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The Length-Biased Topp-Leone Distribution: Properties and Applications

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Abstract. This paper presents an innovative distribution called the length-biased Topp-Leone. Some mathematical properties such as Moments and moment generating function, Measure of uncertainty, The Rnyi entropy, Bonferroni and Lorenz curves, and Order statistics are discussed. To determine the distributions estimated parameters, the maximum likelihood method is employed. Moreover, this novel model is implemented on two existing data sets to showcase its practicality and usefulness.

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1 Introduction

Several univariate continuous distributions find their utilization in various data modeling applications within the contemporary statistical literature. Moreover, the existing array of distributions fails to adequately encompass the diverse nature of data encountered in fields such as medicine, biology, demography, engineering sciences, actuarial science, finance,

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economics, and dependability. Consequently, researchers specializing in statistics and applied mathematics are driven by a profound interest to fabricate novel extended continuous distributions that possess enhanced efficacy for data modeling purposes. Techniques employed to expand well-established distributions include appending additional parameters, compounding elements together, generating fresh structures altogether, or transforming and amalgamating existing ones. The emergence of new lineages containing continuous distributions has intrigued multitudes of statisticians in recent decades resulting in an escalation of groundbreaking models being formulated. The Topp-Leone (TL) distribution, as initially presented by Topp and Leone in 1955, emerges as a captivating probability distribution. Notably, the TL distribution gracefully adheres to limits between 0 and 1 enabling the derivation of concise expressions for both its cumulative distribution function (cdf) and its probability density function (pdf).

The given passage enumerates the extension of Topp-Leone distributions that have garnered recognition in scholarly discourse. Notable examples comprise theTopp-Leone inverse Weibull by [1, 22], Topp Leone Marshall Olkin-Weibull by [2], Topp-Leone Extended Exponential by [3], Sine Topp-Leone-G family by [4], Modified Topp Leone-Chen by [5], odd log-logistic Topp–Leone G family by [8], odd Weibull inverse topp–leone by [10], Transmuted Topp Leone Flexible Weibull by [12], power Topp–Leone generated family by [15], Topp–Leone odd log-logistic family by [16], exponentiated odd weibull-topp-leone-g family by [17], Topp-Leone-Marshall-Olkin-G family by [19], Kumaraswamy Inverted Topp-Leone by [25], Type II generalized Topp-Leone family by [27], alpha power inverted Topp–Leone by [28], Transmuted Topp-Leone Weibull by [29], Topp Leone Burr XII by [30], Topp-Leone Extended Exponential by [31], Type II Topp-Leone Bur XII by [35], Topp–Leone Lomax by [36], Topp-Leone Dagum by [37], Topp Leone odd Lindley-G family by [38], Transmuted Topp-Leone power function by [26], Topp–Leone normal by [39], odd Weibull-Topp-Leone-G power series family by [42], transmuted Topp-Leone G familyby [44]. Also, many distributions have been proposed in the last decades for the data defined in 0-1 intervals. Such as beta power [33], Beta Topp-Leone [43], Generalized beta-generated [6], Beta-normal [20], Kumaraswamy odd log-logistic [7], and Topp-Leone

Kumaraswamy Marshall-Olkin [13].

The Topp–Leone distribution, with its elegant and distinctive J-shaped form, has captivated the minds of countless statisticians seeking alternatives to the Beta distribution. In their insightful work, Topp and Leone provided an expression for the cumulative distribution function (cdf) in terms of a simple equation:

$$G\left(x\right)=x^{\alpha}(2-x)^{\alpha},$$

where $0<x<1$ and $\alpha>0$. The probability density function (pdf) of one-parameter TL is determined as

$$g\left(x\right)=2\alpha x^{\alpha-1}\left(1-x\right)\left(2-x\right)^{\alpha-1}\quad,\ 0<x<1,\ \ \alpha>0, \tag{1}$$

The subsequent portions of this article are arranged in the following manner. In Section 2, readers will be introduced to the elegant and intricate length-biased Topp-Leone distribution (LBTL), where every detail surrounding its characteristics shall artfully unfold. Delving further in Section 3, profound insights regarding the statistical features of LBTL shall captivate attentive minds. The enthralling discourse continues unbidden as Section 4 meticulously dissects the estimation procedure for this remarkable model's parameters. In Section 5, we undertake a simulation study to elucidate our findings. In this section, we also furnish readers with a guideline for selecting the most optimal estimation method. Moreover, in Section 6, we aptly demonstrate the adaptability of the novel distribution through the diligent examination of two real data sets. Finally, Section 7 succinctly concludes our paper with some insightful final remarks.

Theorem 1.1. *Let X be a random variable that follows the Topp-Leone distribution. The k^{th} ordinary moment of X is given by*

$$\mu_k'=E\left(X^k\right)=2\alpha a_{m,n,k},$$

where

$$a_{m,n,k}=\sum_{m=0}^{\infty}\sum_{n=0}^{m+1}\binom{m+1}{n}\frac{\left(-1\right)^n}{m!\left(k+\alpha+n\right)}\frac{\Gamma\left(\alpha\right)}{\Gamma\left(\alpha-m\right)}. \tag{2}$$

Proof. By employing substitution (1) within the definition of the k^{th} ordinary moment, we are able to ascertain that

$$\mu_k' = E\left(X^k\right) = \int_0^1 x^k g\left(x\right) dx = 2\alpha \int_0^1 x^{k+\alpha-1}\left(1-x\right)\left(2-x\right)^{\alpha-1} dx$$

By incorporating the principles of the generalized binomial expansion, we are able to construct the following expression:

$$(2-x)^{\alpha-1} = [1+(1-x)]^{\alpha-1} = \sum_{m=0}^{\infty} \frac{\Gamma(\alpha)}{m!\Gamma(\alpha-m)}(1-x)^m$$

By making a substitution

$$\mu_k' = E\left(X^k\right) = 2\alpha \int_0^1 \sum_{m=0}^{\infty} \frac{\Gamma(\alpha)}{m!\Gamma(\alpha-m)} x^{k+\alpha-1}(1-x)^{m+1} dx$$

since

$$(1-x)^{m+1} = \sum_{n=0}^{m+1} (-1)^n \binom{m+1}{n} x^n,$$

so

$$E\left(X^k\right) = 2\alpha \int_0^1 \sum_{m=0}^{\infty} \sum_{n=0}^{m+1} \frac{(-1)^n \Gamma(\alpha)}{m!\Gamma(\alpha-m)} \binom{m+1}{n} x^{k+\alpha+n-1} dx$$

$$= 2\alpha \sum_{m=0}^{\infty} \sum_{n=0}^{m+1} \binom{m+1}{n} \frac{(-1)^n}{m!\left(k+\alpha+n\right)} \frac{\Gamma\left(\alpha\right)}{\Gamma\left(\alpha - m\right)}$$

hence

$$\mu_k' = E\left(X^k\right) = 2\alpha a_{m,n,k}$$

Where

$$a_{m,n,k} = \sum_{m=0}^{\infty} \sum_{n=0}^{m+1} \binom{m+1}{n} \frac{(-1)^n}{m!\left(k+\alpha+n\right)} \frac{\Gamma\left(\alpha\right)}{\Gamma\left(\alpha - m\right)}$$

□

Corollary 1.2. *Let X represent a random variable that adheres to the Topp-Leone distribution. The mean of X can be derived as follows:*

$$\mu_1' = 2\alpha a_{m,n,1},$$

where

$$a_{m,n,1} = \sum_{m=0}^{\infty} \sum_{n=0}^{m+1} \binom{m+1}{n} \frac{(-1)^n}{m!\,(1+\alpha+n)} \frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}$$

2 The Length-Biased Topp-Leone Distribution

Suppose X represents the random variable in a study. Let $g(x)$ be its probability density function. We can define $f(x)$ as a new probability density function for the Length-Biased Topp-Leone (LBTL) distribution, given by the expression

$$f(x) = \frac{xg(x)}{E(X)} = \frac{x^{\alpha}(1-x)(2-x)^{\alpha-1}}{a_{m,n,1}} \tag{3}$$

where $\alpha > 0$, and

$$a_{m,n,1} = \sum_{m=0}^{\infty} \sum_{n=0}^{m+1} \binom{m+1}{n} \frac{(-1)^n}{m!\,(1+\alpha+n)} \frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}$$

Topp and Leone [41], show the first moment of the topp-Leone distribution is

$$EX = 1 - \frac{4^{\alpha}[\Gamma(1+\alpha)]^2}{\Gamma(2+2\alpha)},$$

So we can define the length-biased Topp-Leone distribution by a closed form as

$$f(x) = \frac{xg(x)}{EX} = \frac{x^{\alpha}(1-x)(2-x)^{\alpha-1}\Gamma(2+2\alpha)}{\Gamma(2+2\alpha) - 4^{\alpha}[\Gamma(1+\alpha)]^2}.$$

The cumulative distribution function of LBTL can be acquired in the following manner

$$F(x) = \int_0^x f(x)\,dx = \int_0^x \frac{x^{\alpha}(1-x)(2-x)^{\alpha-1}}{a_{m,n,1}}dx$$

so

$$F(x) = \sum_{d=0}^{\alpha-1} \binom{\alpha-1}{d} \frac{1}{a_{m,n,1}} \int_0^x x^{\alpha} (1-x)^{d+1} dx$$
$$= \sum_{d=0}^{\alpha-1} \binom{\alpha-1}{d} \frac{1}{a_{m,n,1}} B(\alpha, d, x)$$

Where $a_{m,n,1}$ is given by (2) and

$$B(\alpha, d, x) = \int_0^x x^{\alpha} (1-x)^{d+1} dx. \tag{4}$$

The function of reliability, denoted as rf, can be expressed as

$$R(x) = 1 - F(x) = 1 - \sum_{d=0}^{\alpha-1} \binom{\alpha-1}{d} \frac{1}{a_{m,n,1}} B(\alpha, d, x), \tag{5}$$

The function denoting the rate of occurrence of hazards, commonly known as the hazard rate function (hrf), is defined as

$$h(x) = \frac{f(x)}{R(x)} = \frac{x^{\alpha} (1-x) (2-x)^{\alpha-1}}{a_{m,n,1} - \sum_{d=0}^{\alpha-1} \binom{\alpha-1}{d} B(\alpha, d, x)} \tag{6}$$

Figure 1 exhibits the form of the density function for LBTL(α), varying according to the parameter α. When α takes on a lesser value, this gives rise to a skew distribution that leans towards the right side. Conversely, as α increases, the skew distribution instead veers towards the left. Should α reach an equilibrium at a value of 1, symmetry characterizes its corresponding distribution see [24]; further revealed by its kurtosis level hovering at 1.5 Additionally, when we increase α within the range of 0.1 to 3, it brings about augmenting degrees of skewness within our given distribution; although should we surpass an α greater than 3, any semblance of asymmetry diminishes progressively from thereon out.

3 Statistical Properties

In this section, we delve into acquiring crucial statistical characteristics of LBTL. Additionally, we unveil the moment-generating function (MGF) as well as Rényi and Tsallis's entropies. The ensuing theorem delves into a thorough discussion of the moments pertaining to LBTL.

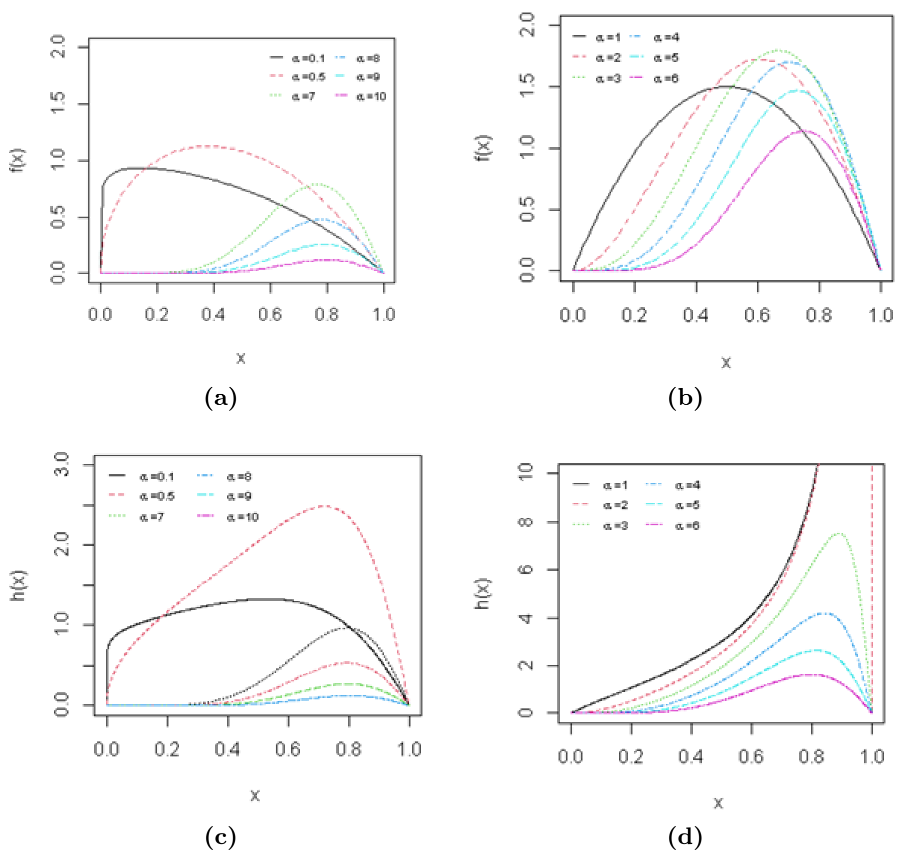

Figure 1: The pdf and hrf of LBTL for selected parameters. (a) Density, (b) Density, (c)Hazard rate, (d) Hazard rate.

3.1 Moments and moment generating function.

Theorem 3.1. *Let's consider X as a random variable that is the LBTL (α) distribution. In this case, the k^{th} moment of X can be expressed as follows:*

$$\mu'_k = \frac{a_{m,n,(k+1)}}{a_{m,n,1}} \tag{7}$$

where $a_{m,n,k}$ is given by equation (2) and $0 \leq n \leq m+1$, $0 \leq m < \infty$.

Proof. We have

$$\mu_k' = E\left(X^k\right) = \int_0^1 x^k f(x)\, dx = \int_0^1 \frac{x^{\alpha+k}(1-x)(2-x)^{\alpha-1}}{a_{m,n,1}} dx$$

By Binomial expansion, we have

$$(2-x)^{\alpha-1} = [1+(1-x)]^{\alpha-1} = \sum_{m=0}^{\infty} \frac{\Gamma(\alpha)}{m!\Gamma(\alpha-m)}(1-x)^m,$$

then

$$\mu_k' = \int_0^1 \sum_{m=0}^{\infty} \frac{\Gamma(\alpha)}{m!\Gamma(\alpha-m)a_{m,n,1}} x^{k+\alpha}(1-x)^{m+1} dx.$$

Consider the power series given by

$$(1-x)^{m+1} = \sum_{n=0}^{m+1} (-1)^n \binom{m+1}{n} x^n,$$

we can write

$$\begin{aligned}
\mu_k' &= \int_0^1 \sum_{m=0}^{\infty} \sum_{n=0}^{m+1} \frac{(-1)^n \Gamma(\alpha)}{m!\Gamma(\alpha-m)a_{m,n,1}} \binom{m+1}{n} x^{k+\alpha+n} dx \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{m+1} \frac{(-1)^n}{k+\alpha+n+1} \binom{m+1}{n} \frac{\Gamma(\alpha)}{m!\Gamma(\alpha-m)a_{m,n,1}},
\end{aligned}$$

by equation (2), we have

$$\mu_k' = \frac{a_{m,n,(k+1)}}{a_{m,n,1}}.$$

We present the first six moments for the LBTL distribution for some selected parameter values. The results are shown in Table 1.

□ [41] show cumulative moments M_k for TL distribution are given by

$$M_k = \int_0^1 x^k [1-F(x)] dx$$

Table 1: The first six moments of LBTL distribution

Moments	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 4$	$\alpha = 10$
μ_1'	0.3459	0.5420	0.4927	0.2000
μ_2'	0.1257	0.4025	0.3202	0.0957
μ_3'	0.0255	0.3713	0.2487	0.0290
μ_4'	0.0696	0.4325	0.2381	0.0104
μ_5'	0.0480	0.6373	0.2851	0.0054
μ_6'	0.0331	1.2024	0.4212	0.0012

where the c.f.f. $F(x)$ is is used. Using M_k and some straight-forward integration, one may obtain for this family of function

$$M_k = \frac{1}{k+1} - \frac{k-1}{\alpha+1} - \frac{1}{(k+1)(\alpha+k-1)} - (k+1)R + (k-1)W$$

where

$$R = \frac{1}{2^{2\alpha+2}} \frac{\Gamma(2\alpha+2)}{\Gamma^2(\alpha+1)}$$

and

$$W = \left(\frac{2\alpha_2}{\alpha + \frac{1}{2}} \right) R.$$

In addition

$$\mu_1' = M_0$$
$$\mu_2' = 2M_1 - M_0^2$$
$$\mu_3' = 3M_2 - 6M_1M_0 + 2M_0^2,$$

where the μ''s are the ordinary moments about the mean for the frequency function $f(x)$. According to Theorem 3.1, it is possible for us to calculate the arithmetic average and standard deviation of the LBTL distribution through the following steps.

$$E(X) = \mu_1'(LBTL) = \frac{a_{m,n,2}}{a_{m,n,1}}$$

and

$$Var(X) = \mu_2'(LBTL) - \mu_1'(LBTL) = \frac{a_{m,n,3}}{a_{m,n,1}} - \frac{a_{m,n,2}}{a_{m,n,1}}$$

Through the application of the formulaic equation(9), it becomes possible to discern and ascertain the measures of skewness and kurtosis by means of the following representation [25].

$$
\begin{aligned}
sk =& \frac{\mu_3' - 2\mu_2'\mu_1' - (\mu_1')^3}{\left(\mu_2' - {\mu_1'}^2\right)^{3/2}} = \frac{a_{m,n,1}a_{m,n,4} - 2a_{m,n,1}a_{m,n,2}a_{m,n,3} - a_{m,n,2}^3}{\left(a_{m,n,1}a_{m,n,3} - a_{m,n,2}^2\right)^3} \\
ku =& \frac{\mu_4' - 3\mu_1'\mu_3' + 6{\mu_1'}^2\mu_2' - 3\left(\mu_1'\right)^4}{\left(\mu_2' - {\mu_1'}^2\right)^2} \\
=& \frac{a_{m,n,5} - 3a_{m,n,2}a_{m,n,4} + 6a_{m,n,2}^2 a_{m,n,3} - a_{m,n,2}^3}{\left(a_{m,n,3} - a_{m,n,2}^2\right)^2}
\end{aligned}
$$

The calculation of the mode for the distribution LBTL involves differentiating the natural log of its probability density function and setting it equal to zero. Referring to equation (3), we find that the expression representing the natural log of said probability density function is as follows:

$$
\ln f(x) = \alpha \ln x + \ln(1-x) \ + (\alpha - 1)\ln(2-x) \ - \ln a_{m,n,1}
$$

The initial derivative of the natural log $f(x)$ with respect to the variable x, is

$$
\frac{\partial \ln f(x)}{\partial x} = \frac{\alpha}{x} - \frac{1}{1-x} - \frac{\alpha - 1}{2-x} = 0,
$$

If we consider the aforementioned condition, it becomes critical value is

$$
x = \frac{4\alpha - 1}{2\alpha}
$$

Through the application of the second derivatives test, one can derive information about the nature and behavior of a function's second derivative

$$
\frac{\partial^2}{\partial x^2} \ln f(x) = \frac{-\alpha}{x^2} - \frac{1}{(1-x)^2} - \frac{(\alpha - 1)}{(2-x)^2}
$$

at $x = \frac{4\alpha-1}{2\alpha}$, then

$$
\begin{aligned}
\frac{\partial^2}{\partial x^2} \ln f(x) &= \frac{-\alpha}{\left(\frac{4\alpha-1}{2\alpha}\right)^2} - \frac{1}{\left(1 - \frac{4\alpha-1}{2\alpha}\right)^2} - \frac{(\alpha - 1)}{\left(2 - \frac{4\alpha-1}{2\alpha}\right)^2} \\
\frac{\partial^2}{\partial x^2} \ln f(x) &= \frac{(2\alpha)^2}{(4\alpha-1)^2(1-2\alpha)^2}\left(-96\alpha^4 - 8\alpha^3 - 44\alpha^2 - 12\alpha - 2\right) < 0
\end{aligned}
$$

Henceforth, it can be deduced that the probability density function for LBTL distribution reaches its pinnacle at the point where it equals $(4\alpha-1)/2\alpha$. Consequently, this engenders $x = (4\alpha-1)/2\alpha$ as the mode for LBTL distribution.

Table 2 presents, the mean, variance, coefficient of variation, skewness, and kurtosis for the LBTL distribution for some selected parameter values.

Table 2: Descriptive statistics of LBTL distributio

Statistic	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 4$	$\alpha = 10$
Mean	0.3459	0.5420	0.4927	0.2000
Variance	0.0061	0.1087	0.0774	0.0557
CV	0.0778	0.6084	0.5648	1.1800
Skewness	-47.0690	0.9838	0.6784	-0.9448
Kurtosis	24.2730	3.6029	3.2608	-1.2698

Theorem 3.2. *The Moment Generating Function (MGF) of the LBTL distribution can be expressed as*

$$M_X(t) = \sum_{s=0}^{\infty} \frac{a_{m,n,(s+1)}}{a_{m,n,1}} \frac{t^s}{s!} \tag{8}$$

Proof. The ensuing equation yields the Moment Generating Function.

$$M_X(t) = E\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} f(x)dx = \int_0^1 e^{tx} \frac{x^{\alpha}(1-x)(2-x)^{\alpha-1}}{a_{m,n,1}} dx$$

Through the utilization of the expansion for the exponential distribution, which is symbolized as $e^{tx} = \sum_{s=0}^{\infty} \frac{(tx)^s}{s!}$, we obtain

$$\begin{aligned} M_X(t) &= \int_0^1 \sum_{s=0}^{\infty} \frac{t^s}{s!} \frac{x^{\alpha+s}(1-x)(2-x)^{\alpha-1}}{a_{m,n,1}} dx \\ &= \sum_{s=0}^{\infty} \frac{t^s}{s!} \int_0^1 \frac{x^{\alpha}(1-x)(2-x)^{\alpha-1}}{a_{m,n,1}} dx \end{aligned}$$

Hence,

$$M_X(t) = \sum_{s=0}^{\infty} \frac{a_{m,n,(s+1)}}{a_{m,n,1}} \frac{t^s}{s!}$$

This completes the proof. □

Theorem 3.3. *The obtainment of the characteristic function of the LBTL can be acquired through the following*

$$\varphi_X(t) = \sum_{s=0}^{\infty} \frac{a_{m,n,(s+1)}}{a_{m,n,1}} \frac{(it)^s}{s!} \tag{9}$$

Proof. We have

$$\varphi_X(t) = E\left(e^{itX}\right) = \int_{-\infty}^{\infty} e^{itx} f(x)dx = \int_0^1 e^{itx} \frac{x^{\alpha}(1-x)(2-x)^{\alpha-1}}{a_{m,n,1}} dx$$

Through the utilization of the expansion for the exponential distribution $e^{itx} = \sum_{s=0}^{\infty} \frac{(itx)^s}{s!}$, then

$$\varphi_X(t) = \sum_{s=0}^{\infty} \frac{(it)^s}{s!} \int_0^1 \frac{x^{\alpha+s}(1-x)(2-x)^{\alpha-1}}{a_{m,n,1}} dx \tag{10}$$

Substituting from (2) into (10), we have

$$\varphi_X(t) = \sum_{s=0}^{\infty} \frac{a_{m,n,(s+1)}}{a_{m,n,1}} \frac{(it)^s}{s!}$$

Hence Theorem 3.3 is proved. □

3.2 Measure of uncertainty

Theorem 3.4. *The Rényi entropy, denoted as $I_\nu(x)$, and the Tsallis entropy, represented by $T_\nu(x)$, of the LBTL distribution are expressed accordingly, as referenced in [40]. So*

$$I_\nu(x) = \frac{1}{1-\nu}\left\{\log a_{(\nu+m),n,\nu} - \log a_{m,n,1}\right\}, \tag{11}$$

and

$$T_\nu(x) = \frac{1}{1-\nu}\left[\frac{a_{(\nu+m),n,\nu}}{a_{m,n,1}} - 1\right] \tag{12}$$

Proof. By [23, 40], we have

$$I_{\nu}(x) = \frac{1}{1-\nu} \log \int_0^{\infty} f^{\nu}(x) dx \tag{13}$$

Substituting from (3) into (13), we can write

$$I_{\nu}(x) = \frac{1}{1-\nu} \log \frac{1}{a_{m,n,1}} \int_0^1 x^{\alpha\nu} (1-x)^{\nu} (2-x)^{\nu(\alpha-1)} dx$$

By employing the principle of the generalized binomial expansion, one is able to present the mathematical expression as follows:

$$(2-x)^{\nu(\alpha-1)} = \sum_{m=0}^{\infty} \frac{\Gamma(\nu\alpha-\nu+1)}{m!\Gamma(\nu\alpha-\nu-m+1)} (1-x)^m$$

Through the utilization of binomial expansion, one can articulate the following expression:

$$I_{\nu}(x) = \frac{1}{1-\nu} \log \frac{1}{a_{m,n,1}} \int_0^1 x^{\alpha\nu} \sum_{m=0}^{\infty} \frac{\Gamma(\nu\alpha-\nu+1)}{m!\Gamma(\nu\alpha-\nu-m+1)} (1-x)^{\nu+m} dx$$

hence

$$I_{\nu}(x) = \frac{1}{1-\nu} \log \frac{1}{a_{m,n,1}} \sum_{m=0}^{\infty} \sum_{n=0}^{\nu+m} \binom{\nu+m}{n} \frac{(-1)^n \Gamma(\nu\alpha-\nu+1)}{m!\Gamma(\nu\alpha-\nu-m+1)} \int_0^1 x^{\alpha\nu+n} dx$$

then

$$I_{\nu}(x) = \frac{1}{1-\nu} \log \frac{1}{a_{m,n,1}} \sum_{m=0}^{\infty} \sum_{n=0}^{\nu+m} \binom{\nu+m}{n} \frac{(-1)^n \Gamma(\nu\alpha-\nu+1)}{m!\Gamma(\nu\alpha-\nu-m+1)(\alpha\nu+n+1)}$$

By using (2), we have

$$I_{\nu}(x) = \frac{1}{1-\nu} \log \frac{a_{(\nu+m),n,\nu}}{a_{m,n,1}}$$

then

$$I_{\nu}(x) = \frac{1}{1-\nu} \left\{ \log a_{(\nu+m),n,\nu} - \log a_{m,n,1} \right\}$$

By [40], we have

$$T_{\nu}\left(x\right)=\frac{1}{1-\nu}\left[\int_{0}^{\infty}f^{\nu}\left(x\right)dx-1\right]$$

in a similar steps,hence

$$T_{\nu}\left(x\right)=\frac{1}{1-\nu}\left[\int_{0}^{1}f^{\nu}\left(x\right)dx-1\right]=\frac{1}{1-\nu}\left[\frac{a_{(\nu+m),n,\nu}}{a_{m,n,1}}-1\right]$$

□

3.3 Inequality measures

Theorem 3.5. *The Bonferroni curves denoted as $B(p)$ and the Lorenz curves represented by $L(p)$ illustrate of the LBTL distribution in the ensuing discourse.*

$$B\left(p\right)=\frac{q^{\alpha+n+2}}{p}$$

and

$$L\left(p\right)=\frac{q^{\alpha+n+2}\ a_{m,n,2}}{2\alpha a_{m,n,1}^{2}}$$

Proof. The Bonferroni and Lorenz curves [21], hold significant prevalence across various disciplines. These intricately crafted curves are derived through the manipulation of a parameter, p, which is $0\leq p\leq 1$.

$$B\left(p\right)=\frac{1}{p\mu}\int_{0}^{F^{-1}(p)}xf\left(x\right)dx$$

By taking $q=F^{-1}\left(p\right)$, and substituting (3) and (7), we have

$$B\left(p\right)=\frac{1}{p\ a_{m,n,2}}\int_{0}^{q}x^{\alpha+1}\left(1-x\right)\left(2-x\right)^{\alpha-1}dx$$

By applying binomial expansion,

$$B\left(p\right)=\frac{1}{pa_{m,n,2}}\sum_{m=0}^{\infty}\sum_{n=0}^{m+1}\binom{m+1}{n}\frac{\left(-1\right)^{n}}{m!}\frac{\Gamma\left(\alpha\right)}{\Gamma\left(\alpha-m\right)}\int_{0}^{q}x^{\alpha+n+1}dx$$

hence

$$B\left(p\right)=\frac{1}{p\ a_{m,n,2}}\sum_{m=0}^{\infty}\sum_{n=0}^{m+1}\binom{m+1}{n}\frac{\left(-1\right)^{n}}{m!\left(\alpha+n+2\right)}\frac{\Gamma\left(\alpha\right)}{\Gamma\left(\alpha-m\right)}q^{\alpha+n+2}$$

By substituting (2),

$$B\left(p\right)=\frac{q^{\alpha+n+2}}{p}$$

and

$$L\left(p\right)=\frac{1}{\mu}\int_{0}^{F^{-1}\left(p\right)}xf\left(x\right)dx$$

then

$$L\left(p\right)=q^{\alpha+n+2}$$

□

3.4 Order statistics

Order statistics hold a paramount position within the realm of nonparametric statistics and inference, serving as foundational tools. Let us consider $X1;\ X2;\ldots;Xn$ as a randomly selected complete sample from Eq. (3).

Theorem 3.6. *Let us denote the sequence of $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$, as the order statistics [18]. It is widely acknowledged that the probability density function (pdf) for the r^{th} order statistic can be expressed as follows:*

$$\begin{aligned}f_{X_{(r)}}\left(x\right)=&\frac{n!}{\left(r-1\right)!\left(n-r\right)!}\left(\frac{x^{\alpha}\left(1-x\right)\left(2-x\right)^{\alpha-1}}{a_{m,n,1}}\right)\\&\times\left(\sum_{m=0}^{\alpha-1}\binom{\alpha-1}{m}\frac{1}{a_{m,n,1}}B(\alpha,m,x)\right)^{r-1}\\&\times\left(1-\sum_{m=0}^{\alpha-1}\binom{\alpha-1}{m}\frac{1}{a_{m,n,1}}B(\alpha,m,x)\right)^{n-r}.\end{aligned}\tag{14}$$

Proof. We have

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!\,(n-r)!} f_X(x)(F_X(x))^{r-1}(1-F_X(x))^{n-r}$$

By utilizing Equations (3) and (4), it is possible to demonstrate that the density function pertaining to the n^{th} and first-order statistics of any LBTL distribution can be eloquently expressed.

$$f_{X_{(n)}}(x) = \frac{nx^{\alpha}(1-x)(2-x)^{\alpha-1}}{a_{m,n,1}} \left(\sum_{m=0}^{\alpha-1} \binom{\alpha-1}{m} \frac{1}{a_{m,n,1}} B(\alpha,m,x) \right)^{n-1}$$

By taking $n = 1$, we have

$$f_{X_{(1)}}(x) = \frac{nx^{\alpha}(1-x)(2-x)^{\alpha-1}}{a_{m,n,1}} \left(1 - \sum_{m=0}^{\alpha-1} \binom{\alpha-1}{m} \frac{1}{a_{m,n,1}} B(\alpha,m,x) \right)^{n-1}$$

□

4 Parameters Estimation

In this section, the discussion revolves around the estimation of parameters in LBTL. The method for obtaining these estimations is maximum likelihood estimation (MLE), derived from a complete sample.

4.1 Maximum likelihood estimators

Consider a complete sample of size n, denoted by $x1, x2, \ldots, xn$, extracted from the $LBTL(\alpha)$ distribution. It is imperative to determine the likelihood function for this.

$$L(\alpha \mid \underline{x}) = \prod_{i=1}^{N} f(x_i;\alpha) = \frac{\prod_{i=1}^{N} x_i^{\alpha} \prod_{i=1}^{N}(1-x_i) \prod_{i=1}^{N}(2-x_i)^{\alpha-1}}{a_{m,n,1}}$$

The log-likelihood function may be formulated as

$$\ell = \ln L = \alpha \sum_{i=1}^{N} \ln x_i + \sum_{i=1}^{N} \ln(1-x_i) + (\alpha-1) \sum_{i=1}^{N} \ln(2-x_i) - N \ln a_{m,n,1}$$

Formulating the derivatives stemming from the aforementioned equation with regards to α, and subsequently equating the ensuing outcomes to a value of zero.

$$\frac{\partial \ell}{\partial \alpha} = \sum_{i=1}^{N} \ln x_i + \sum_{i=1}^{N} \ln(2-x_i)$$
$$-\frac{N}{a_{m,n,1}} \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{m+1} \frac{(-1)^n}{m!} \binom{m+1}{n} \left[\frac{-1}{(1+n+\alpha)^2} \frac{\Gamma(\alpha)}{\Gamma(\alpha-m)} + \frac{1}{1+n+\alpha} \Theta_\alpha \right] \right\}$$
$$=0$$

Where

$$\Theta_\alpha = \frac{d}{d\alpha} \frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}$$

5 Simulation Study

In this simulation, the utilization of R software delves into a thorough examination involving 1000 samples of considerable magnitude N, encapsulating the LBTL(α) distribution. For every individual case, the paramount task at hand is to estimate the parameter α via both the maximum likelihood method as well as Newton-Raphson method. Subsequently, an assessment is made regarding both the estimated value and MSE (Mean Squared Error) pertaining to said parameter α.

$$\widehat{\alpha} = \frac{1}{1000} \sum_{i=1}^{1000} \widehat{\alpha}_i$$
$$MSE = Var(\widehat{\alpha}_i) + (\widehat{\alpha} - \alpha)^2$$

In every instance, the experiment was conducted using a total of 1000 samples. These samples were divided into different sizes: N =30, 50, 100, 150, and 200. Additionally, varying significance levels of α were employed (ranging from 0.1 to 10 with increments of 1). The data presented in Table 3 was derived from an extensive execution of the Newton-Raphson algorithm - specifically carried out a staggering number of times: totaling at least sixty thousand iterations. For example, the initial entry under Table 3 signifies the averaged results obtained from one thousand separate instances where thirty individual LBTL samples were analyzed (significance level set at $\alpha = 0.1$).

Table 3: MSE, Bias, and MLE of the LBTL distribution for various parameters and sample sizes.

	$n = 30$		$n = 50$		$n = 100$		$n = 150$		$n = 200$	
α	$\widehat{\alpha}$	MSE Bias	$\widehat{\alpha}$	MSE Bias	$\widehat{\alpha}$	MSE Bias	$\widehat{\alpha}$	MSE Bias	$\widehat{\alpha}$	MSE Bias
$\alpha = .5$	1.0002	0.2502	1.0001	0.2501	1.0000	0.2500	0.9977	0.2482	0.9895	0.2459
		-0.5002		-0.5001		-0.500		-0.4977		-0.4895
$\alpha = 1$	1.0594	0.0626	1.0280	0.0275	1.0019	0.0010	1.0018	0.0010	1.0008	0.0000
		-0.0594		-0.0280		-0.0019		-0.0018		-0.0008
$\alpha = 2$	2.0910	0.5529	2.0071	0.3138	1.9988	0.1880	1.9972	0.1138	1.9981	0.0762
		-0.0910		-0.0071		0.0012		0.0028		0.0019
$\alpha = 3$	3.1617	0.8366	3.1008	0.4504	3.0647	0.2093	3.0402	0.1437	3.0502	0.0952
		-0.1617		-0.1008		-0.0647		-0.0402		-0.0502
$\alpha = 4$	4.2021	1.1871	4.0881	0.6035	4.0387	0.3045	4.0473	0.1908	4.0158	0.1371
		-0.2021		-0.0881		-0.0387		-0.0473		-0.0158
$\alpha = 5$	5.2701	1.8296	5.1064	1.0091	5.0527	0.4675	5.0325	0.3291	5.0158	0.2366
		-0.2701		-0.1064		-0.0527		-0.0325		-0.0158
$\alpha = 6$	6.2048	2.4199	6.0809	1.6578	5.9718	0.9520	5.9419	0.6885	5.9722	0.6779
		-0.2048		-0.0809		0.0282		0.0581		0.0278
$\alpha = 7$	7.1177	3.1753	7.1260	2.0418	6.9764	1.4649	6.9780	1.4614	6.9937	1.1618
		-0.1177		-0.1260		0.0236		0.0220		0.0063
$\alpha = 8$	8.3377	4.4307	8.1191	2.2626	8.0614	1.2837	8.0627	0.9392	8.0210	0.7255
		-0.3377		-0.1191		-0.0614		-0.0627		-0.0210
$\alpha = 9$	9.3099	4.6088	9.1081	2.5521	9.0929	1.2866	9.1556	0.8699	9.0412	0.5480
		-0.3099		-0.1081		-0.0929		-0.1556		-0.0412
$\alpha = 10$	10.3962	5.1104	10.1594	3.1253	10.1106	1.4180	10.1116	0.9357	10.0412	0.6698
		-0.3962		-0.1594		-0.1106		-0.1116		-0.0412

Table 3 displays a notable trend: as the sample size increases, the MSE values consistently diminish across all α values. This indicates the estimators' consistency and dependability. Conversely, when examining each sample size individually, it becomes apparent that an escalation in parameter value corresponds with an amplified MSE.

6 Applications

Within this particular segment, a meticulous examination of two authentic sets of data ensues with the intent to confirm the versatility and practicality encompassed within the LBTL distribution. Specialized analytical metrics have been systematically employed to ascertain the optimal distribution amidst fierce contention among various alternatives.

6.1 Data set 1

The data set 1 pertains to the cumulative milk yield of 107 SINDI breed cows during their initial calving. These cows are under the ownership of Agropecuária Manoel Dantas Ltda (AMDA) and reside on the Carnaba farm located in Taperoá City, Paraná, Brazil. The necessity for the following alteration arose as the original figures exceeded the prescribed range of values between 0 and 1.

$$x_i = \frac{y_i - \min(y_i)}{\max(y_i) - \min(y_i)}, \quad i = 1, \ldots, 107$$

According [44], the author provides a list of inherent values denoted as y_i. Additionally, the renowned scholar also presents a series of x_i values associated with these aforementioned elements.

0.5140, 0.6907, 0.7471, 0.2605, 0.6196, 0.8781, 0.4990, 0.6058, 0.6891, 0.5770, 0.5394, 0.1479, 0.2356, 0.6012, 0.1525, 0.5483, 0.6927, 0.7261, 0.3323, 0.0671, 0.2361, 0.4800, 0.5707, 0.7131, 0.5853, 0.6768, 0.5350, 0.4151, 0.6789, 0.4576, 0.3259, 0.2303, 0.7687, 0.4371, 0.3383, 0.6114, 0.3480, 0.4564, 0.7804, 0.3406, 0.4823, 0.5912, 0.5744, 0.5481, 0.1131, 0.7290, 0.0168, 0.5529, 0.4530, 0.3891, 0.4752, 0.3134, 0.3175, 0.1167, 0.6750, 0.5113, 0.5447, 0.4143, 0.5627, 0.5150, 0.0776, 0.3945, 0.4553, 0.4470, 0.5285, 0.5232, 0.6465, 0.0650, 0.8492, 0.8147, 0.3627, 0.3906, 0.4438, 0.4612, 0.3188, 0.2160, 0.6707, 0.6220, 0.5629, 0.4675, 0.6844, 0.3413, 0.4332, 0.0854, 0.3821, 0.4694, 0.3635, 0.4111, 0.5349, 0.3751, 0.1546, 0.4517, 0.2681, 0.4049, 0.5553, 0.5878, 0.4741, 0.3598, 0.7629, 0.5941, 0.6174, 0.6860, 0.0609, 0.6488, 0.2747.

Table 4 presents the comprehensive statistical analysis of this data collection. Figure 2 visually portrays the balanced distribution of data with a symmetric density value of 0.48, as derived from the median and mean statistics.

The data density portrayed in Figure 2 conveys the harmonious symmetry of the LBTL(α) distribution at 1, while the information presented

Table 4: Descriptive statistics of the data set (1)

Min.	Q1	MEDIAN	Mean	Q3	MAX
0.0609	0.3874	0.4881	0.4872	0.5959	0.8781

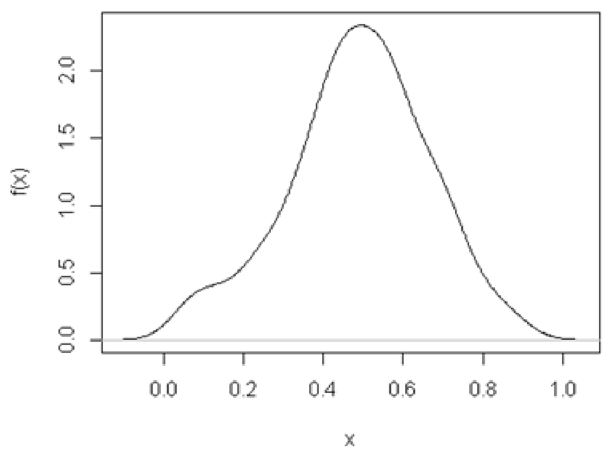

Figure 2: The density of the data set (1)

in Table 5 demonstrates that a likelihood estimate of 1 has been determined for this particular dataset.

Table 5 shows The LBTL distribution fits these data better than the Beta Topp-Leone Generated (BTLG) [43], modified Kies exponential (MKE) [9], Type II Top-Leone Inverse Lomax($TIITLIL$) [34], Beta, Kumaraswamy and Truncated exponential Topp Leone exponential (TETLE) [11].

In Figure 3, it becomes apparent that the LBTL model, with an α value of 1.001217, exhibits a superior alignment with the data at hand. Conversely, when considering the range of LBTL(α) distributions, it can be deduced that the LBTL(α=1) variant is particularly well-suited for symmetrical data sets.

Table 5: Parameter estimates (the standard errors in parentheses) and goodness of fit criterion for data set 1

Model	Parameter estimates (S.E)	AIC	BIC	-log-likelihood	K-S
LBTL	$\hat{\alpha} = 1.0012$ (0.0261)	44.9531	-42.3888	23.4766	0.4740
BTLG	$\hat{a} = 0.0716$ (0.0148) $\hat{b} = 0.0563$ (0.0191) $\hat{c} = 0.7562$ (0.0011) $\hat{\alpha} = 3.749e - 8$ (¡0.0001) $\hat{\lambda} = 6.5937$ (0.0005)	57.8319	-40.3702	28.5802	0.8778
MKE	$\hat{\alpha} = 2.7137$ (0.4491) $\hat{\lambda} = 0.2351$ (0.0049)	63.2607	-36.5184	33.1024	0.9180
$TIITLIL$	$\hat{\xi} = 0.0103$ (0.0041) $\hat{b} = 0.4179$ (0.0429) $\hat{\lambda} = 0.7362$ (0.2001) $\hat{\nu} = 6.6219$ (6.4343)	53.8625	-39.7399	26.6887	0.7739
TETLE	$\hat{\alpha} = 17.0429$ (2.4730) $\hat{\theta} = 0.3254$ (0.2140) $\hat{\lambda} = 14.3812$ (11.9537)	59.7701	-41.0065	36.1252	0.9941
Beta	$\hat{\alpha} = 3.2539$ (0.0206) $\hat{\beta} = 4.9876$ (0.8041)	48.7251	-41.2138	25.1463	0.6788
Kw	$\hat{a} = 5.2901$ (0.0017) $\hat{b} = 0.7619$ (0.0382)	49.6312	-38.8119	24.0538	0.7185

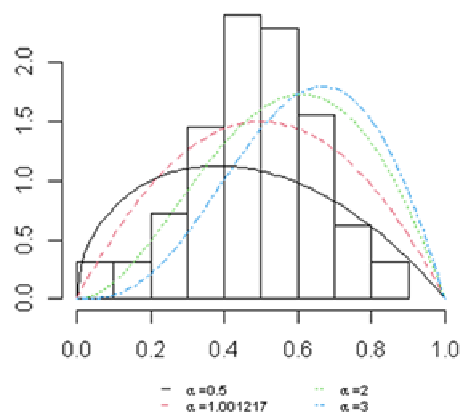

Figure 3: Histigram of data set 1.

6.2 Data set 2

This segment employs a series of actual datasets to execute the proposed technique. These datasets can be found in the medical data package within R software, specifically under the category of income. It is important to highlight that we specifically utilize the conc variable within this dataset, which consists of a total of 66 observations. To transform this data into a range between 0 and 1, we have implemented the minimum and maximum transformation methods. It should be noted that this transformation method does not alter the distribution's shape; rather, it simply places the data within an interval ranging from 0 to 1. Consequently, any values that are converted to either zero or one through this process are excluded from further analysis. Based on The Minimax Formula, these two extreme values (i.e., min and max) serve as primary references for transformating all other data points:

$$z_{\min\max} = \frac{x - \min(x)}{\max(x) \ - \min(x)}$$

orginal data

[1]	1.50	0.94	0.78	0.48	0.37	0.19	0.12	0.11	0.08	0.07	2.03	1.63	0.71	0.70
[15]	0.64	0.36	0.32	0.20	0.25	0.12	0.08	1.49	1.16	0.80	0.80	0.39	0.22	0.12
[29]	0.11	0.08	0.08	1.85	1.39	1.02	0.89	0.59	0.40	0.16	0.11	0.10	0.07	0.07
[43]	2.05	1.04	0.81	0.39	0.30	0.23	0.13	0.11	0.08	0.10	0.06	2.31	1.44	1.03
[57]	0.84	0.64	0.42	0.24	0.17	0.13	0.10	0.09						

¿ converted data with minimax

[1]	0.543071161	0.333333333	0.273408240	0.161048689	0.119850187	0.052434457
[7]	0.026217228	0.022471910	0.011235955	0.007490637	0.741573034	0.591760300
[13]	0.247191011	0.243445693	0.220973783	0.116104869	0.101123596	0.056179775
[19]	0.074906367	0.026217228	0.011235955	0.539325843	0.415730337	0.280898876
[25]	0.280898876	0.127340824	0.063670412	0.026217228	0.022471910	0.011235955
[31]	0.011235955	0.674157303	0.501872659	0.363295880	0.314606742	0.202247191
[37]	0.131086142	0.041198502	0.022471910	0.018726592	0.007490637	0.007490637
[43]	0.749063670	0.370786517	0.284644195	0.127340824	0.093632959	0.067415730
[49]	0.029962547	0.022471910	0.011235955	0.018726592	0.003745318	0.846441948
[55]	0.520599251	0.367041199	0.295880150	0.220973783	0.138576779	0.071161049
[61]	0.044943820	0.029962547	0.018726592	0.014981273		

Figure 4 depicts the density function of their pristine and standardized data, with exclusions made for observations possessing values of zero or one in abbreviated statements. Additionally, the inclusion of exact values within introductory phrasing has been eliminated. As common knowledge affirms, there exists no discernable variance between the distribution patterns exhibited by both the genuine and transformated data sets.

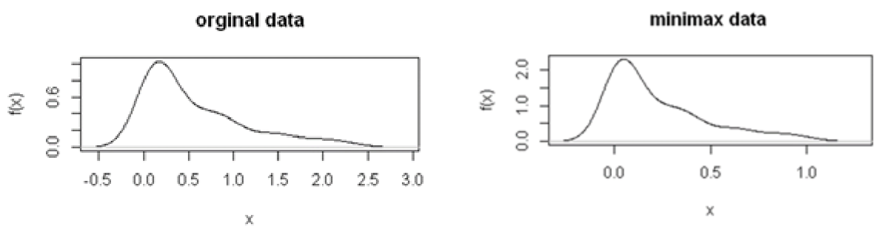

Figure 4: The distribution of the original and transformated data set 2

In Figure 4, it is evident that when the distribution is skewed to the right, the alpha parameter assumes a value smaller than 0.5. This statistical observation is further supported by Table 6.

Table 6 shows the novel distribution fits this data set better than the transmuted Topp-Leone beta (TTLB) [44], beta Weibull(BW) [14], beta, Kumaraswamy, inverted length-biased exponential (ILBE) [11], and Type II Topp-Leone inverse length biased exponential (TIITL-ILBE) [11]. The goodness-of-fit measures for the fitted LBTL model and other fitted distributions to both data sets are presented in Table 5 and Table 6, respectively.

Where the standard errors are obtained by taking the square root of the diagonal elements of the covariance matrix. we can also perform these computations by minimizing the negative of the log-likelihood function.

The LBTL(α) distribution possesses the remarkable capability to aptly capture both skewness and symmetry. It proves well-suited for data that exhibit a right-sided, as well as symmetric, nature with an utmost elongation or height of 1.5. In addition, it proficiently accommodates left-sided data of maximum magnitude equaling 2.

Table 6: Parameter estimates (the standard errors in parentheses) and goodness of fit criterion for data set 1

Model	Parameter estimates (S.E)	AIC	BIC	-log-likelihood	K-S
LBTL	$\hat{\alpha} = 0.02198$ (0.0026)	0.4794	10.8514	12.8616	0.2370
TTLB	$\hat{\alpha} = 0.8239$ (0.3976) $\hat{\lambda} = 0.5389$ (0.3895) $\hat{a} = 0.3184$ (0.0143) $\hat{b} = 2.531$ (0.7528)	29.6540	26.7867	14.0198	0.8873
BW	$\hat{\alpha} = 0.5720$ (0.0278) $\hat{\beta} = 5.4520$ (0.0621) $\hat{\lambda} = 5.8769$ (0.2741) $\hat{\theta} = 3.7361$ (0.1792)	27.3158	29.2791	13.7682	0.6022
ILBE	$\hat{\alpha} = 5.7980$ (0.8351)	31.4792	34.7103	17.4582	0.9953
TIITL-ILBE	$\hat{\eta} = 18.1831$ (7.7422) $\hat{\alpha} = 3.0056$ (0.6381)	26.3300	29.6191	13.9775	0.7232
Beta	$\hat{\alpha} = 6.9108$ (0.0018) $\hat{\beta} = 8.2375$ (0.0546)	20.9351	12.4913	16.5129	0.3136
Kw	$\hat{a} = 3.5912$ (0.1684) $\hat{b} = 1.7838$ (0.0251)	21.5918	22.6873	18.4914	0.8500

7 Conclusion

Within this paper, we put forth and meticulously examine a novel classification of distributions named Length-Biased Topp-Leone. Throughout our discourse, we introduce certain distinctive distributions and delve into an exploration of numerous structural properties related to this class. This investigation encompasses the expansions for the density function as well as explicit expressions for various measures such as the quantile function, ordinary and incomplete moments, a generating function, Rényi entropy, and distribution of order statistics.To facilitate accuracy in our findings, we employ the maximum likelihood method to estimate the pertinent parameters. Furthermore, to ascertain the veracity and reliability of these estimations through empirical means, we conduct a Monte Carlo simulation study on two specific cases which allows us to evaluate their performance within finite samples. Real data examples are also included in this paper to provide tangible evidence regarding the significance and potential inherent within our proposed family. Upon careful examination and comparison with existing models of Topp-Leone distributions showcased herein, this proposed distribution demonstrates superior efficacy.

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