

$$\sup_{(x,y)\in\varrho} D\big(w(x,y), w_{n,m}(x,y)\big) \longrightarrow 0.$$

□

5 Examples

In this section, we are going to solve several examples, and then compare their approximate and exact solutions from the graphical and numerical point of view. The results of numerical solution of these examples are obtained by the mentioned method. All the numerical computations have been done using mathematics (9).
The presented method represents the absolute error of $w(x,y)$, as shown in Tables $1, 2$, and 3, and Figures $1, 2$, and 3 show the exact and approximate solution and absolute error functions obtained by the present method for different M and N.

Example 5.1. Consider the following two-dimensional Fredholm integral equation

$$w(x,y) = \frac{1}{1+x+y} - \frac{x}{1+y} + \int_0^1 \int_0^1 \frac{x(1+t+s)}{1+y} w(t,s)dsdt. \qquad (6)$$

The exact solution of (6) is given by

$$w(x,y) = \frac{1}{1+x+y}.$$

Table 1: The approximate solution and absolute error of $W_{n,m}(x,y)$ and $|W(x,y) - W_{n,m}(x,y)|$ of example (5.1) for $n = 4, 7, 10$ and $m = 6, 10, 12$.

(x,y)	Approximate Solution			Absolute Error		
$(2^{-v}, 2^{-v})$	$W_{4,6}(x,y)$	$W_{7,10}(x,y)$	$W_{10,12}(x,y)$	$\lvert W_{4,6}(x,y)\rvert$	$\lvert W_{7,10}(x,y)\rvert$	$\lvert W_{10,12}(x,y)\rvert$
$(2^{-1}, 2^{-1})$	0.00000000	0.00000000	0.00000000	0.50000000	0.50000000	0.50000000
$(2^{-2}, 2^{-2})$	$2.18559E^{-3}$	$-1.74330E^{-4}$	0.00000000	$6.64481E^{-1}$	$6.64481E^{-1}$	$6.66667E^{-1}$
$(2^{-3}, 2^{-3})$	$-4.13002E^{-3}$	$6.79738E^{-4}$	$4.08567E^{-4}$	$8.04130E^{-1}$	$7.99320E^{-1}$	$7.99591E^{-1}$
$(2^{-4}, 2^{-4})$	$-1.05617E^{-2}$	$-2.99003E^{-3}$	$-1.28354E^{-3}$	$8.99451E^{-1}$	$8.91879E^{-1}$	$8.90172E^{-1}$
$(2^{-5}, 2^{-5})$	$-9.43441E^{-4}$	$-5.61998E^{-3}$	$-4.18069E^{-3}$	$9.50611E^{-1}$	$9.46796E^{-1}$	$9.45357E^{-1}$
$(2^{-6}, 2^{-6})$	$-6.11826E^{-3}$	$-4.80454E^{-3}$	$-4.20148E^{-3}$	$9.75815E^{-1}$	$9.74502E^{-1}$	$9.73898E^{-1}$
$(2^{-7}, 2^{-7})$	$-3.46305E^{-3}$	$-3.07991E^{-3}$	$-2.88780E^{-3}$	$9.88078E^{-1}$	$9.87695E^{-1}$	$9.87503E^{-1}$

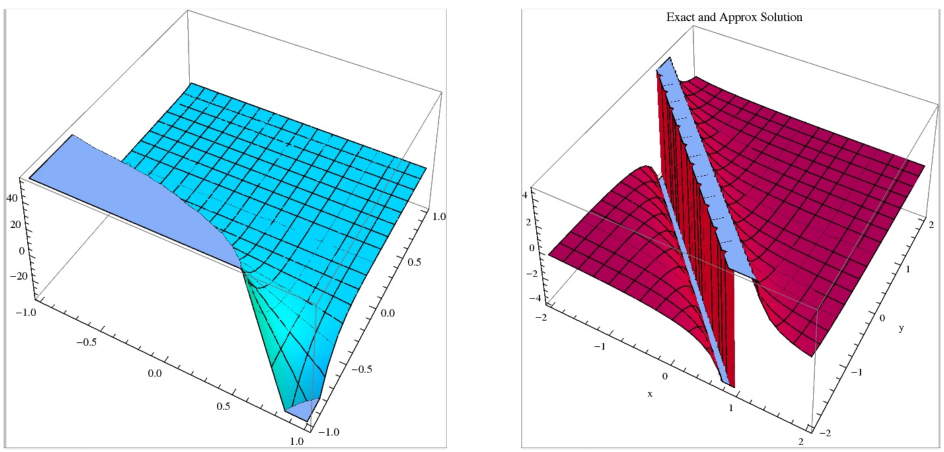

Figure 1: Exact and Approximate Solutions and Absolute Error functions obtained by the present method for $n = 4$ and $m = 6$ of example 5.1

Example 5.2. Consider the following two-dimensional Fredholm integral equation

$$w(x,y) = \frac{x^2}{3} + y^2 - \frac{2}{3}y - \frac{131}{180} + \int_0^1 \int_0^1 (x^2 + y + s^2 + t) w(t,s) ds dt. \qquad (7)$$

The exact solution of (7) is given by

$$w(x,y) = x^2 + y^2.$$

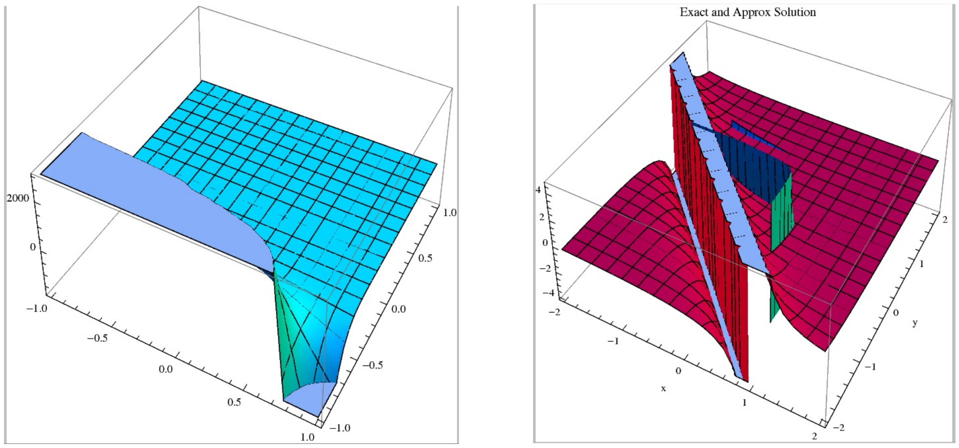

Figure 2: Exact and Approximate Solutions and Absolute Error functions obtained by the present method for $n = 7$ and $m = 10$ of example 5.1

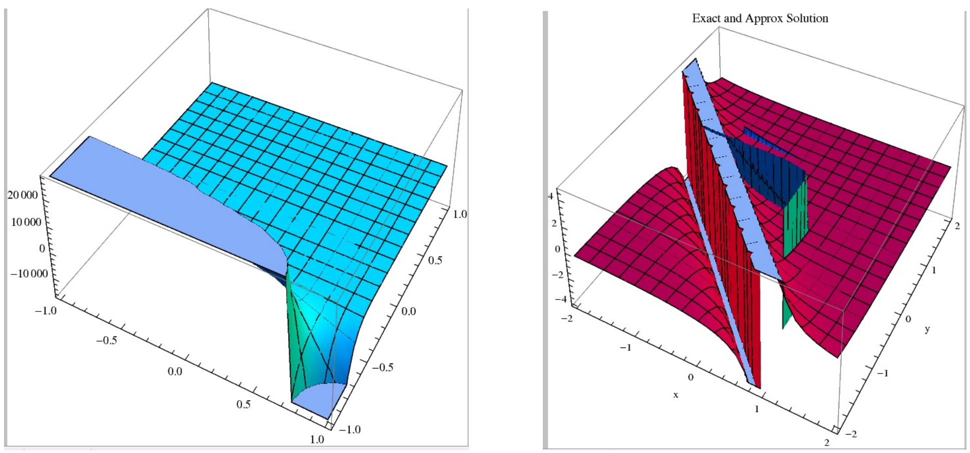

Figure 3: Exact and Approximate Solutions and Absolute Error functions obtained by the present method for $n = 10$ and $m = 12$ of example 5.1

Example 5.3. Consider the following two-dimensional Fredholm integral equation

$$w(x,y) = xe^{-y} - \frac{1}{2}y + \frac{1}{3}(e^{-1} - \frac{7}{4})x + \int_0^1 \int_0^1 (tx + ye^s) w(t,s) ds dt. \quad (8)$$

The exact solution of (8) is given by

$$w(x,y) = xe^{-y} + y.$$

Table 2: The approximate solution and absolute error of $W_{n,m}(x,y)$ and $|W(x,y) - W_{n,m}(x,y)|$ of example (5.2) for $n = m = 4, 8, 12$

(x,y)	Approximate Solution			Absolute Error		
$(2^{-v}, 2^{-v})$	$W_{4,4}(x,y)$	$W_{8,8}(x,y)$	$W_{12,12}(x,y)$	$\lvert W_{4,4}(x,y)\rvert$	$\lvert W_{8,8}(x,y)\rvert$	$\lvert W_{12,12}(x,y)\rvert$
$(2^{-1}, 2^{-1})$	0.00000000	0.00000000	0.00000000	0.50000000	0.50000000	0.50000000
$(2^{-2}, 2^{-2})$	0.00000000	0.00000000	0.00000000	$1.25000E^{-1}$	$1.25000E^{-1}$	$1.25000E^{-1}$
$(2^{-3}, 2^{-3})$	$1.41422E^{-2}$	0.00000000	$-3.87504E^{-4}$	$1.71078E^{-1}$	$3.12500E^{-1}$	$3.16375E^{-2}$
$(2^{-4}, 2^{-4})$	$1.37624E^{-2}$	$5.07467E^{-3}$	$1.10633E^{-3}$	$5.94992E^{-3}$	$2.73783E^{-3}$	$6.70617E^{-3}$
$(2^{-5}, 2^{-5})$	$9.01682E^{-3}$	$6.01239E^{-3}$	$3.43764E^{-3}$	$7.06370E^{-3}$	$4.05927E^{-3}$	$1.48451E^{-3}$
$(2^{-6}, 2^{-6})$	$5.10332E^{-3}$	$4.33847E^{-3}$	$3.37459E^{-3}$	$4.61504E^{-3}$	$3.85019E^{-3}$	$2.88630E^{-3}$
$(2^{-7}, 2^{-7})$	$2.70817E^{-2}$	$2.57588E^{-3}$	$2.29242E^{-3}$	$2.58610E^{-2}$	$2.45381E^{-3}$	$2.17035E^{-2}$

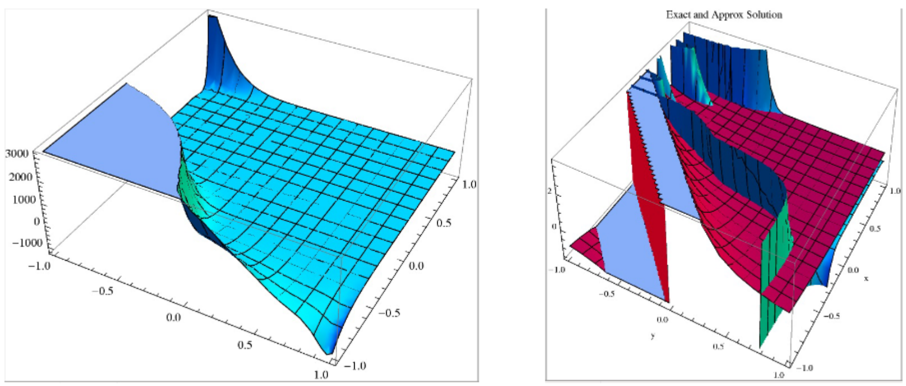

Figure 4: Exact and Approximate Solutions and Absolute Error functions obtained by the present method for $n = m = 4$ of example 5.2

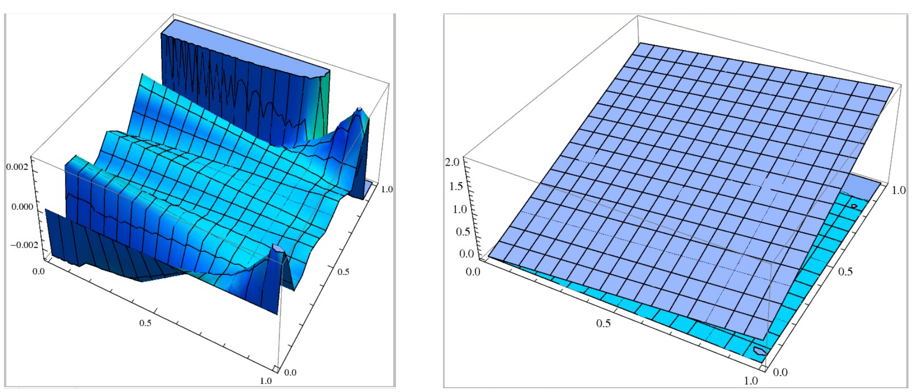

Figure 7: Exact and Approximate Solutions and Absolute Error functions obtained by the present method for $n = m = 4$ of example 5.3

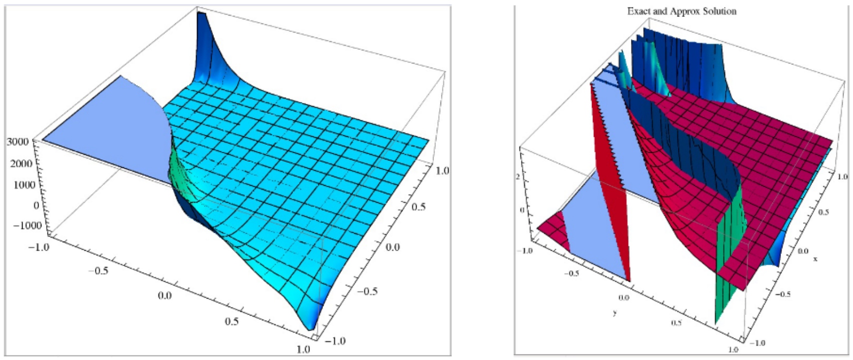

Figure 5: Exact and Approximate Solutions and Absolute Error functions obtained by the present method for $n = m = 8$ of example 5.2

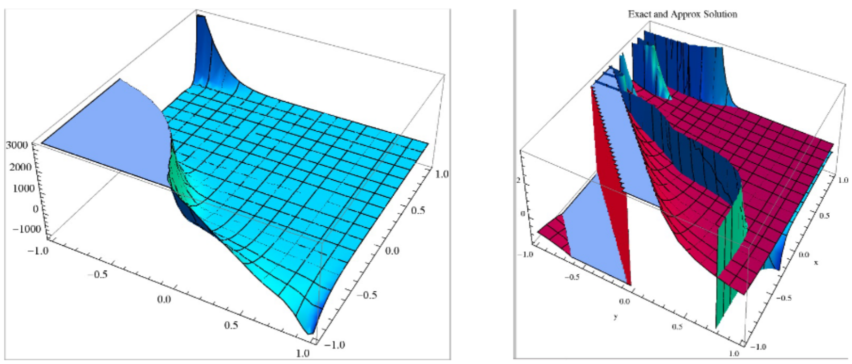

Figure 6: Exact and Approximate Solutions and Absolute Error functions obtained by the present method for $n = m = 12$ of example 5.2

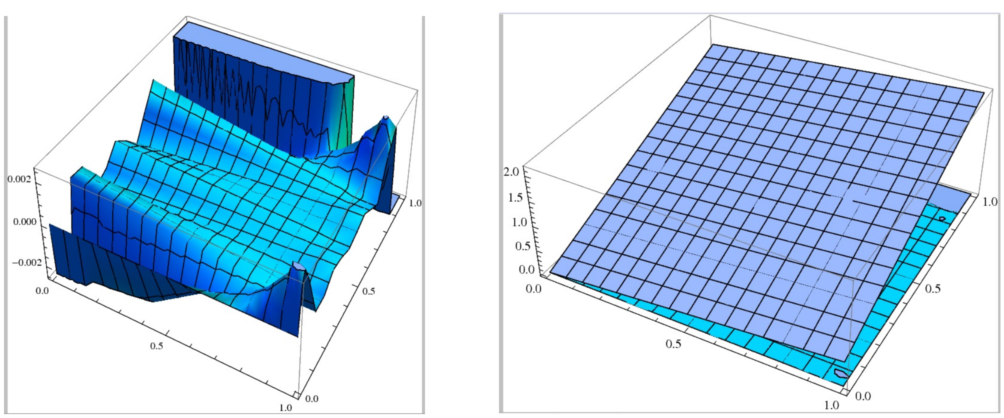

Figure 8: Exact and Approximate Solutions and Absolute Error functions obtained by the present method for $n = m = 8$ of example 5.3

Table 3: The approximate solution and absolute error of $W_{n,m}(x,y)$ and $|W(x,y) - W_{n,m}(x,y)|$ of example (5.3) for $n = m = 4, 8, 12$.

(x,y)	Approximate Solution			Absolute Error		
$(2^{-v}, 2^{-v})$	$W_{4,4}(x,y)$	$W_{8,8}(x,y)$	$W_{12,12}(x,y)$	$\|W_{4,4}(x,y)\|$	$\|W_{8,8}(x,y)\|$	$\|W_{12,12}(x,y)\|$
$(2^{-1}, 2^{-1})$	$-2.22045E^{-15}$	$6.14263E^{-11}$	$-1.38835E^{-6}$	1.07436	1.07436	1.07436
$(2^{-2}, 2^{-2})$	$-2.92301E^{-16}$	$1.99761E^{-13}$	$3.49357E^{-10}$	$3.83506E^{-1}$	$3.83506E^{-1}$	$3.83506E^{-1}$
$(2^{-3}, 2^{-3})$	$1.93587E^{-2}$	$-1.56536E^{-15}$	$-5.10878E^{-4}$	$1.37910E^{-1}$	$1.57269E^{-1}$	$1.57779E^{-1}$
$(2^{-4}, 2^{-4})$	$1.84283E^{-2}$	$6.57967E^{-3}$	$1.42763E^{-3}$	$5.20089E^{-2}$	$6.38575E^{-2}$	$6.90095E^{-2}$
$(2^{-5}, 2^{-5})$	$1.19740E^{-2}$	$7.73150E^{-3}$	$4.39959E^{-3}$	$2.12445E^{-2}$	$2.54870E^{-2}$	$2.88189E^{-2}$
$(2^{-6}, 2^{-6})$	$6.75352E^{-3}$	$5.55946E^{-3}$	$4.30379E^{-3}$	$9.36168E^{-3}$	$1.05557E^{-2}$	$1.18114E^{-2}$
$(2^{-7}, 2^{-7})$	$3.57825E^{-3}$	$3.29555E^{-3}$	$2.91897E^{-3}$	$4.35656E^{-3}$	$4.63926E^{-3}$	$5.01584E^{-3}$

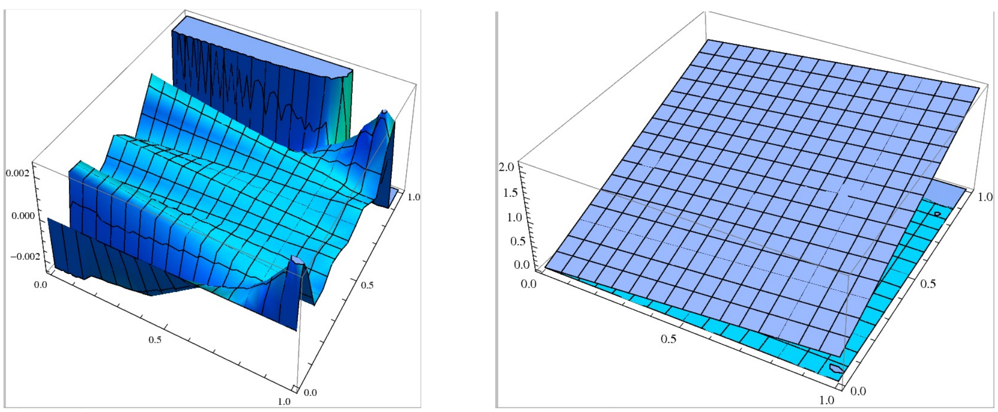

Figure 9: Exact and Approximate Solution and Absolute Error functions obtained by the present method for $n = m = 12$ of example 5.3

6 Conclusion

The main purpose of this paper is to introduce a numerical method to solve two-dimensional Fredholm integral equations. The features of this method can be used simply to solve the method as well as numerical methods and to use the linear equation system to obtain the set response that increases the accuracy and stability of this method compared with other numerical methods. One significant advantage of this approach is the two-dimensional Fredholm integral equation, which can be calculated easily using a computer program. Another advantage of this method is that, considering that the support points are obtained approximately, the approximate answer has better accuracy. Our examples demonstrate the validity and application of techniques on the computer using a program written in *Mathematica* 9. The correctness of the solutions obtained by integrating them in the original equation with the help of the test can be examined to ensure more certainty about the results.

Conflict of interest. The authors declare that they have no conflict of interest.

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References

[1] P. Balland and E. Süli, Analysis of the cell-vertex finite volume method for hyperbolic problems with variable coefficients, *SIAM J. Numer. Anal*, 34: 1127–1151, 1997.

[2] A. Alipanah, S. Esmaeili, Numerical solution of the two-dimensional Fredholm integral equations using Gaussian radial basis function, *J. Comput. Appl. Math.* 235: 5342–5347, 2011.

[3] Z. Avazzadeh, M. Heydari, Chebyshev polynomials for solving two dimensional linear and nonlinear integral equations of the second kind, *Comput. Appl. Math.* 31: 127–142, 2012.

[4] Z. Avazzadeh, M. Heydari, G. B. Loghmani, A comparison between solving two dimensional integral equations by the traditional collocation method and radial basis functions, *Appl. Math. Sci.* 5:1145--1152, 2011.

[5] I. Bhat, L. Mishra. Numerical solutions of Volterra integral equations of third kind and its convergence analysis., *Symmetry* . 14:(12)2600, 2022.

[6] L. M. Delves, J. L. Mohamed, Computational Method for Integral Equations, *Cambridge University Press, New York* 1985.

[7] G. Gnecco, V. Keurkova, M. Sanguineti, Accuracy of approximations of solutions to Fredholm equations by kernel methods, *Appl. Math. Comput.* 218: 7481–7497, 2012

[8] A. Golbabai, S. Seifollahi, Numerical solution of the second kind integral equations using radial basis function networks, *Appl. Math. Comput.* 174: 877–883, 2006.

[9] G. Han, R. Wang, Richardson extrapolation of iterated discrete Galerkin solution for two-dimensional Fredholm integral equations, *Comput. Appl. Math.* 139: 49–63, 2002

[10] S. Hatamzadeh-Varmazyar, Z. Masouri, Numerical method for analysis of one- and two-dimensional electromagnetic scattering based on using linear Fredholm integral equation models, *Math.Comput. Mode.* 54:2199–2210, 2011.

[11] F. Khan, T. Arshad, A. Ghaffar, K. Sooppy Nisar, D. Kumar, Numerical solutions of 2D Fredholm integral equation of first kind by discretization technique, *AIMS Mathematics*, 5:2295–2306, 2020

[12] F. Khan, M. Omar, Z. Ullah, Discretization method for the numerical solution of 2D Volterra integral equation based on two-dimensional Bernstein polynomial, *AIP Adv*, 8: 1–9, 2018.

[13] Y. Ma, J. Huang, H. Li, A Novel Numerical Method of Two-Dimensional Fredholm Integral Equations of the Second Kind, *Math. Probl. Eng.* 2015.

[14] K. Maleknejad, M. Soleiman Dehkordi, Numerical solutions of two-dimensional nonlinear integral equations via Laguerre Wavelet method with convergence analysis, *Applied Mathematics-A Journal of Chinese Universities*, 36:83–98, 2021.

[15] S. T. Micula, Numerical Solution of Two-Dimensional Fredholm?Volterra Integral Equations of the Second Kind, *Symmetry.* 13: 8, 2021.

[16] F. Mirzaee, E. Hadadiyan, Numerical solution of linear Fredholm integral equations via two-dimensional modification of hat functions, *Appl. Math. Comput.* 250: 805–816, 2015.

[17] F. Mirzaee, S. Piroozfar, Numerical solution of the linear two-dimensional Fredholm integral equations of the second kind via two-dimensional triangular orthogonal functions, *JKSUES.* 22: 185–193, 2010.

[18] M. Mohammad A numerical solution of Fredholm integral equations of the second kind based on tight framelets generated by the oblique extension principle. , *Symmetry.* 11:(7) 854, 2019.

[19] G. A. Mosa, M. A. Abdou, A. S. Rahby. Numerical solutions for nonlinear Volterra-Fredholm integral equations of the second kind with a phase lag. *AIMS Mathematics*, 6: 8525–8543, 2021.

[20] M. A. Fariborzi Araghi, N. Parandin, Numerical Solution of Fuzzy Fredholm Integral Equations by the Lagrange Interpolation based on the Extension Principle, *Soft Comput.* 15:(12) 2449–2456, 2021.

[21] M. Y. Rahimi, S. Shahmorad, F. Talati, A. Tari, An Operational Method for The Numerical Solution of Two Dimensional Linear Fredholm Integral Equations with an Error Estimation, *BIMS* 36:(2) 119–132, 2010.

[22] E. Shoukralla, N. Saber, A.Sayed Computational method for solving weakly singular Fredholm integral equations of the second kind using an advanced barycentric Lagrange interpolation formula. , *Advanced Modeling and Simulation in Engineering Sciences.* 8: 1–22, 2021.

[23] J. Stoer and R. Bulirsch, Introduction to Numerical Analysis, *Springer-Verlag New York.* 1993.

[24] A. Tari, S. Shahmorad, A Computational Method for solving two-dimensional linear Fredholm integral equations of the second kind, *j. ANZIAM.* 49: 543–549, 2008.

[25] E. Tohidi, Taylor matrix method for solving linear two-dimensional Fredholm integral equations with Piecewise Intervals, *Comput. Appl. Math.* 34: 1117–1130, 2015.

Gholam Reza Farahmand
PhD Candidate of Mathematics
Department of Mathematics
Hamedan Branch, Islamic Azad University
Hamedan, Iran
E-mail: g.r_ farahmand@yahoo.com

Nouraldin Parandin
Associate Professor of Applied Mathematics
Department of Mathematics
Kermanshah Branch, Islamic Azad University
Kermanshah, Iran
E-mail: nparandin54@gmail.com

Nasrin Karamikabir
Assistant Professor of Mathematics
Department of Mathematics
Hamedan Branch, Islamic Azad University
Hamedan, Iran
E-mail: n.karamikabir@iauh.ac.ir