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A Study on Fuzzy Fractional Equation using Laplace Transform in Quantum environment

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Abstract. This paper explores quantum q-fractional differential equation in the Caputo sense. The primary focus is on an equation incorporating a q-derivative and an unknown function $f(x)$. The existence of solutions is established using the q-Laplace transform and q-Mittag-Leffler function. The study also incorporates a fuzzy-valued function in the Caputo q-fractional differential equation, solving it with the q-Laplace transform. Theoretical findings are supported by numerical results. Furthermore, we examine the Hyers-Ulam-Rassias stability of the equation, confirming their stability.

AMS Subject Classification: 34A07, 34A08, 44A10
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1 Introduction

Quantum calculus, commonly referred to as q-calculus is a generalization of conventional calculus that allows for the consideration of non-

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commutative and non-differentiable elements. It is mostly employed in the discipline of quantum mechanics, where the fundamental laws of quantum physics prevent the straightforward application of standard calculus. Extending the ideas of differentiation, integration, and other basic operations to non-commutative spaces is the major objective of quantum calculus [1]. This has the advantage of making it possible to create mathematical techniques that are more suited for researching quantum phenomena and systems. In this work, the quantum concept incorporated with a fractional differential equation is solved along with numerical simulation.

In the early period of quantum calculus, it was represented as the linkage of physics and mathematics. q-operator by Jackson created the renowned illustration of quantum calculus. Recently, the q-calculus has been employed in various scientific fields such as the theory of relativity, particle physics, quantum theory, computing, electroanalytical chemistry, neurology and it is used in the mathematical fields such as combinatorics, statistics, control theory and orthogonal polynomials. The quantum calculus was first put into application by Jackson, who also popularized the derivative's q-analogue and q-integral. The development in quantum calculus is growing enormously and expanding widely in numerous areas. In addition, the quantum field in computing has escalated the requirement of mathematics in the computer science domain. Quantum calculus became a crucial component in quantum mechanics, which has a major aspect in quantum computing. In approximation theory, various operators are described using q-calculus [2]. Along with the numerous benefits of quantum calculus, a new category of harmonic functions was constructed using the concept of Le Roy-type Mittag Leffler functions in [3]. Several academics have lately applied q-calculus to the domain of geometric functions, resulting in the creation of new classes and also harmonic functions. Further, q-calculus has its imprint in image processing to increase the efficiency and precision of the algorithms. The quantum calculus based local fractional entropy was consistent when compared to other operators in [4].

The generalization of classical calculus is fractional calculus and is useful in formulating and finding the solution for variational problems. Many real-world phenomena exhibit behavior that cannot be captured by integer-order derivatives. However, Caputo q-fractional differential equations allow for a more flexible description of these fractional-order dynamics. By the progressive work of many pioneers such as Bernoulli, Riemann, Leibniz, Liouville, Euler and so on, the field of fractional calculus has attracted a large number of researchers [5]. The fractional calculus, which is the integration and differentiation of arbitrary order, is used in Engineering fields such as Control Engineering, Signal Processing, financial risk management etc.,[6], [7] and [8]. Notably, the fractional differential equation is utilized in many areas such as Biology, Chemistry and Physics. The benefit of fractional derivatives in conjunction with Artificial Neural Network and its effect on performance indices are emphasized in [9]. Also in [10], the potential of a fractional deterministic framework as a viable place to begin for developing a suitable model representing tumor evolution is illustrated.

Fuzzy sets and fuzzy logic have a remarkable influence on the development of various concepts in numerous scientific fields. The name fuzzy differential equation first arose in literature in 1978, and depending on the fuzzy derivative (Dubois- Prade derivative) the fuzzy differential equation known present-day emerged [11]. Later various fuzzy derivative definitions were considered. In 2010, R.P. Agarwal et.al. [12] suggested the fractional calculus incorporated with the fuzzy concept. The possible combination of derivatives such as Caputo, Riemann-Liouville, Conformable, Modified R-L, etc., along with the fuzzy derivatives gave the origin to the fuzzy fractional differential equation. Since then, the topic of uncertain fractional differential equations has seen significant advancement. The uncertain fractional equations are resolved with the application of fuzzy Laplace transform in [13]-[15]. Koca [16] suggested an analytical method to find the solution of the fractional differential equation.

Zhang et.al. solved quantum related fractional equation by utilising the difference formula [17]. Notably, various methods are involved in solving q-differential equation of fractional order. In [18] Noeiaghdam et.al. combined the quantum calculus with fractional derivative and gH derivative. The q-Mittag-Leffler function and method of successive approximation are used in solving the fractional q-differential equation. The q-fractional derivative along with the proportional derivative [19] is proposed and solved by Laplace transform in the quantum sense. The q-differential equation of fractional order class is converted to its equivalent integer-order differential equation in quantum sense [20] and its solution is established. For delay q-fractional difference equation, its solution and Hyers-Ulam and Hyers-Ulam-Rassias stability are discussed in [21].

Inspired by the above thoughts, the intent of this paper is to employ the q-Laplace transform to solve the fuzzy fractional differential equation of quantum calculus in the Caputo sense. Further, the fuzzy-valued Caputo q-fractional differential equation is solved by employing the q-Laplace transform. The numerical examples are solved using the above method and the stability of the equation is analyzed.

This paper is organized as the following: Section (2) is enclosed by preliminaries, properties and basic results. The Caputo q-fractional initial value problem is solved by using q-Laplace transform in section (3). In section (4), two examples are considered and solved numerically. The fuzzy q-fractional initial value problem for the fuzzy valued function is solved using q-Laplace transform in section (5). In section (6) the numerical example for the above method is provided. Finally, the section (7) is devoted to examining the stability of the Caputo q-fractional differential equation using Hyers-Ulam-Rassias stability.

2 Preliminary Results and Notations

Before going into the main part and proving the fuzzy q-fractional differential equation, the basic results required are provided. Here are a few properties and definitions of fractional calculus, which are necessary to derive a fuzzy q-fractional solution for the proposed equation.

Definition 1. [18] The Riemann Liouville fractional integral for the function $y(t)$ of order $\alpha > 0$ is defined as,

$$I_a^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s) ds, \quad I_a^0 y(t) = y(t).$$

where, Γ denotes Euler gamma function.

Definition 2. [18] The function $y(t)$ under fractional order denotes the

Caputo derivative as,

$$^{C}D^{\alpha}y(t) = \frac{1}{\Gamma(k-\alpha)}\int_{a}^{t}(s-t)^{k-\alpha-1}D^{k}y(s)ds, \quad k-1<\alpha<k, \quad t>a, \quad k\in\mathbb{N}$$

where the function $y:[a,b]\to\mathbb{R}$ and $D^{k}y(s)$, for all k, are integrable.

The following are some characteristics of Caputo fractional differential equations.

1. $^{C}D^{\alpha}y(t) = I_{a}^{k-\alpha}D^{k}y(s)ds,$
2. $I_{a}^{\alpha C}D^{\alpha}y(t) = y(t)-y(a) \quad \text{for } 0<\alpha<1.$

For other results and properties related to fuzzy fractional calculus, refer [22]. Now, some of the required results and the fundamental definition of q-calculus, that is, calculus without limits are given here.

Quantum calculus is a branch of mathematics that explores alternate approaches for differentiation and integration when compared to classical calculus. It is also known as "calculus without limits". The fundamental properties required in solving the quantum fractional differential equation are stated here concisely.

The familiar classical calculus derivative given by $\frac{dy}{dx}$ is obtained from

$$\frac{y(x)-y(x_0)}{x-x_0}$$

provided, if there exist the limit as x approaches x_0.

The timescale $\mathbb{T}_q$ [23], for the condition $0<q<1$

$$\mathbb{T}_q = \{q^n : n\in\mathbb{Z}\}\cup\{0\},$$

where $\mathbb{Z}$ denotes the set of integers. For non negative real number α, [18]

$$\mathbb{T}_q^{\alpha} = \{q^{\alpha+n} : n\in\mathbb{Z}\}\cup\{0\}.$$

Definition 3. [1] In quantum calculus, the term "quantum derivative" usually refers to the q-derivative, also called the Jackson derivative. It is an alternate definition of differentiation that does not rely on limits, compared to the classical derivative. The q-derivative of the function $y(x)$ is given by

$$D_q y(x) = \frac{d_q y(x)}{d_q x} = \frac{y(qx) - y(x)}{(q-1)x}$$

Various results and properties related to quantum calculus can be referred to from [1]. The q-fractional integral and Caputo derivative are provided here briefly,

Definition 4. [18] The q-fractional integral for the function $y(t)$ that contains order $\alpha > 0$ is defined as,

$$_qI_a^{\alpha} y(t) = \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} y(s) d_q s, \quad I_a^0 y(t) = y(t). \tag{1}$$

Definition 5. [18] The Caputo q-derivative of $y(t)$ under fractional order is termed as follows

$$^C_q D^{\alpha} y(t) = \frac{1}{\Gamma_q(k-\alpha)} \int_a^t (t - qs)_q^{k-\alpha-1} D_q^k y(s) d_q s, \tag{2}$$

where $\alpha \notin \mathbb{N}$ and D_q^k are q-integrable and continuous function for all k.

The q-gamma function, denoted by $\Gamma_q(\alpha)$, can be calculated by

$$\Gamma_q(\alpha) = \frac{(1-q)_q^{\alpha-1}}{(1-q)^{\alpha-1}}, \quad \alpha \in \mathbb{R}/\{0\} \cup \mathbb{Z}_-, \quad 0 < q < 1,$$

which satisfies the following recurrence relation:

$$\Gamma_q(\alpha+1) = \frac{1-q^{\alpha}}{1-q} \Gamma_q(\alpha), \quad \Gamma_q(1) = 1, \quad \alpha > 0.$$

Theorem 2.1. *[23] Consider $\mu \in (0, \infty)$, then*

$$_qI_a^{\alpha} (t-a)_q^{\mu} = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\alpha+\mu+1)} (t-a)_q^{\gamma+\alpha}$$

where $0 < a < x < b$. As a particular case, when $\gamma = 0$, $_qI_a^{\alpha}(1) = \frac{1}{\Gamma_q(\alpha+1)} (t-a)_q^{\alpha}$.

Theorem 2.2. *[25] Consider* $0 < a < x < b$, *then*

$$\int_a^t (t-qs)_q^{\alpha-1} d_q s = \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha+1)}(t-a)_q^{\alpha}.$$

where $0 < q < 1$.

3 Caputo q-Fractional Differential Equation and q-Laplace Transform

The Laplace transform and its different variants are discussed in [24]. The benefit of utilizing the q-Laplace transform lies in its ability to find the solution and transform the q-fractional differential equation to a simpler form.

Definition 6. The q-Laplace transform for $f(x)$ is defined by the following

$$_qL_s[f(x)] = \int_0^{\infty} e_q(-sx) f(x) d_q x, s > 0 \tag{3}$$

where the notation e_q denotes q-exponential formula which is defined by,

$$e_q(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma_q(k+1)}$$

on the timescale $\mathbb{T}_q$.

Similar to the Laplace transform, the q-Laplace transform for derivatives can be obtained by applying the definition of the q-Laplace transform. The q-Laplace transform for $_qD^m f(x)$ is given as

$$_qL_s(_qD^m f(x)) = s^m q^{-\binom{m+1}{2}} F(q^{-m}s) - \sum_{i=0}^{m-1} s^{m-1-i} q^{-\binom{m-i}{2}} f^{(i)}(0)$$

where $0 < q < 1$. Assuming the function g and h as piecewise continuous, we get the q-convolution property as

$$(g * h)(t) = \int_0^t g(\tau) h(t - q\tau) d_q t,$$

$$\int_a^x (x-qt)_q^{m-\alpha-1} g(x) d_q t = (x)_q^{m-\alpha-1} g(x). \tag{4}$$

Definition 7. [23] Let $y, y_0 \in \mathbb{T}_q$, the q-Mittag-Leffler function where $\alpha > 0$ is

$$_qE_{\alpha,\beta}(\lambda, y-y_0) = \sum_{l=0}^{\infty} \lambda^l \frac{(y-y_0)_q^{\alpha l}}{\Gamma_q(\alpha l+\beta)},$$

when $\beta = 1$, the equation $_qE_{\alpha,1}(\lambda, y-y_0) =_q E_\alpha(\lambda, y-y_0)$ is obtained.

Some of the properties required in solving the equations are proposed below. For other properties and remarks connected with q-Laplace transform, refer [24]. For $\alpha > -1$ and $\alpha \in \mathbb{R}$,

$$\begin{aligned} {}_qL(1) &= \frac{1}{s} \\ {}_qL(t) &= \frac{1}{s^2} \\ {}_qL(t^\alpha) &= \frac{1}{s^{\alpha+1}}\Gamma_q(\alpha+1) \end{aligned} \tag{5}$$

The principal part of this section is applying the q-Laplace transform to solve a fuzzy q-fractional equation.

Theorem 3.1. *Assume $0 < \alpha \le 1$ and $0 < q < 1$. Let $f(x)$ be q-differentiable function, then there exist solution for the q-differential equation of fractional order in Caputo sense,*

$$^C D_q^\alpha f(x) = \frac{1}{\lambda} f(x), \tag{6}$$

where $\lambda \in \mathbb{R}$ with the initial condition $f^i(0) = f_0 \in \mathbb{R}$.

Proof. The q-differentiable function $f(x)$ involves q-derivatives of non-integer order and plays a vital role in solving q-fractional differential equations. The q-Laplace transform is applied to both sides of (6) which can be solved as,

$$\begin{aligned} {}_qL_s({}^C D_q^\alpha f(x)) &= {}_qL_s[{}_qI_a^{m-\alpha}({}_qD^m)f(x)] \\ &= {}_qL_s[\frac{1}{\Gamma_q(m-\alpha)}\int_a^x (x-qt)_q^{m-\alpha-1}({}_qD^m f(x))d_qt] \\ &= {}_qL_s[\frac{1}{\Gamma_q(m-\alpha)}\int_a^x (x-qt)_q^{m-\alpha-1} g(x)d_qt] \\ &= {}_qL_s[\frac{1}{\Gamma_q(m-\alpha)}(x)_q^{m-\alpha-1} g(x)] \end{aligned}$$

where $g(x) =_q D^m f(x)$. Additionally, by using (5), the following equations are derived,

$$\begin{aligned}
{}_qL_s({}^C D_q^\alpha f(x)) &= \frac{1}{\Gamma_q(m-\alpha)}\frac{1}{s^{m-\alpha}}\Gamma_q(m-\alpha){}_qL_s g(x) \\
&= \frac{1}{s^{m-\alpha}}{}_qL_s({}_qD^m f(x)) \\
&= \frac{1}{s^{-\alpha}}[q^{-\binom{m+1}{2}}F(q^{-m}s) - \sum_{i=0}^{m-1} s^{-1-i}q^{-\binom{m-i}{2}}f_0] \\
{}_qL_s[\frac{1}{\lambda}f(x)] &= \frac{1}{\lambda s}F(s)
\end{aligned}$$

Further, the equations are computed as follows

$$\begin{aligned}
s^{\alpha+1}\lambda[q^{-\binom{m+1}{2}}F(q^{-m}s) - \sum_{i=0}^{m-1} s^{-1-i}q^{-\binom{m-i}{2}}f_0] &= F(s) \\
\lambda q^{-\binom{m+1}{2}}F(q^{-m}s) - \sum_{i=0}^{m-1}\frac{\lambda}{s^{1+i}}q^{-\binom{m-i}{2}}f_0 &= \frac{F(s)}{s^{\alpha+1}} \\
\lambda q^{-\binom{m+1}{2}}{}_qL_s f(q^{-m}x) - \sum_{i=0}^{m-1}\frac{\lambda}{s^{1+i}}q^{-\binom{m-i}{2}}f_0 &= \frac{1}{\Gamma_q(\alpha+1)}{}_qL_s(x^\alpha){}_qL_s f(x)
\end{aligned} \tag{7}$$

By applying the inverse Laplace transform,

$$\begin{aligned}
\lambda q^{-\binom{m+1}{2}}f(q^{-m}x) - \sum_{i=0}^{m-1}{}_qE_{i,1}(\lambda,x)q^{-\binom{m-i}{2}}f_0 &= \frac{1}{\Gamma_q(\alpha+1)}(x^\alpha)f(x) \\
\lambda q^{-\binom{m+1}{2}}f(q^{-m}x) - \sum_{i=0}^{m-1}\frac{\lambda x^\alpha}{\Gamma_q(i+1)}q^{-\binom{m-i}{2}}f_0 &= \frac{1}{\Gamma_q(\alpha+1)}(x^\alpha)f(x)
\end{aligned}$$

For solving numerically, the equation (7) can be rewritten as,

$$\begin{aligned}
\lambda q^{-\binom{m+1}{2}}(q^m\sum_{j=0}^{\infty}(q^{m^j}){}_qL_s f(x)) - \sum_{i=0}^{m-1}\frac{\lambda}{s^{1+i}}q^{-\binom{m-i}{2}}f_0 &= \frac{1}{\Gamma_q(\alpha+1)}{}_qL_s(x^\alpha){}_qL_s f(x) \\
\lambda q^{-\binom{m+1}{2}}(q^m\sum_{j=0}^{\infty}(q^{m^j})f(x)) - \sum_{i=0}^{m-1}\frac{\lambda x^\alpha}{\Gamma_q(i+1)}q^{-\binom{m-i}{2}}f_0 &= \frac{1}{\Gamma_q(\alpha+1)}(x^\alpha)f(x)
\end{aligned}$$

$$\lambda q^{-\binom{m+1}{2}}(\sum_{j=0}^{\infty}(q^{m^{j+1}})f(x)) - \sum_{i=0}^{m-1}\frac{\lambda x^{\alpha}}{\Gamma_q(i+1)}q^{-\binom{m-i}{2}}f_0 = \frac{1}{\Gamma_q(\alpha+1)}(x^{\alpha})f(x) \quad (8)$$

Due to the complex nature of fuzzy q-fractional differential equations, obtaining analytical solutions is often difficult. The q-Mittag-Leffler function provides a framework for articulating approximate or numerical solutions to these equations. □

4 Numerical Illustration on Caputo q-Fractional Differential Equation

The main objective of this section is to solve numerically the proposed q-fractional equation and show the efficacy of the above method.

Example 4.1. Considering the Caputo q-fractional initial value problem

$$^{C}D_q^{0.1}f(x) = \frac{1}{2}f(x), \quad f^i(0) = f_0. \quad (9)$$

The parameters are observed as $\lambda = 2$, $m = 100$ in (8) and $q = 0.2$. The initial condition is $x_0 = 0$ and the numerical solution can be acquired as:

$$\lambda q^{-\binom{m+1}{2}}(\textstyle\sum_{j=0}^{\infty}(q^{m^{j+1}})f(x)) - \sum_{i=0}^{m-1}\dfrac{\lambda x^{\alpha}}{\Gamma_q(i+1)}q^{-\binom{m-i}{2}}f_0$$
$$= \frac{1}{\Gamma_q(\alpha+1)}(x^{\alpha})f(x)$$
$$2*(0.2)^{-\binom{100+1}{2}}(\textstyle\sum_{j=0}^{\infty}((0.2)^{100^{j+1}})f(x)) - \sum_{i=0}^{m-1}\dfrac{2x^{0.1}}{\Gamma_q(i+1)}(0.2)^{-\binom{100-i}{2}}f_0$$
$$= \frac{1}{\Gamma_q(0.1+1)}(x^{0.1})f(x)$$
$$[1.594231012991422 \times 10^{3460} - 1.02209x^{0.1}]f(x)$$
$$= -2.783246126369888 \times 10^{116484}x^{0.1}f_0$$

Example 4.2. Considering the Caputo q-fractional initial value problem

$$^{C}D_q^{0.2}f(x) = 4f(x), \quad f^i(0) = f_0, \quad (10)$$

Apply $\lambda = 1/4$ in (8) and $q = 0.3$, for m=10, the solution is obtained as,

$$\lambda q^{-\binom{m+1}{2}}(\sum_{j=0}^{\infty}(q^{m^{j+1}})f(x)) - \sum_{i=0}^{m-1}\frac{\lambda x^{\alpha}}{\Gamma_q(i+1)}q^{-\binom{m-i}{2}}f_0 = \frac{1}{\Gamma_q(\alpha+1)}(x^{\alpha})f(x)$$

$$[8.46226 \times 10^{22} - 1.04683x^{0.2}]f(x) = 2.10663 \times 10^{86}x^{0.2}f_0$$

From the examples, we can deduce that as m increases, the power of t decreases and the coefficient of t also decreases.

5 Solving Fuzzy-valued Function in q-Fractional Initial Value Problem

The fuzzy number is crucial for ranking fuzzy sets and their computations [22]. The fuzzy set is an ordered pair, which mostly involves the set and its membership function.

Definition 8. [14] The fuzzy number $f(r)$ is composed of the functions $(f_L(r), f_U(r))$ which satisfy the conditions listed below,

- $f_L(r)$ is less than or equal to $f_U(r)$, where $0 \leq r \leq 1$.
- $f_L(r)$ and $f_U(r)$ are left continuous on $(0, 1]$ and are right continuous at 0.
- $f_L(r)$ and $f_U(r)$ are bounded functions.
- $f_L(r)$ is monotonically increasing and $f_U(r)$ is monotonically decreasing function.

Let $\mathbb{R}_{\mathbb{F}}$ be the set of all fuzzy numbers. Consider the q-fractional differential equation in Caputo sense consisting of fuzzy number $f : \mathbb{R} \to \mathbb{R}_{\mathbb{F}}$ with α being $0 < \alpha \leq 1$,

$$^{C}D_q^{\alpha}f(x) = \frac{1}{\lambda}f(x), \quad f^i(0) = f(0) \in \mathbb{R}. \tag{11}$$

where $\lambda \in \mathbb{R}$. By the application q-Laplace transform on the equation (11) we get:

$$\begin{aligned}
{}_qL_s({}^C D_q^\alpha [f(x)]_r) &= {}_qL_s[{}_qI_a^{m-\alpha}({}_qD^m)[f(x)]_r] \\
&= {}_qL_s[\frac{1}{\Gamma_q(m-\alpha)} \int_a^x (x-qt)_q^{m-\alpha-1}({}_qD^m[f_L(x;r), f_U(x;r)])d_qt]
\end{aligned}$$

With the application of convolution property, we obtain,

$$\begin{aligned}
{}_qL_s({}^C D_q^\alpha [f(x)]_r) &= {}_qL_s[\frac{1}{\Gamma_q(m-\alpha)}(x)_q^{m-\alpha-1}[g_L(x;r), g_U(x;r)]] \\
&= \frac{1}{\Gamma_q(m-\alpha)} \frac{1}{s^{m-\alpha}} \Gamma_q(m-\alpha) {}_qL_s[g_L(x;r), g_U(x;r)]
\end{aligned}$$

where $[g_L(x;r), g_U(x;r)]$ represents ${}_qD^m[f_L(x;r), f_U(x;r)]$,

$$= \frac{1}{s^{m-\alpha}} {}_qL_s({}_qD^m[f_L(x;r), f_U(x;r)])$$

$$\begin{aligned}
{}_qL_s({}_qD^m[f_L(x;r)]) &= [s^m q^{-\binom{m+1}{2}} F_L(q^{-m}s;r) - \sum_{i=0}^{m-1} s^{m-1-i} q^{-\binom{m-i}{2}} f_L^{(i)}(0)] \\
{}_qL_s({}_qD^m[f_U(x;r)]) &= [s^m q^{-\binom{m+1}{2}} F_U(q^{-m}s;r) - \sum_{i=0}^{m-1} s^{m-1-i} q^{-\binom{m-i}{2}} f_U^{(i)}(0)] \\
\frac{1}{s^{m-\alpha}} {}_qL_s({}_qD^m[f_L(x;r)]) &= \frac{1}{s^{-\alpha}}[q^{-\binom{m+1}{2}} F_L(q^{-m}s;r) - \sum_{i=0}^{m-1} s^{-1-i} q^{-\binom{m-i}{2}} f_L(0)] \\
\frac{1}{s^{m-\alpha}} {}_qL_s({}_qD^m[f_U(x;r)]) &= \frac{1}{s^{-\alpha}}[q^{-\binom{m+1}{2}} F_U(q^{-m}s;r) - \sum_{i=0}^{m-1} s^{-1-i} q^{-\binom{m-i}{2}} f_U(0)]
\end{aligned}$$

where the Laplace transform of f_L and f_U are denoted as F_L and F_U. In a similar manner, we acquire the following equations

$$\begin{aligned}
{}_qL_s[\frac{1}{\lambda}[f(x)]_r] &= \frac{1}{\lambda} {}_qL_s(1) {}_qL_s[f(x)]_r \\
&= \frac{1}{\lambda s}[F_L(s;r), F_U(s;r)]
\end{aligned}$$

Equating the both sides, we obtain the equations as follows,

$$\frac{1}{s^{-\alpha}}[q^{-\binom{m+1}{2}}F_L(q^{-m}s;r)-\sum_{i=0}^{m-1}s^{-1-i}q^{-\binom{m-i}{2}}f_L(0)]=\frac{1}{\lambda s}[F_L(s;r)]$$
$$\frac{1}{s^{-\alpha}}[q^{-\binom{m+1}{2}}F_U(q^{-m}s;r)-\sum_{i=0}^{m-1}s^{-1-i}q^{-\binom{m-i}{2}}f_U(0)]=\frac{1}{\lambda s}[F_U(s;r)]$$
$$s^{\alpha+1}[\lambda q^{-\binom{m+1}{2}}F_L(q^{-m}s;r)-\sum_{i=0}^{m-1}\frac{\lambda}{s^{1+i}}q^{-\binom{m-i}{2}}f_L(0)]=F_L(s;r)$$
$$s^{\alpha+1}[\lambda q^{-\binom{m+1}{2}}F_U(q^{-m}s;r)-\sum_{i=0}^{m-1}\frac{\lambda}{s^{1+i}}q^{-\binom{m-i}{2}}f_U(0)]=F_U(s;r)$$

In addition, we get the equation as given below:

$$\lambda q^{-\binom{m+1}{2}}{}_qL_sf_L(q^{-m}x;r)-\sum_{i=0}^{m-1}\frac{\lambda}{s^{1+i}}q^{-\binom{m-i}{2}}f_L(0)$$
$$=\frac{1}{\Gamma_q(\alpha+1)}{}_qL_s(x^{\alpha})_qL_sf_L(x;r) \tag{12}$$
$$\lambda q^{-\binom{m+1}{2}}{}_qL_sf_U(q^{-m}x;r)-\sum_{i=0}^{m-1}\frac{\lambda}{s^{1+i}}q^{-\binom{m-i}{2}}f_U(0)$$
$$=\frac{1}{\Gamma_q(\alpha+1)}{}_qL_s(x^{\alpha})_qL_sf_U(x;r) \tag{13}$$

By applying inverse Laplace transform,

$$\lambda q^{-\binom{m+1}{2}}f_L(q^{-m}x;r)-\sum_{i=0}^{m-1}{}_qE_{i,1}(\lambda,x)q^{-\binom{m-i}{2}}f_L(0)=\frac{1}{\Gamma_q(\alpha+1)}(x^{\alpha})f_L(x;r)$$
$$\lambda q^{-\binom{m+1}{2}}f_U(q^{-m}x;r)-\sum_{i=0}^{m-1}{}_qE_{i,1}(\lambda,x)q^{-\binom{m-i}{2}}f_U(0)=\frac{1}{\Gamma_q(\alpha+1)}(x^{\alpha})f_U(x;r)$$
$$\lambda q^{-\binom{m+1}{2}}f_L(q^{-m}x;r)-\sum_{i=0}^{m-1}\frac{\lambda x^{\alpha}}{\Gamma_q(i+1)}q^{-\binom{m-i}{2}}f_L(0)=\frac{1}{\Gamma_q(\alpha+1)}(x^{\alpha})f_L(x;r)$$
$$\lambda q^{-\binom{m+1}{2}}f_U(q^{-m}x;r)-\sum_{i=0}^{m-1}\frac{\lambda x^{\alpha}}{\Gamma_q(i+1)}q^{-\binom{m-i}{2}}f_U(0)=\frac{1}{\Gamma_q(\alpha+1)}(x^{\alpha})f_U(x;r)$$

For solving numerically, the equation (12) and (13) can be written as,

$$\lambda q^{-\binom{m+1}{2}}(q^m \sum_{j=0}^{\infty}(q^{m^j})_q L_s f_L(x;r)) - \sum_{i=0}^{m-1} \frac{\lambda}{s^{1+i}} q^{-\binom{m-i}{2}} f_L(0) = \frac{1}{\Gamma_q(\alpha+1)} {}_qL_s(x^\alpha)_q L_s f_L(x;r)$$

$$\lambda q^{-\binom{m+1}{2}}(q^m \sum_{j=0}^{\infty}(q^{m^j})_q L_s f_U(x;r)) - \sum_{i=0}^{m-1} \frac{\lambda}{s^{1+i}} q^{-\binom{m-i}{2}} f_U(0) = \frac{1}{\Gamma_q(\alpha+1)} {}_qL_s(x^\alpha)_q L_s f_U(x;r)$$

$$\lambda q^{-\binom{m+1}{2}}(\sum_{j=0}^{\infty}(q^{m^{j+1}}) f_L(x;r)) = f_L(0) \sum_{i=0}^{m-1} \frac{\lambda x^\alpha}{\Gamma_q(i+1)} q^{-\binom{m-i}{2}} + \frac{1}{\Gamma_q(\alpha+1)}(x^\alpha) f_L(x;r) \tag{14}$$

$$\lambda q^{-\binom{m+1}{2}}(\sum_{j=0}^{\infty}(q^{m^{j+1}}) f_U(x;r)) = f_U(0) \sum_{i=0}^{m-1} \frac{\lambda x^\alpha}{\Gamma_q(i+1)} q^{-\binom{m-i}{2}} + \frac{1}{\Gamma_q(\alpha+1)}(x^\alpha) f_U(x;r) \tag{15}$$

$$\lambda q^{-\binom{m+1}{2}} \sum_{j=0}^{\infty}(q^{m^{j+1}})[f(x)]_r = [f(0)]_r \sum_{i=0}^{m-1} \frac{\lambda x^\alpha}{\Gamma_q(i+1)} q^{-\binom{m-i}{2}} + \frac{1}{\Gamma_q(\alpha+1)}(x^\alpha)[f(x)]_r \tag{16}$$

6 Numerical Example for Fuzzy-valued Function on q-Fractional Initial Value Problem

This section contains two numerical examples and all computations are performed using Mathematica 10.2.

Example 6.1. Considering fuzzy valued q-fractional initial value problem under Caputo sense, as follows

$$^C D_q^{0.1} f(x) = \frac{1}{3} f(x), \quad [f^i(0)]_r = [1+r, 3-r], \quad 0 \le r \le 1 \tag{17}$$

The parameters take the values as $m = 20$ and $q = 0.5$. By eqn (16),

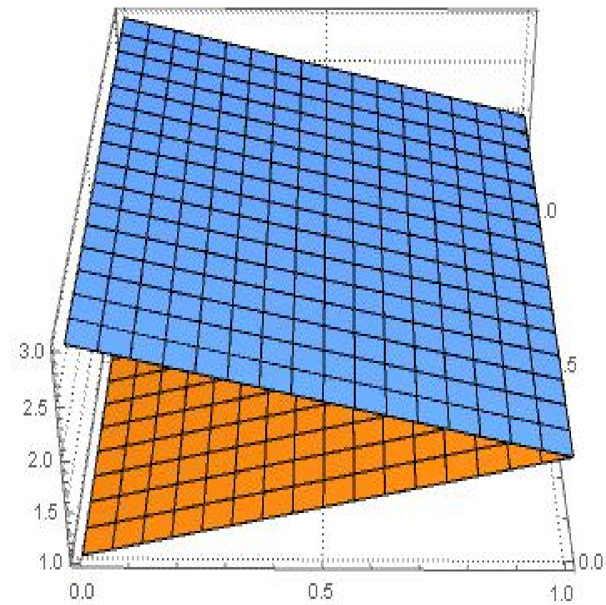

Figure 1: Lower and upper branch of $f(x)$ in Example. (6.1)

$$[f(0)]_r \sum_{i=0}^{m-1} \frac{\lambda x^{\alpha}}{\Gamma_q(i+1)} q^{-\binom{m-i}{2}} + \frac{1}{\Gamma_q(\alpha+1)}(x^{\alpha})[f(x)]_r = \lambda q^{-\binom{m+1}{2}} \sum_{j=0}^{\infty} (q^{m^{j+1}})[f(x)]_r$$

$$[f(0)]_r \sum_{i=0}^{20-1} \frac{3 * x^{0.1}}{\Gamma_q(i+1)} (0.5)^{-\binom{20-i}{2}} + \frac{1}{\Gamma_q(0.1+1)}(x^{0.1})[f(x)]_r = 3 * 0.5^{-\binom{20+1}{2}} \sum_{j=0}^{\infty} (0.5^{20^{j+1}})[f(x)]_r$$

$$[1+r, 3-r] \sum_{i=0}^{20-1} \frac{3 * x^{0.1}}{\Gamma_q(i+1)} (0.5)^{-\binom{20-i}{2}} + \frac{1}{\Gamma_q(0.1+1)}(x^{0.1})[f(x)]_r = 3 * 0.5^{-\binom{20+1}{2}} \sum_{j=0}^{\infty} (0.5^{20^{j+1}})[f(x)]_r$$

$$[1+r, 3-r] 2.434643478090655 \times 10^{401} x^{0.1} + 1.03648 x^{0.1} f[x]_r = 4.70783 \times 10^{57} f[x]_r.$$

As m increases and the power of x decreases, we obtain an approximate solution for (17). The graph plotted in Fig. (1) provides the approximate solution of the eqn (17).

Example 6.2. The fuzzy valued q-fractional initial value problem under Caputo sense is considered as follows

$$^{C}D_q^{0.3}f(x) = 5 * f(x), \quad [f^i(0)]_r = [3.5 + r, 5.5 - r], \quad 0 \leq r \leq 1 \quad (18)$$

The parameters observe the values as $m = 30$ and $q = 0.1$. By eqn (18),

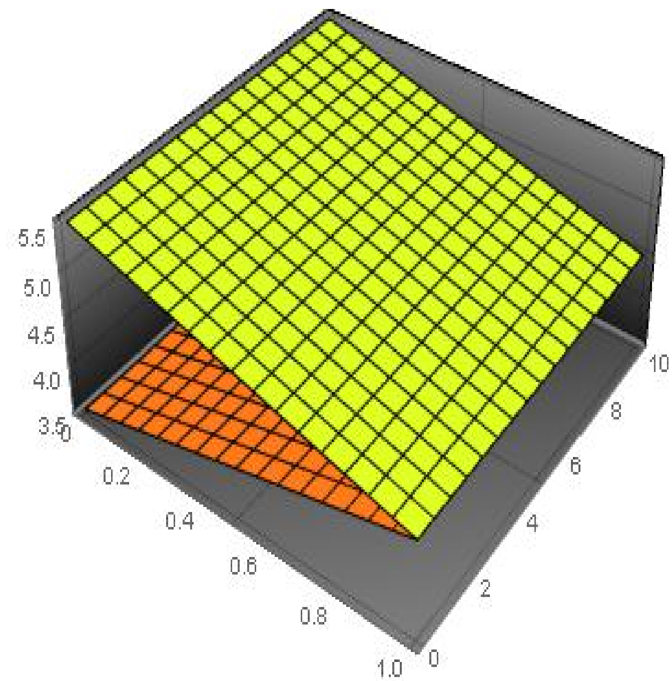

Figure 2: Lower and upper branch of $f(x)$ in Example (6.2)

$$[f(0)]_r \sum_{i=0}^{m-1} \frac{\lambda x^{\alpha}}{\Gamma_q(i+1)} q^{-\binom{m-i}{2}} + \frac{1}{\Gamma_q(\alpha+1)}(x^{\alpha})[f(x)]_r = \lambda q^{-\binom{m+1}{2}} \sum_{j=0}^{\infty}(q^{m^{j+1}})[f(x)]_r$$

$$[f(0)]_r \sum_{i=0}^{30-1} \frac{\frac{1}{5} * x^{0.3}}{\Gamma_q(i+1)} (0.1)^{-\binom{30-i}{2}} + \frac{1}{\Gamma_q(0.3+1)}(x^{0.3})[f(x)]_r = \frac{1}{5} * (0.1)^{-\binom{30+1}{2}} \sum_{j=0}^{\infty}(0.1^{30^{j+1}})[f(x)]_r$$

$$[3.5+r, 5.5-r]\sum_{i=0}^{30-1}\frac{\frac{1}{5}*x^{0.3}}{\Gamma_q(i+1)}(0.1)^{-\binom{30-i}{2}}+\frac{1}{\Gamma_q(0.3+1)}(x^{0.3})[f(x)]_r$$
$$=\frac{1}{5}*(0.1)^{-\binom{30+1}{2}}\sum_{j=0}^{\infty}(0.1^{30^{j+1}})[f(x)]_r$$
$$[3.5+r, 5.5-r]2.124721887537431\times 10^{4495}x^{0.3}+1.02831x^{0.3}f[x]_r$$
$$=2.\times 10^{179}f[x]_r.$$

From the above examples, we can conclude that when m increases, the power and coefficient of x decrease and we get an approximate solution. The graph plotted in Fig. (2) provides an approximate solution of the eqn (18).

7 Hyers-Ulam-Rassias Stability of Caputo q-Fractional Differential Equation

The Hyers-Ulam stability is a flourishing research area in this contemporary period. It originated at Wisconsin University in 1940 and later, between 1982 and 1998 Rassias improved the stability analysis by considering the stability for unbounded Cauchy differences, which led to the term "Hyers-Ulam-Rassias" stability.

Definition 9. The q-fractional system [6] is Hyers-Ulam stable if for all $\epsilon > 0$ and there is a real number $\mathcal{C}$ satisfying

$$|{}_q^C D^{\alpha} g(x) - \frac{1}{\lambda} g(x)| \leq \epsilon, \tag{19}$$

for all g defined on $[a, b] \to \mathbb{R}$, there exists the solution $f(x)$ of [6] satisfying

$$|g(x) - f(x)| \leq \mathcal{C}\epsilon, \tag{20}$$

Definition 10. The q-fractional differential equation (6) is Hyers-Ulam-Rassias stable if for all $\epsilon > 0$, there is a function $\phi(x)$ where $\phi : \mathbb{T}_q \to \mathbb{R}$ and a real number $\mathcal{C}$ satisfying

$$|{}_q^C D^{\alpha} g(x) - \frac{1}{\lambda} g(x)| \leq \epsilon\phi(x), \tag{21}$$

for all g defined on $[a,b] \to \mathbb{R}$, there exists the solution $f(x)$ of [6] satisfying

$$|g(x) - f(x)| \leq \mathcal{C}\epsilon\phi(x), \tag{22}$$

Lemma 7.1. *Let $0 < \alpha \leq 1$ and $g(x)$ be the solution of (19), then $g(x)$ holds for the inequality,*

$$|q^{-\binom{m+1}{2}} g(q^{-m}x) - \sum_{i=0}^{m-1} \frac{x^{\alpha}}{\Gamma_q(i+1)} q^{-\binom{m-i}{2}} g_0 - \frac{1}{\lambda\Gamma_q(\alpha+1)} x^{\alpha} g(x)| \leq \epsilon \frac{|x^{\alpha-1}|}{\Gamma_q(\alpha)} \tag{23}$$

Proof. The solution $g(x)$ satisfies the inequality only if there exists a function $h(x)$ that holds $h(x) \leq \epsilon$.

$${}_q^C D^{\alpha} g(x) - \frac{1}{\lambda} g(x) = h(x) \tag{24}$$

Solving (24) through Laplace transform process,

$$\frac{1}{s^{-\alpha}}[q^{-\binom{m+1}{2}} G(q^{-m}s) - \sum_{i=0}^{m-1} s^{-1-i} q^{-\binom{m-i}{2}} g_0] - \frac{1}{\lambda s} G(s) = H(s)$$

$$q^{-\binom{m+1}{2}} g(q^{-m}x) - \sum_{i=0}^{m-1} \frac{x^{\alpha}}{\Gamma_q(i+1)} q^{-\binom{m-i}{2}} g_0 = \frac{1}{\lambda\Gamma_q(\alpha+1)} x^{\alpha} g(x) + \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} h(x)$$

$$|q^{-\binom{m+1}{2}} g(q^{-m}x) - \sum_{i=0}^{m-1} \frac{x^{\alpha}}{\Gamma_q(i+1)} q^{-\binom{m-i}{2}} g_0 - \frac{1}{\lambda\Gamma_q(\alpha+1)} x^{\alpha} g(x)| \leq |\frac{x^{\alpha-1}}{\Gamma_q(\alpha)} h(x)|$$

$$|q^{-\binom{m+1}{2}} g(q^{-m}x) - \sum_{i=0}^{m-1} \frac{x^{\alpha}}{\Gamma_q(i+1)} q^{-\binom{m-i}{2}} g_0 - \frac{1}{\lambda\Gamma_q(\alpha+1)} x^{\alpha} g(x)| \leq \epsilon \frac{|x^{\alpha-1}|}{\Gamma_q(\alpha)}$$

□

Theorem 7.2. *Let the inference of the lemma hold and $z(x) = g(x) - f(x)$. Assume that $\mathcal{K} = \lambda\epsilon \frac{1}{\Gamma_q(\alpha)}$. Then the q- fractional differential equation (6) is Hyers-Ulam Rassias Stable.*

Proof. Let $g(x)$ and $f(x)$ be the solution of ${}_q^C D^{\alpha} g(x) - \frac{1}{\lambda} g(x)$ and (6) respectively. For $\epsilon > 0$, and by application of Laplace transform provides,

$$\begin{aligned}
g(x)-f(x) &= \lambda q^{-\binom{m+1}{2}} g(q^{-m}x) - \sum_{i=0}^{m-1} \frac{\lambda x^{\alpha}}{\Gamma_q(i+1)} q^{-\binom{m-i}{2}} g_0 - \frac{1}{\Gamma_q(\alpha+1)}(x^{\alpha})g(x) \\
&\quad -\lambda q^{-\binom{m+1}{2}} f(q^{-m}x) + \sum_{i=0}^{m-1} \frac{\lambda x^{\alpha}}{\Gamma_q(i+1)} q^{-\binom{m-i}{2}} f_0 + \frac{1}{\Gamma_q(\alpha+1)}(x^{\alpha})f(x) \\
g(x)-f(x) &= \lambda q^{-\binom{m+1}{2}}[g(q^{-m}x)-f(q^{-m}x)] - \sum_{i=0}^{m-1} \frac{\lambda x^{\alpha}}{\Gamma_q(i+1)} q^{-\binom{m-i}{2}}[g_0-f_0] \\
&\quad -\frac{1}{\Gamma_q(\alpha+1)}(x^{\alpha})[g(x)-f(x)] \\
g(x)-f(x) &= \lambda q^{-\binom{m+1}{2}}[g(q^{-m}x)-f(q^{-m}x)] - \sum_{i=0}^{m-1} \frac{\lambda x^{\alpha}}{\Gamma_q(i+1)} q^{-\binom{m-i}{2}}[g_0-f_0] \\
&\quad -\frac{1}{\Gamma_q(\alpha+1)}(x^{\alpha})[g(x)-f(x)] \\
z(x) &= \lambda q^{-\binom{m+1}{2}}[z(q^{-m}x)] - \sum_{i=0}^{m-1} \frac{\lambda x^{\alpha}}{\Gamma_q(i+1)} q^{-\binom{m-i}{2}}[z_0] - \frac{1}{\Gamma_q(\alpha+1)}(x^{\alpha})[z(x)] \\
\|z(x)\| &= sup|\lambda q^{-\binom{m+1}{2}}[z(q^{-m}x)] - \sum_{i=0}^{m-1} \frac{\lambda x^{\alpha}}{\Gamma_q(i+1)} q^{-\binom{m-i}{2}}[z_0] \\
&\quad -\frac{1}{\Gamma_q(\alpha+1)}(x^{\alpha})[z(x)]| \\
\|z(x)\| &\leq \lambda\epsilon \frac{1}{\Gamma_q(\alpha)} x^{\alpha-1} \\
\|g(x)-f(x)\| &\leq \mathcal{K}\Psi(x)
\end{aligned}$$

where $\mathcal{K} = \lambda\epsilon\frac{1}{\Gamma_q(\alpha)}$. Hence, the Caputo q-fractional differential equation is Hyers-Ulam Rassias Stable □

8 Conclusion

Quantum differential equations can be used in fields such as quantum information processing, quantum chemistry, and condensed matter physics in addition to quantum mechanics. Understanding how to solve and analyse these equations enables researchers to create new models and technologies in these disciplines. To put it briefly, learning about these differential equations lays the groundwork for comprehending quantum mechanics, investigates quicker computational techniques, and leads to novel findings across a range of scientific fields. In this study, the Caputo fractional differential equation combined with quantum derivative is considered and solved with the assistance of q-Laplace transform and q-Mittag-Leffler functions. Furthermore, the Caputo q-fractional derivative for the fuzzy valued function has been studied in this work. The numerical examples are solved, and their graphical representation is presented in this paper. Finally, the Hyers-Ulam-Rassias stability is analysed for the q-fractional differential equation. The future direction of our work will focus on the following forms:

- The h-calculus flourished from the Quantum Calculus, is a finite difference calculus. Exploring h-calculus and its derivatives for the Caputo fractional differential equation can help with future research.
- Investigating the q-fractional differential equation using different methods, particularly Fourier Transform and also solving them for various other derivatives can lead a new path towards future research.
- The combination of the fuzzy fractional model with quantum calculus is a remarkable scope for further study.

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