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Asymptotically Almost Automorphic Mild Solutions for Semilinear Integro-differential Evolution Equations

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Abstract. In this paper, we investigate the existence of asymptotically almost automorphic mild solution for a class of integro-differential equations. The existence results are established through the application of Mönch's fixed point theorem and the utilization of measures of non-compactness. Additionally, we present an illustrative example to showcase the obtained outcomes.

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1 Introduction

The study of integro-differential equations has gained significant attention in recent years, primarily due to their wide range of applications in fields such as electrical engineering, mechanics, and medical biology. In the past few decades, numerous researchers have focused on exploring the existence, uniqueness, stability, controllability, and other qualitative and quantitative characteristics of solutions to these equations. They have achieved this by employing the fixed point technique and relying on the theory of the resolvent operator, which holds significant significance in solving integro-differential equations; see for example [1, 8–12, 33, 35, 46]. More generalized results can be found in [23–25, 29].

In recent times, numerous researchers have been investigating various phenomena in nature, biology, finance, and environmental sciences by employing differential, integro-differential, and fractional equations. In [47], the authors investigated a class of the singular fractional integro-differential quantum equations with multi-step methods. Baleanu *et al.* [5] studied the existence of solutions for a three step crisis integro-differential equation. In [43], a nonlinear quantum boundary value problem formulated in the sense of quantum Caputo derivative, with fractional q-integro-difference conditions along with its fractional quantum-difference inclusion are investigated. For more recent studies on these type of equations, we suggest the publications [2–4, 6, 32, 44].

The concept of almost automorphy is an important generalization of the classic almost periodicity by Bohr, which was first introduced by Bochner in [13] in relation to some aspects of differential geometry. Since then, several advancements and practical implementations have emerged in the realm of various mathematical equations. These include ordinary differential equations, partial differential equations, functional differential equations, integro-differential equations, fractional differential equations, and stochastic differential equations. To explore these topics, one can refer to sources such as [14–16, 20, 22, 26, 36, 37, 40, 41, 48], as well as the references provided therein. Moreover, the concept of asymptotically almost automorphic functions was introduced by N'Guérékata [39], which has led to several intriguing, natural, and robust generalizations. These functions have found numerous applications in the field of differential equations. For further insights and results on this topic, readers can explore references such as [16, 21, 30, 31, 34, 45], along with the monographs by N'Guérékata [42], which delve into the recent theory and applications of asymptotically almost automorphic functions.

In this paper, we consider the uniqueness of mild solutions on a semi-infinite positive real interval $[0, +\infty)$ for a class of integro-differential equations in the abstract form

$$\begin{cases} v'(t) = \mathcal{A}v(t) + \displaystyle\int_0^t \mathcal{B}(t-\zeta)v(\zeta)d\zeta \\ \qquad\qquad +f\left(t, v(t), \displaystyle\int_0^t g(t,\zeta,v(\zeta))d\zeta\right), \ t \geq 0, \\ v(0) = v_0. \end{cases} \tag{1}$$

In this text, $\mathcal{V}$ is a Banach space endowed with a norm $|\cdot|$, $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{V} \to \mathcal{V}$ is the infinitesimal generator of a C_0-semigroup $(\Psi(t))_{t\geq 0} \in \mathcal{V}$. Here $\mathcal{B}(t)$ is a closed linear operator on $\mathcal{V}$, with domain $D(\mathcal{A}) \subset D(\mathcal{B}(t))$ which is independent of t. The nonlinear function $f : \mathbb{R}^+ \times \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, and $g : \mathcal{D} \times \mathcal{V} \to \mathcal{V}$, $\mathcal{D} = \{(t, \jmath) \in \mathbb{R}^+ \times \mathbb{R}^+ : \jmath \leq t\}$, are given functions to be specified later. It is noteworthy that our study can be viewed as a natural extension and continuation of the research outlined in the publications [16,21,30,31,34,39,42,45]. This contribute to the advancement of theories related to integro-differential equations through the incorporation of the concept of almost automorphy.

We will now proceed to a description of the work. In Section 2, we recall some basic concepts and properties of continuous evolution family and measure of noncompactness. In addition, notations about almost automorphic functions and asymptotically almost automorphic functions are also introduced in this section. The results are based on Monch's fixed point theorem under some appropriate assumptions, which we give in Section 3. In Section 4, we provide an example to illustrate the validity of our primary findings.

2 Preliminary Notions

In this section, we review some basic concepts, notations, and properties needed to establish our main results.
Throughout the paper, we assume that $(\mathcal{V}, |\cdot|)$, $(\mathcal{U}, |\cdot|)$ are two real Banach spaces.
To enhance the subsequent discussion, we introduce the following :
▶ $\mathfrak{C}(\mathbb{R}^+, \mathcal{V})$: the space of all continuous $\mathcal{V}$-valued functions on interval $\mathbb{R}^+$.
▶ $B\mathfrak{C}(\mathbb{R}^+ \times \mathcal{U} \times \mathcal{U}, \mathcal{V}))$: the Banach space of bounded continuous functions from $\mathbb{R}^+ \times \mathcal{U} \times \mathcal{U}$ to $\mathcal{V}$ equipped with the norm

$$\|v\|_{B\mathfrak{C}} = \sup_{t \in \mathbb{R}^+} |v(t)|.$$

▶ $C_0(\mathbb{R}^+, \mathcal{V})$: the space of all continuous functions $h : \mathbb{R}^+ \to \mathcal{V}$ such that $\lim_{t \to \infty} h(t) = 0$.
▶ $C_0(\mathbb{R}^+ \times \mathcal{U} \times \mathcal{U}, \mathcal{V})$; the space of all continuous functions from $\mathbb{R}^+ \times \mathcal{U} \times \mathcal{U}$ to $\mathcal{V}$ satisfying $\lim_{t \to \infty} h(t, v, \vartheta) = 0$ in t and uniformly for all $(v, \vartheta) \in K$, where K is any bounded subset of $\mathcal{U} \times \mathcal{U}$.
▶ $L^p(\mathbb{R}^+, \mathcal{V})$ denotes the space of $\mathcal{V}$-valued Bochner functions on $\mathbb{R}^+$ with the norm

$$\|v\|_{L^p} = \left(\int_0^{+\infty} |v(t)|^p dt \right)^{\frac{1}{p}}, \qquad p \geq 1.$$

▶ $B(\mathcal{V}, \mathcal{U})$ the Banach space of bounded linear operators from $\mathcal{V}$ into $\mathcal{U}$, equipped with the usual operator norm $\|\cdot\|_{B(\mathcal{V})}$. In particular, we write $B(\mathcal{V})$ when $\mathcal{V} = \mathcal{U}$.
First, let's recall some basic definitions and results on the strong continuous evolution family which will be used later.

We consider the following Cauchy problem

$$\begin{cases} v'(t) = \mathcal{A}(t)v(t) + \displaystyle\int_0^t \mathcal{B}(t-\zeta)v(\zeta)d\zeta \;\; t \geq 0, \\ v(t) = v_0. \end{cases} \tag{2}$$

Definition 2.1 ([19,27]). A resolvent for Equation (2) is a bounded linear operator valued function $\Phi(t) \in B(\mathcal{V})$ for $t \geq 0$, having the following properties:

(a) For any $t \in \mathbb{R}^+$, $\Phi(t) = 0$ and $\|\Phi(t)\|_{B(\mathcal{V})} \leq \eta e^{-\lambda(t-\jmath)}$ for some constants η and λ.

(b) For each $v \in \mathcal{V}, \Phi(t)v$ is strongly continuous for $t \geq 0$.

(c) For $v \in \mathcal{V}$, $\Phi(\cdot)v \in C^1([0,+\infty),\mathcal{V}) \cap C([0,+\infty),\mathcal{U})$ and

$$\begin{aligned}\Phi'(t)v &= \mathcal{A}\Phi(t)v + \int_0^t \mathcal{B}(t-\zeta)\Phi(\zeta)v d\zeta \\ &= \Phi(t)\mathcal{A}v + \int_0^t \Phi(t-\zeta)\mathcal{B}(\zeta)v d\zeta.\end{aligned}$$

Theorem 2.2 ([19, 27]). *Assume that:*

(a) *$\mathcal{A}$ is the infinitesimal generator of a strongly continuous semigroup $(\Phi(t))_{t\geq 0}$ on $\mathcal{V}$.*

(b) *For all $t \geq 0$, $\mathcal{B}(t)$ is closed linear operator from $D(\mathcal{A})$ to $\mathcal{V}$ and $\mathcal{B}(t) \in B(\mathcal{U},\mathcal{V})$. For any $v \in \mathcal{V}$, the map $t \to \mathcal{B}(t)v$ is bounded, differentiable and the derivative $t \to \mathcal{B}'(t)v$ is bounded uniformly continuous on $\mathbb{R}^+$.*

Then there exists a unique resolvent operator for the Cauchy problem (2).

Definition 2.3 ([13, 41]). A continuous function $f : \mathbb{R} \to \mathcal{V}$ is almost automorphic if for every sequence of real numbers $\{\varkappa'_n\}$, there exists a subsequence $\{\varkappa_n\}$ where

$$\widehat{f}(t) = \lim_{n\to\infty} f(t + \varkappa_n)$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} \widehat{f}(t - \varkappa_n) = f(t) \quad \text{for each } t \in \mathbb{R}.$$

Denote by $AA(\mathbb{R},\mathcal{V})$ the set of all these functions.

Example 2.4. The following is a typical example of almost automorphic function:

$$f(t) = \sin\left(\frac{1}{2 + \cos t + \cos\sqrt{2}t}\right), \quad t \in \mathbb{R}.$$

Lemma 2.5 ([40]). *$AA(\mathbb{R},\mathcal{V})$ is a Banach space with the norm*

$$\|f\|_\infty = \sup_{t\in\mathbb{R}} |f(t)|.$$

Definition 2.6 ([41]). A continuous function $f:\mathbb{R}\times\mathcal{U}\times\mathcal{U}\to\mathcal{V}$ is said to be almost automorphic in $t\in\mathbb{R}$ uniformly for all $(v,\vartheta)\in k$, where K is any bounded subset of $\mathcal{U}\times\mathcal{U}$, if for every sequence of real numbers $\{\varkappa'_n\}$, there exists a subsequence $\{\varkappa_n\}$ such that

$$\lim_{n\to\infty} f(t+\varkappa_n, v,\vartheta) = \widehat{f}(t,v,\vartheta)$$

is well defined for each $t\in\mathbb{R}$ and each $(v,\vartheta)\in K$ and

$$\lim_{n\to\infty} \widehat{f}(t-\varkappa_n, v,\vartheta) = f(t,v,\vartheta)$$

for each $t\in\mathbb{R}$ and each $(v,\vartheta)\in K$.
The set of those functions is denoted by $AA(\mathbb{R}\times\mathcal{U}\times\mathcal{U},\mathcal{V})$.

Example 2.7. The function $f:\ \mathbb{R}\times\mathcal{V}\times\mathcal{V}\to\mathcal{V}$ given by

$$f(t,v,\vartheta) = \sin\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right)(\sin v+\vartheta)$$

is almost automorphic in $t\in\mathbb{R}$ uniformly for all $(v,\vartheta)\in K$, where K is any bounded subset of $\mathcal{V}\times\mathcal{V}$; $\quad \mathcal{V}=L^2([0,1])$.

Lemma 2.8 ([16]). *$f:\mathbb{R}\times\mathcal{V}\times\mathcal{V}\to\mathcal{V}$ is almost automorphic, and assume that $f(t,\cdot,\cdot)$ is uniformly continuous on each bounded subset $K\subset\mathcal{V}$ uniformly for $t\in\mathbb{R}$, that is for any $\varepsilon>0$, there exists $\varrho>0$ such that $v_1,v_2,\vartheta_1,\vartheta_2\in K$ and $|v_2(t)-v_1(t)|+|\vartheta_2(t)-\vartheta_1(t)|<\varrho$ imply that $|f(t,v_1(t);v_2(t))-f(t,\vartheta_1(t),\vartheta_2(t))|<\varepsilon$ for all $t\in\mathbb{R}$. Let $\phi,\psi:\mathbb{R}\to\mathcal{V}$ be almost automorphic. Then the function $\Pi:\mathbb{R}\to\mathcal{V}$ defined by $\Pi(t)=f(t,\phi(t),\psi(t))$ is almost automorphic.*

Remark 2.9. If $f(t,v,\vartheta)$ satisfies a local Lipschitz condition with respect to v and ϑ uniformly in $t\in\mathbb{R}$, i.e., for each pair $v_1,v_2,\vartheta_1,\vartheta_2\in\mathcal{V}$, $t\in\mathbb{R}$

$$|f(t,v_1(t),v_2(t))-f(t,\vartheta_1(t),\vartheta_2(t))| \le \gamma(t)\,|v_2(t)-v_1(t)|+|\vartheta_2(t)-\vartheta_1(t)|\,.$$

where $\gamma(t)\in B\mathfrak{C}(\mathbb{R},\mathbb{R})$, then $f(t,\vartheta(t),\vartheta(t))$ is uniformly continuous on K uniformly for $t\in\mathbb{R}$, where K is any bounded subset of $\mathcal{V}\times\mathcal{V}$.

Definition 2.10 ([41]). A continuous function $f : \mathbb{R}^+ \to \mathcal{V}$ is asymptotically almost automorphic if it can be decomposed as

$$f(t) = \widehat{f}(t) + \nabla(t),$$

where

$$\widehat{f}(t) \in AA(\mathbb{R}, \mathcal{V}),\ \nabla(t) \in C_0(\mathbb{R}^+, \mathcal{V}).$$

Denote by $AAA(\mathbb{R}^+, \mathcal{V})$ the set of all such functions.

Lemma 2.11 ([40]). *$AA(\mathbb{R}^+, \mathcal{V})$ is also a Banach space with the supremum norm $\|f\|_\infty$.*

Example 2.12. The function $f : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$f(t) = \sin\left(\frac{1}{2 + \cos t + \cos\sqrt{2}t}\right) + e^{-t}$$

is an asymptotically almost automorphic function.

Definition 2.13 ([41]). A continuous function $f : \mathbb{R}^+ \times \mathcal{U} \times \mathcal{U} \to \mathcal{V}$ is asymptotically almost automorphic if it can be decomposed as

$$f(t, v, \vartheta) = \widehat{f}(t, v, \vartheta) + \nabla(t, v, \vartheta),$$

where

$$\widehat{f}(\cdot, \cdot, \cdot) \in AA(\mathbb{R} \times \mathcal{U} \times \mathcal{U}, \mathcal{V}), \quad \nabla(\cdot, \cdot, \cdot) \in C_0(\mathbb{R}^+ \times \mathcal{U} \times \mathcal{U}, \mathcal{V}).$$

Denote by $AAA(\mathbb{R}^+ \times \mathcal{U} \times \mathcal{U}, \mathcal{V})$ the set of all such functions.

Example 2.14. The function $f :\ \mathbb{R}^+ \times \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ given by

$$f(t, v, \vartheta) = \sin\left(\frac{1}{2 + \cos t + \cos\sqrt{2}t}\right)(\sin v + \vartheta) + e^{-t}|v + \sin\vartheta|$$

is asymptotically almost automorphic in $t \in \mathbb{R}^+$ uniformly for all $(v, \vartheta) \in K$, where K is any bounded subset of $\mathcal{V} \times \mathcal{V}$, $\mathcal{V} = L^2([0, 1]$, and the functions $\widehat{f} \in AA$, $\nabla \in C_0$ are defined by

$$\widehat{f}(t, v, \vartheta) = \sin\left(\frac{1}{2 + \cos t + \cos\sqrt{2}t}\right)(\sin v + \vartheta) \in AA(\mathbb{R} \times \mathcal{V} \times \mathcal{V}, \mathcal{V}),$$

$$\nabla(t, v, \vartheta) = e^{-t}|v + \sin\vartheta| \in C_0(\mathbb{R}^+ \times \mathcal{V} \times \mathcal{V}, \mathcal{V}).$$

Now, we introduce the Kuratowski measure of noncompactness Ψ given by

$$\Psi(\Theta) = \inf\{\kappa > 0 : \Theta \text{ has a finite cover by sets of diameter } \leq \kappa\},$$

for a bounded set Θ in any space $\mathcal{V}$. Some basic properties of $\Psi(\cdot)$ are given in the following lemma. For more details, please see [7].

Lemma 2.15 ([7]). *Let $\mathcal{V}$ be a Banach space and $\Theta_1, \Theta_2 \subset \mathcal{V}$ be bounded, and the following properties are satisfied:*

($\imath_1$) *Θ is pre-compact if and only if $\Psi(\Theta) = 0$,*

($\imath_2$) *$\Psi(\Theta) = \Psi(\overline{\Theta}) = \Psi(Conv\Theta)$, where $\overline{\Theta}$ and $conv\Theta$ are the closure and the convex hull of Θ, respectively,*

($\imath_3$) *$\Psi(\Theta_1) \leq \Psi(\Theta_2)$ when $\Theta_1 \subset \Theta_2$,*

($\imath_4$) *$\Psi(\Theta_1 + \Theta_2) \leq \Psi(\Theta_1) + \Psi(\Theta_2)$,*

($\imath_5$) *$\Psi(k\Theta) = |k|\Psi(\Theta)$ for any $k \in \mathbb{R}$,*

($\imath_6$) *$\Psi(\Theta) = \Psi(\Theta)$,*

($\imath_6$) *$\Psi(\Theta_2 + \Theta_1) \leq \Psi(\Theta_2) + \Psi(\Theta_2)$ where*

$$\Theta_2 + \Theta_1 = \{v + \vartheta : v \in \Theta, \vartheta \in \Theta_2\},$$

($\imath_6$) *$\Psi(\Theta_2 \cup \Theta_1) \leq \max\left(\Psi(\Theta_2), \Psi(\Theta_2)\right)$,*

($\imath_6$) *if $\Gamma : \mathcal{V} \to \mathcal{V}$ is a Lipschitz continuous map with constant k, then $\Psi(\Gamma(\Theta)) \leq k\Psi(\Theta)$ for all bounded subset Θ of $\mathcal{V}$.*

Lemma 2.16. *([17]) Let $\mathcal{V}$ be a Banach space, $\Theta \subset \mathcal{V}$ be bounded. Then there exists a countable set $\Theta_0 \subset \Theta$, such that*

$$\Psi(\Theta) \leq 2\Psi(\Theta_0).$$

Lemma 2.17 ([28]). *Let $\mathcal{V}$ be a Banach space, and let $\Theta = \{v_n\} \subset \mathfrak{C}([c, d], \mathcal{V})$ be a bounded and countable set for constants $-\infty < c < d < +\infty$. Then $\Psi(v(t))$ is Lebesgue integral on $[c, d]$, and*

$$\Psi\Big(\Big\{\int_c^d v_n(t)dt : n \in \mathbb{N}\Big\}\Big) \leq 2\int_c^d \Psi(\Theta(t))dt.$$

Now, we recall a useful compactness criterion.

Lemma 2.18 (Corduneanu [18]). *A set $\mathcal{C} \subset B\mathfrak{C}(\mathbb{R}^+, \mathcal{V})$ is relatively compact if the following conditions hold*

(i) *$\mathcal{C}$ is bounded in $B\mathfrak{C}(\mathbb{R}^+, \mathcal{V})$,*

(ii) *$\mathcal{C}$ is a locally equicontinuous family of function, i.e., for any constant $d > 0$, the functions in $\mathcal{C}$ are equicontinuous in $[0, d]$,*

(iii) *the set $\mathcal{C}(t) := \{v(t) : v \in \mathcal{C}\}$ is relatively compact on any compact interval of $\mathbb{R}^+$,*

(iv) *the functions from C are equiconvergent, i.e For each $\varepsilon > 0$,there exists $d(\varepsilon) > 0$ such that $|v(t) - v(+\infty)| < \varepsilon$ for all $t \geq d(\varepsilon)$ and for all $v \in \mathcal{C}$.*

3 The Main Results

In this section, we discuss the existence of mild solutions for system (1). Firstly, let us propose the definition of the mild solution of system (1).

Definition 3.1. A function $v \in B\mathfrak{C}(\mathbb{R}^+, \mathcal{V})$ is called a mild solution to the problem (1) if v satisfies the integral equation

$$v(t) = \Phi(t)v_0 + \int_0^t \Phi(t-\jmath) f\left(\jmath, v(\zeta), \int_0^{\jmath} g(\jmath, \zeta, v(\zeta))d\zeta\right) d\jmath, t \in \mathbb{R}^+. \quad (3)$$

In order to obtain the results, we need the following conditions:

($\mathbb{H}_1$) $\mathcal{A}$ is an infinitesimal generator which generates a C_0-semigroup $(\Psi(t))_{t\geq 0}$ such that

$$\|\Phi(t-\jmath)\|_{B(E)} \leq \eta e^{-\lambda(t-\jmath)}.$$

with $\eta > 0$ and $\lambda > 0$ for all $t \geq 0$.

($\mathbb{H}_2$) The function $f : \mathbb{R}^+ \times \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ satisfies:

(i) For a.e. $t \in \mathbb{R}^+$, the function $f(t, \cdot, \cdot) : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ is continuous, and for each $(v, \vartheta) \in \mathcal{V} \times \mathcal{V}$, the function $f(\cdot, v, \vartheta) : \mathbb{R}^+ \to \mathcal{V}$ is strongly measurable.

(i) The function $f(t,v,\vartheta)$ asymptotically almost automorphic i.e., $f(t,v,\vartheta)=\widehat{f}(t,v,\vartheta)+\nabla(t,v,\vartheta)$ with

$$\widehat{f}(t,v,\vartheta)\in AA(\mathbb{R}\times\mathcal{V}\times\mathcal{V},\mathcal{V}),\quad \nabla(t,v,\vartheta)\in C_0(\mathbb{R}^+\times\mathcal{V}\times\mathcal{V},\mathcal{V}),$$

and $\widehat{f}(t,v,\vartheta)$ is uniformly continuous on any bounded subset $K\subset\mathcal{V}\times\mathcal{V}$ uniformly for $t\in\mathbb{R}$.

(ii) There exists a function $\hbar\in L^{\frac{1}{p_1}}(\mathbb{R}^+,\mathbb{R}^+)$, for a constant $p_1\in(0,1)$ such that:

$$|f(t,v,\vartheta)|\le\hbar(t)(|v|+|\vartheta|) \text{ for a.e } t\in\mathbb{R}^+ \text{ and each } v,\vartheta\in\mathcal{V}.$$

(iii) There exists a function $\rho\in L^{\frac{1}{p_2}}(\mathbb{R}^+,\mathbb{R}^+)$, for a constant $p_2\in(0,1)$ such that:

$$\Psi(f(t,V_1,V_2))\le\rho(t)\left(\Psi(V_1)+\Psi(V_2)\right)$$

for a.e $t\in\mathbb{R}^+$ and $V_1,V_2\subset\mathcal{V}$.

$(\mathbb{H}_3)$ The function $g:\mathcal{D}\times\mathcal{V}\to\mathcal{V}$ satisfies the following:

(i) There exists a positive function $\beta(t,\jmath)\in L^1(\mathcal{D},\mathbb{R}^+)$ such that:

$$|g(t,\jmath,v)|\le\beta(t,\jmath)\,|v|\,, \text{ for a.e } (t,\jmath)\in\mathcal{D} \text{ and each } v\in\mathcal{V}.$$

(ii) There exists a positive function $\chi(t,\jmath)\in L^1(\mathcal{D},\mathbb{R}^+)$ such that:

$$\Psi(g(t,\jmath,V)\le\chi(t,\jmath)\Psi(V)\quad \text{for a.e } (t,\jmath)\in\mathcal{D} \text{ and } V\subset\mathcal{V}.$$

For brevity of notations, we denote

$$\beta^*=\sup_{t\in\mathbb{R}^+}\int_0^t\beta(t,\jmath)dt,\qquad \chi^*=\sup_{t\in\mathbb{R}^+}\int_0^t\chi(t,\jmath)dt.$$

We need the following technical lemma.

Lemma 3.2. *Assume that the hypotheses* $(\mathbb{H}_1)$ *is satisfied and let* $V\in AA(\mathbb{R},\mathcal{V})$. *If* $\aleph_1$ *is the function defined by*

$$\aleph_1(t)=\int_{-\infty}^t\Phi(t-\jmath)V(\jmath)\mathrm{d}\jmath,\ t\in\mathbb{R},$$

then $\aleph_1\in AA(\mathbb{R},\mathcal{V})$.

Proof. By $(\mathbb{H}_1)$, we deduce that $\aleph$ is well-defined and continuous on $\mathbb{R}$. Since $V(t) \in AA(\mathbb{R}, \mathcal{V})$, then for every sequence $\{\varkappa'_n\}$, we can extract a subsequence $\{\varkappa_n\}$ where

(Cd_1) $\lim_{n\to\infty} V(t+\varkappa_n) - \tilde{V}(t) = 0$ for each $t \in \mathbb{R}$ and,

(Cd_2) $\lim_{n\to\infty} \tilde{V}(t-\varkappa_n) - V(t) = 0$ for each $t \in \mathbb{R}$.

Notes that $\tilde{V}$ is also bounded on $\mathbb{R}$, and measurable. Define

$$\tilde{\aleph}_1(t) = \int_{-\infty}^{t} \Phi(t-\jmath)\tilde{V}(\jmath)\mathrm{d}\jmath, \quad t \in \mathbb{R}.$$

For $t \in \mathbb{R}$, Since $\tilde{V}$ is measurable, $\tilde{\aleph}_1$ is well-defined.
Using $(\mathbb{H}_1)$, it yields

$$\begin{aligned}
&\Big|\aleph_1 v)(t+\varkappa_n) - (\tilde{\aleph}_1 v)(t)\Big| \\
&= \left|\int_{-\infty}^{t+\varkappa_n} \Phi(t+\varkappa_n-\jmath)V(\jmath)d\jmath - \int_{-\infty}^{t} \Phi(t-\jmath)\tilde{V}(\jmath)d\jmath\right| \\
&= \left|\int_{-\infty}^{t} \Phi(t-\jmath)V(\jmath+\varkappa_n)d\jmath - \int_{-\infty}^{t} \Phi(t-\jmath)\tilde{V}(\jmath)d\jmath\right| \\
&\leq \int_{-\infty}^{t} \|\Phi(t-\jmath)\|_{B(\mathcal{V})} \Big|V(\jmath+\varkappa_n) - \tilde{V}(\jmath))\Big| d\jmath \\
&\leq \eta \int_{-\infty}^{t} e^{-\lambda(t-\jmath)} d\jmath \sup_{t\in\mathbb{R}} \Big|V(t+\varkappa_n) - \tilde{V}(t))\Big| \\
&\leq \frac{\eta}{\lambda} \sup_{t\in\mathbb{R}} \Big|V(t+\varkappa_n) - \tilde{V}(t))\Big|
\end{aligned}$$

Using (Cd_1), we obtain that for $n \to \infty$,

$$\aleph_1(t+\varkappa_n) \to \tilde{\aleph}_1(t).$$

Similarly, it is possible to demonstrate that,

$$\tilde{\aleph}_1(t-\varkappa_n) \to \aleph(t)_1 \text{ for each } t \in \mathbb{R} \text{ as } n \to \infty.$$

Therefore

$$\aleph_1 \in AA(\mathbb{R}, \mathcal{V}).$$

Lemma 3.3. *Assume that the hypotheses* $(\mathbb{H}_1)$ *is satisfied and let* $U \in C_0(\mathbb{R}^+, \mathcal{V})$. *If* $\aleph_2$ *is the function defined by*

$$\aleph_2(t) = \int_0^t \Phi(t-\jmath)U(\jmath)\mathrm{d}\jmath, \ t \in \mathbb{R}^+,$$

then $U \in C_0(\mathbb{R}^+, \mathcal{V})$.

Proof. By $(\mathbb{H}_1)$ we have that $\aleph_2$ is well-defined and continuous on $\mathbb{R}$. Since $U \in C_0(\mathbb{R}^+, \mathcal{V})$, one can choose a $T > 0$ such that

$$\|U\|_\infty \leq \varepsilon.$$

This enables us to conclude that for all $t > T$,

$$\begin{aligned}
|(\aleph_2 v)(t)| &\leq \int_0^t \|\Phi(t-\jmath)\|_{B(\mathcal{V})}|U(\jmath)|d\jmath \\
&\leq \int_0^{\frac{t}{2}} \|\Phi(t-\jmath)\|_{B(\mathcal{V})}|U(\jmath)|d\jmath \\
&+ \int_{\frac{t}{2}}^t \|\Phi(t-\jmath)\|_{B(\mathcal{V})}|U(\jmath)|d\jmath \\
&\leq \eta \sup_{t\in\mathbb{R}^+} |U(t)| \int_0^{\frac{t}{2}} e^{-\lambda(t-\jmath)}d\jmath \\
&+ \eta\varepsilon \int_{\frac{t}{2}}^t e^{-\lambda(t-\jmath)}d\jmath \\
&\leq \eta\|U\|_\infty \frac{e^{-\frac{t}{2}\lambda} - e^{-\lambda t}}{\lambda} + \frac{\eta\varepsilon(1-e^{-\frac{t}{2}\lambda})}{\lambda} \to \frac{\eta\varepsilon}{\lambda} \quad \text{as} \quad t \to +\infty.
\end{aligned}$$

Since ε is arbitrary, we get that $\aleph \in C_0(\mathbb{R}^+, \mathcal{V})$.

Theorem 3.4. *Assume that the hypotheses* $(\mathbb{H}_1) - (\mathbb{H}_3)$ *are satisfied. Then the problem (1) has a asymptotically almost automorphic mild solution If*

$$\max\left(2\eta(1+\beta^*)\|\hbar\|_{L^{\frac{1}{p_1}}}, 4\eta(1+2\chi^*)\|\rho\|_{L^{\frac{1}{p_2}}}\right) < 1. \tag{4}$$

Proof. Consider the operator $\mathcal{T} : B\mathfrak{C}(\mathbb{R}^+, \mathcal{V}) \to B\mathfrak{C}(\mathbb{R}^+, \mathcal{V})$ defined by

$$(\mathcal{T}v)(t) = \Phi(t)v_0 + \int_0^t \Phi(t-\jmath) f\left(\jmath, v(\zeta), \int_0^\jmath g(\jmath, \zeta, v(\zeta))d\zeta\right) d\jmath, t \in \mathbb{R}^+. \tag{5}$$

Our aim is to show that $\mathcal{T}$ admits at least one fixed point in a $AAA(\mathbb{R}^+ \times \mathcal{V} \times \mathcal{V}, \mathcal{V})$.

Step 1. We demonstrate that $\mathcal{T} : B\mathfrak{C}(\mathbb{R}^+, \mathcal{V}) \to B\mathfrak{C}(\mathbb{R}^+, \mathcal{V})$ is well defined.
For $t \in \mathbb{R}^+$, thus, from the hypotheses $(\mathbb{H}_1)$-$(\mathbb{H}_3)$, we get

$$\begin{aligned}
|(\mathcal{T}v)(t)| &\le \|\Phi(t)\|_{\mathfrak{B}(\mathcal{V})} |v_0| \\
&\quad + \int_0^t \|\Phi(t-\jmath)\|_{\mathfrak{B}(\mathcal{V})} \hbar(\jmath) \left(|v(\jmath)| + \int_0^\jmath \xi(\jmath,\zeta)|v(\jmath)|d\zeta\right) d\jmath \\
&\le \eta\,|v_0| + \eta \int_0^t e^{-\lambda(t-\jmath)}\hbar(\jmath) \left(|v(\jmath)| + \int_0^\jmath \xi(\jmath,\varkappa)|v(\zeta)|d\varkappa\right) d\jmath \\
&\le \eta|v_0| + \eta \int_0^t e^{-\lambda(t-\jmath)}\hbar(\jmath) \left(\sup|v(\jmath)| + \beta^* \sup|v(\zeta)|)d\zeta\right) d\jmath \\
&\le \eta|v_0| + \eta \int_0^t e^{-\lambda(t-\jmath)}\hbar(\jmath) \left((1+\beta^*) \sup|v(\jmath)|\right) d\jmath \\
&\le \eta|v_0| + \eta(1+\beta^*) \int_0^t e^{-\lambda(t-\jmath)}\hbar(\jmath) d\jmath \|v\|_{B\mathfrak{C}} \\
&\le \eta|v_0| + \eta(1+\beta^*)\|\hbar\|_{L^{\frac{1}{p_1}}} \left(\int_0^t e^{-\frac{\lambda}{1-p_1}(t-\jmath)} d\jmath\right)^{1-p_1} d\jmath \|v\|_{B\mathfrak{C}} \\
&\le \eta|v_0| + \eta(1+\beta^*)\|\hbar\|_{L^{\frac{1}{p_1}}} \left(1 - e^{-\frac{\lambda t}{1-p_1}}\right) d\jmath \|v\|_{B\mathfrak{C}} \\
&\le \eta|v_0| + \eta(1+\beta^*)\|\hbar\|_{L^{\frac{1}{p_1}}} \|v\|_{B\mathfrak{C}},
\end{aligned}$$

which implies that $\mathcal{T} : B\mathfrak{C}(\mathbb{R}^+, \mathcal{V}) \to B\mathfrak{C}(\mathbb{R}^+, \mathcal{V})$ is well defined.
In what follows, we need to demonstrate that all assumptions of Mönch's fixed point theorem [38] are fulfilled by the operator $\mathcal{T}$. For any $\varrho > 0$, set

$$\overline{\mathfrak{A}_\varrho} = \left\{v \in B\mathfrak{C}(\mathbb{R}^+, \mathcal{V}) : \|v\| \le \varrho\right\}.$$

Clearly, the subset $\overline{\mathfrak{A}_\varrho}$ is a bounded, closed and convex subset of $B\mathfrak{C}(\mathbb{R}^+, \mathcal{V})$.

Step 2. We demonstrate that for $\varrho_0 > 0$ we have $\mathcal{T}\left(\overline{\mathfrak{A}_{\varrho_0}}\right) \subset \overline{\mathfrak{A}_{\varrho_0}}$.
If this condition fails, then for every positive constant $\varrho > 0$ and $t \geq 0$, there exists a function $\widehat{v} \in \overline{\mathfrak{A}_\varrho}$ but $\mathcal{T}(\widehat{v}) \notin \overline{\mathfrak{A}_\varrho}$, i.e $|(\mathcal{T}\widehat{v})(t)| > \varrho$. Thus, by the Hölder inequality, the condition $(\mathbb{H}_1) - (\mathbb{H}_3)$, and thus, we can show that

$$|(\mathcal{T}v)(t)| \quad \leq \quad \eta|v_0| + \eta(1+\beta^*)\|\hbar\|_{L^{\frac{1}{p_1}}}\varrho.$$

Thus,

$$\varrho \quad \leq \quad \eta|v_0| + \eta(1+\beta^*)\|\hbar\|_{L^{\frac{1}{p_1}}}\varrho.$$

Dividing on both sides by ϱ and taking the lower limit as $\varrho \to +\infty$, we can obtain that

$$1 \quad \leq \quad \eta(1+\beta^*)\|\hbar\|_{L^{\frac{1}{p_1}}},$$

which contradicts the assumption (4). Hence, there is a positive constant ϱ_0 such that $\mathcal{T}(\overline{\mathfrak{A}_\varrho}) \subset \overline{\mathfrak{A}_\varrho}$.

Step 3. $\mathcal{T}$ is continuous on $\overline{\mathfrak{A}_{\varrho_0}}$.
To demonstrate the continuity of $\mathcal{T}$, we assume that there exists a sequence $v_n \to v$ in $\overline{\mathfrak{A}_{\varrho_0}}$.
Case 1. If $t \in [0,d]$, $d > 0$, and $v_n, v \in \overline{\mathfrak{A}_{\varrho_0}}$, we have

$$\begin{aligned}|(\mathcal{T}v_n)(t) - (\mathcal{T}v)(t)| \leq \eta \int_0^t \Bigg| f\left(t, v_n(t), \int_0^t g(t,\zeta,v_n(\zeta))d\zeta\right) \\ - f\left(t, v(t), \int_0^t g(t,\zeta,v(\zeta))d\zeta\right)\Bigg| d\jmath.\end{aligned}$$

By the Lebesgue dominated convergence theorem accompanying with $(\mathbb{H}_2)(i)$, we get

$$\|\mathcal{T}v_n - \mathcal{T}v\| \to 0 \qquad \text{as} \qquad n \to +\infty.$$

Case 2. If $t \in (d, +\infty)$, $d > 0$, By $(\mathbb{H}_2)(i)$, we can see that

$$\left| f\left(t, v_n(t), \int_0^t g(t,\zeta,v_n(\zeta))d\zeta\right) - f\left(t, v(t), \int_0^t g(t,\zeta,v(\zeta))d\zeta\right)\right| \leq \frac{\lambda\varepsilon}{\eta}, \quad (6)$$

for $t \geq d$.
Hence, according to the dominated convergence theorem and (6), we obtain that for every $t \geq 0$,

$$\begin{aligned} |(\mathcal{T}v_n)(t) - (\mathcal{T}v)(t)| &\leq \int_0^t \|\Phi(t-\jmath)\|_{B(E)} \left| f\left(t, v_n(t), \int_0^t g(t,\zeta,v_n(\zeta))d\zeta\right) \right. \\ &\qquad \left. - f\left(t, v(t), \int_0^t g(t,\zeta,v(\zeta))d\zeta\right) \right| d\jmath \\ &\leq \frac{\eta\lambda\varepsilon}{\eta} \int_0^t e^{-\lambda(t-\jmath)} d\jmath \\ &\leq \frac{\eta}{\lambda}\frac{\lambda\varepsilon}{\eta}(1-e^{-\lambda t}) \\ &\leq \varepsilon. \end{aligned} \tag{7}$$

Then the inequality (7) reduces to

$$\|\mathcal{T}(v_n) - \mathcal{T}(v)\|_{BC} \to 0 \quad \text{as } n \to \infty.$$

Thus we conclude that $\mathcal{T}$ is continuous in $\overline{\mathfrak{A}_{\varrho_0}}$.
Next, we show that $\mathcal{T}$ is equi-continuous on every compact interval $[0, d]$ of $[0, +\infty)$, for $d > 0$ and is equi-convergent in $v \in \overline{\mathfrak{A}_{\varrho_0}}$.

Step 4. $\mathcal{T}(\overline{\mathfrak{A}_{\varrho_0}})$ is equicontinuous.
Let $0 < d < +\infty$ be an arbitrary constant. Generally, let $0 \leq t_1 \leq t_2 \leq d$, for any $v \in \overline{\mathfrak{A}_{\varrho_0}}$, we know that

$$\begin{aligned} &|(\mathcal{T}v)(t_2) - (\mathcal{T}v)(t_1)| \\ &\quad = \left| \Phi(t_2)v_0 + \int_0^{t_2} \Phi(t-\jmath) f\left(\jmath, v(\zeta), \int_0^{\jmath} g(\jmath,\zeta,v(\zeta))d\zeta\right) d\jmath \right. \\ &\qquad \left. - \Psi(t_1)v_0 + \int_0^{t_1} \Phi(t-\jmath) f\left(\jmath, v(\zeta), \int_0^{\jmath} g(\jmath,\zeta,v(\zeta))d\zeta\right) d\jmath \right| \\ &\quad \leq |\Phi(t_2)v_0 - \Phi(t_1)v_0| \\ &\qquad + \left| \int_0^{t_1} (\Phi(t_2,\jmath) - \Phi(t_1,\jmath)) f\left(\jmath, v(\zeta), \int_0^{\jmath} g(\jmath,\zeta,v(\zeta))d\zeta\right) d\jmath \right. \\ &\qquad \left. + \int_{t_1}^{t_2} \Phi(t_2,\varkappa) f\left(\jmath, v(\zeta), \int_0^{\jmath} g(\jmath,\zeta,v(\zeta))d\zeta\right) d\jmath \right| \end{aligned}$$

$$\begin{aligned}&\leq |\Phi(t_2)v_0 - \Phi(t_1)v_0| \\ &\quad + \int_0^{t_1} \|\Phi(t_2,\varkappa) - \Phi(t_1,\varkappa)\|_{B(\mathcal{V})}\ \hbar(\jmath)\left(|v(\jmath)| + \int_0^{\jmath} \beta(\jmath,\zeta)|v(\zeta)|d\zeta\right)d\jmath \\ &\quad + \eta\int_{t_1}^{t_2} e^{-\lambda(t-\jmath)}\hbar(\jmath)\left(|v(\jmath)| + \int_0^{\jmath}\beta(\jmath,\zeta)|v(\zeta)|d\zeta\right)d\jmath.\end{aligned}$$

It follows from the Hölder's inequality that

$$\begin{aligned}&|(\mathcal{T}v)(t_2) - (\mathcal{T}v)(t_1)| \\ &\leq \|\Phi(t_2) - \Phi(t_1)\|_{B(\mathcal{V})}\,|v_0| \\ &+(1+\beta^*)\varrho\int_0^{t_1}\|\Phi(t_2-\jmath) - \Phi(t_1-\jmath)\|_{B(\mathcal{V})}\ \hbar(\jmath)d\jmath \\ &+\eta\|\hbar\|_{L^{\frac{1}{p_1}}}(1+\beta^*)\varrho\|\chi\|_{L^{\frac{1}{p_1}}}\left(\int_0^t e^{-\frac{\lambda}{1-p_1}(t-\jmath)}d\jmath\right)^{1-p_1}.\end{aligned}$$

It follows that

$$\begin{aligned}&|(\mathcal{T}v)(t_2) - (\mathcal{T}v)(t_1)| \\ &\leq \|\Phi(t_2) - \Phi(t_1\|_{B(\mathcal{V})}\,|v_0| \\ &+(1+\beta^*)\varrho\int_0^{t_1}\|\Phi(t_2-\jmath) - \Phi(t_1-\jmath)\|_{B(\mathcal{V})}\ \hbar(\jmath)d\jmath \\ &+\frac{\eta\|\hbar\|_{L^{\frac{1}{p_1}}}(1+\beta^*)\varrho(1-p_1)^{1-p_1}}{\lambda^{1-p_1}}\left(e^{-\frac{\lambda}{1-p_1}(t-t_2)} - e^{-\frac{\lambda}{1-p}(t-t_1)}\right)^{1-p_1}.\end{aligned}$$

As $t_2 \to t_1$, the right-hand side of the above inequality tends to zero. Therefore, $\mathcal{T}(\overline{\mathfrak{A}_{\varrho_0}})$ is equicontinuous.

Step 5. $\overline{\mathfrak{A}_{\varrho_0}}(t) = \left\{(\mathcal{T}v)(t) : v \in \overline{\mathfrak{A}_{\varrho}}\right\}$ is a relatively compact subset of $\mathcal{V}$ in each $t \in \mathbb{R}^+$.

Let $\mathcal{M}$ be a subset of $\overline{\mathfrak{A}_{\varrho_0}}$ such that $\mathcal{M} \in \overline{conv}(\mathcal{T}(\mathcal{M}) \cup \{0\})$. In addition, by Lemma 2.16, we know that there is a countable set $\{v\}_{n=0}^{n=+\infty} \subset \Theta$ such that $\Psi((\mathcal{T}(\Theta)) \leq 2\Psi((\mathcal{T}(\{v\}_{n=0}^{n=+\infty})$ for any bounded set Θ. Thus for $\{v_n\}_{n=0}^{+\infty} \subset \mathcal{M}$, for the appropriate choice of $\mathcal{M}$. For every $t \in [0,d]$, by utilizing Lemma 2.17 and conditions and the properties of the measure Ψ, we obtain

$$\begin{aligned}&\Psi((\mathcal{T}(\mathcal{M}(t)) \\ &\leq 2\Psi((\mathcal{T}(\{v_n(t)\}_{n=0}^{\infty})\end{aligned}$$

$$\leq 2\Psi\left(\left\{\Phi(t)v_0 + \int_0^t \Phi(t-\jmath)f\left(\jmath, v_n(\jmath), \int_0^\jmath g(\jmath,\zeta,v_n(\zeta))d\zeta\right)d\jmath\right\}_{n=0}^{\infty}\right)$$
$$\leq 2\Psi\left(\left\{\int_0^t \Phi(t-\jmath)f\left(\jmath, v_n(\jmath), \int_0^\jmath g(\jmath,\zeta,v_n(\zeta)d\zeta\right)d\jmath\right\}_{n=0}^{\infty}\right)$$
$$\leq 2\Psi\left(\int_0^t \Phi(t-\jmath)f\left(\jmath, \{v_n(\jmath)\}_{n=0}^{\infty}, \int_0^\jmath g(\jmath,\zeta,\{v_n(\zeta)\}_{n=0}^{\infty})d\varkappa\right)d\jmath\right)$$
$$\leq 4\eta\int_0^t e^{-\lambda(t-\jmath)}\rho(t)$$
$$\times\left(\sup_{\jmath\in[0,t]}\Psi(\{v_n(\jmath)\}_{n=0}^{\infty}) + 2\int_0^\jmath \chi(\jmath,\zeta)\sup_{\varkappa\in[0,\jmath]}\Psi(\{v_n(\zeta)\}_{n=0}^{\infty})d\varkappa\right)d\jmath$$
$$\leq 4\eta\int_0^t e^{-\lambda(t-\jmath)}\rho(t)$$
$$\times\left(\sup_{\jmath\in[0,t]}\Psi(\{v_n(\jmath)\}_{n=0}^{\infty}) + 2\sup_{\varkappa\in[0,\jmath]}\Psi(\{v_n(\zeta)\}_{n=0}^{\infty})\int_0^\jmath \chi(\jmath,\zeta)d\zeta\right)d\jmath$$
$$\leq 4\eta(1+2\chi^*)\int_0^t e^{-\lambda(t-\jmath)}\rho(\jmath)\sup_{\jmath\in[0,t]}\Psi(\{v_n(\jmath)\}_{n=0}^{\infty})d\jmath$$
$$\leq 4\eta(1+2\chi^*)\int_0^t e^{-\lambda(t-\jmath)}\rho(\jmath)d\jmath\Psi(\{v_n\}_{n=0}^{\infty})$$
$$\leq 4\eta(1+\chi^*)\|\rho\|_{L^{\frac{1}{p_2}}}\left(\int_0^t e^{-\frac{\lambda}{1-p_2}(t-\jmath)}d\jmath\right)^{1-p_2}\Psi(\{v_n\}_{n=0}^{\infty})$$
$$\leq 4\eta(1+2\chi^*)\|\rho\|_{L^{\frac{1}{p_2}}}\left(1-e^{-\frac{\lambda t}{1-p_2}}\right)\Psi(\{v_n\}_{n=0}^{\infty})$$
$$\leq 4\eta(1+2\chi^*)\|\rho\|_{L^{\frac{1}{p_2}}}\Psi(\{v_n\}_{n=0}^{\infty}),$$

which ensures that

$$\Psi((\mathcal{T}(\mathcal{M})(t)) \leq 4\eta(1+2\chi^*)\|\rho\|_{L^{\frac{1}{p_2}}}\Psi(\mathcal{M}(t)).$$

Then,

$$\Psi(\mathcal{M}) \leq \Psi((\mathcal{T}(\Theta)(t) \leq 4\eta\|\rho\|_{L^{\frac{1}{p_2}}}(1+2\chi^*)\Psi(\mathcal{M}).$$

That is to say

$$\left(1-4\eta(1+2\chi^*)\|\rho\|_{L^{\frac{1}{p_2}}}\right)\Psi(\mathcal{M}) \leq 0.$$

From (4), we observe that $\Psi(\mathcal{M}) = 0$.

Step 6. $\mathcal{T}(\overline{\mathfrak{A}_{\varrho_0}})$ is equiconvergent.
Let $v \in \overline{\mathfrak{A}_{\varrho_0}}$. For $t \in \mathbb{R}^+$, we have

$$\begin{aligned}
|(\mathcal{T}v)(t)| &\leq \Phi(t)v_0 + \int_0^t \Phi(t-\jmath) f\left(\jmath, v(\zeta), \int_0^{\jmath} g(\jmath,\zeta,v(\zeta))d\zeta\right) d\jmath \\
&\leq \eta|v_0|e^{-\lambda t} + \eta(1+2\beta^*)\int_0^t e^{-\lambda(t-\jmath)}\hbar(\jmath)d\jmath \\
&\leq \eta|v_0|e^{-\lambda t} + \eta(1+\beta^*)\|\hbar\|_{L^{\frac{1}{p_1}}}\left(\int_0^t e^{-\frac{\lambda}{1-p_1}(t-\jmath)}d\jmath\right)^{1-p_1} \\
&\leq \eta|v_0|e^{-\lambda t} + \eta(1+\beta^*)\|\hbar\|_{L^{\frac{1}{p_1}}}\left(1-e^{-\frac{\lambda t}{1-p_1}}\right).
\end{aligned}$$

Then, we get

$$|\mathcal{T}(t)| \to \eta(1+\beta^*)\|\hbar\|_{L^{\frac{1}{p_1}}} \quad as \quad t \to +\infty.$$

Hence, as $t \to +\infty$, we have $|(\mathcal{T}v)(t) - (\mathcal{T}v)(+\infty)| \to 0$.

Step 7. We prove that $\mathcal{T}(AAA(\mathbb{R}^+ \times \mathcal{V} \times \mathcal{V}, \mathcal{V})) \subset (AAA(\mathbb{R}^+ \times \mathcal{V} \times \mathcal{V}, \mathcal{V})$.
Let $v, \vartheta \in AAA(\mathbb{R}^+, \mathcal{V})$ with $v = \omega + \delta$ and $\vartheta = \sigma + \wp$, where ω, σ is the principal term and δ, ϱ the corrective term of v, ϑ.

Let

$$\mathcal{E}(t) = \Phi(t)v_0,$$

then

$$|\mathcal{E}(t)| = |\Phi(t)v_0| \leq |\Phi(t)v_0| \leq \eta e^{-\lambda t}|v_0|.$$

Since $\lambda > 0$, we get $\lim_{t\to+\infty} |(\mathcal{E}(t)| = 0$. That is

$$\mathcal{E} \in C_0(\mathbb{R}^+, \mathcal{V}). \tag{8}$$

We can have

$$f(t, v(t), \vartheta(t)) = \widehat{f}(t, \omega(t), \sigma(t)) + f(t, v(t), \vartheta(t)) - f(t, \omega(t), \sigma(t))$$

$$+ \nabla(t, \omega(t), \sigma(t))$$
$$= \widehat{f}(t, \omega(t), \sigma(t)) + \Bbbk(t, v(t), \vartheta(t)), \tag{9}$$

In view of (9), we have

$$\begin{aligned}
\varpi(t) &= \int_0^t \Phi(t-\jmath) f(t, v(\jmath), \vartheta(\jmath)) d\jmath \\
&= \int_0^t \Phi(t-\jmath) \widehat{f}(t, \omega(\jmath), \sigma(\jmath)) d\jmath + \int_0^t \Phi(t,\jmath) \Bbbk(t, v(\jmath), \vartheta(\jmath)) d\jmath \\
&= \int_{-\infty}^t \Phi(t-\jmath) \widehat{f}(t, \omega(\jmath), \sigma(\jmath)) d\jmath - \int_{-\infty}^0 \Phi(t-\jmath) \widehat{f}(t, \omega(\jmath), \sigma(\jmath)) d\jmath \\
&\quad + \int_0^t \Phi(t-\jmath) \Bbbk(t, v(\jmath), \vartheta(\jmath)) d\jmath \\
&= (\Lambda_1 v)(t) + (\Lambda_2 v)(t),
\end{aligned}$$

where

$$(\Lambda_1 v)(t) = \int_{-\infty}^t \Phi(t-\jmath) \widehat{f}(t, \omega(\jmath), \sigma(\jmath)) \mathrm{d}\jmath,$$

$$\begin{aligned}
(\Lambda_2 v)(t) &= \int_0^t \Phi(t-\jmath) \Bbbk(t, v(\jmath), \vartheta(\jmath)), \vartheta(\jmath)) d\jmath \\
&- \int_{-\infty}^0 \Phi(t-\jmath) \widehat{f}(t, \omega(\jmath), \sigma(\jmath)) d\jmath.
\end{aligned}$$

Let

$$(\mathcal{N}_1 v)(t) = \int_0^t \Phi(t-\jmath) \Bbbk(t, v(\jmath), \vartheta(\jmath)) d\jmath,$$

$$(\mathcal{N}_2 v)(t) = \int_{-\infty}^t \Phi(t-\jmath) \widehat{f}(t, \omega(\jmath), \sigma(\jmath)) \mathrm{d}\jmath.$$

Using $(\mathbb{H}_2)$ and Lemma 2.8, $\jmath \to \widehat{f}(t, \omega(\jmath), \sigma(\jmath))$ is in $AA(\mathbb{R}, \mathcal{V})$. Thus, by Lemma 3.2 we obtain

$$\Lambda_1 \in AA(\mathbb{R}, \mathcal{V}). \tag{10}$$

Let's prove that $\mathcal{N}_1 \in C_0(\mathbb{R}^+, \mathcal{V}), \mathcal{N}_2 \in C_0(\mathbb{R}^+, \mathcal{V})$.
Indeed by definition $\Bbbk \in C_0(\mathbb{R}^+, \mathcal{V})$, Thus, by Lemma 3.4 we obtain

$$\mathcal{N}_1 \in C_0(\mathbb{R}^+, \mathcal{V}). \tag{11}$$

Demonstrating that $\mathcal{N}_2 \in C_0(\mathbb{R}^+, \mathcal{V})$.

$$\begin{aligned}|(\mathcal{N}_2 v)(t)| &\leq \int_{-\infty}^{0} \|\Phi(t-\jmath)\|_{B(\mathcal{V})} |\widehat{f}(t, \omega(\jmath), \sigma(\jmath))| d\jmath \\ &\leq \eta \sup_{t\in\mathbb{R}} |\widehat{f}(t, \omega(\jmath), \sigma(\jmath))| \int_{-\infty}^{0} e^{-\lambda(t-\jmath)} d\jmath \\ &\leq \eta \|\widehat{f}\|_\infty \frac{e^{-\lambda t}}{\lambda} \qquad \to 0 \text{ as } t \to \infty.\end{aligned}$$

So,

$$\mathcal{N}_2 \in C_0(\mathbb{R}^+, \mathcal{V}). \tag{12}$$

Finally combining (8),(10), (11) and (12) proves our claim that

$$\mathcal{T}(AAA(\mathbb{R}^+, \mathcal{V})) \subset (AAA(\mathbb{R}^+, \mathcal{V}).$$

Thus, from the above results, we have that

$$\mathcal{T} : \overline{\mathfrak{A}_{\varrho_0} \cap AAA(\mathbb{R}^+, \mathcal{V})} \to \overline{\mathfrak{A}_{\varrho_0} \cap AAA(\mathbb{R}^+, \mathcal{V})}$$

is a continuous mapping and the assumption

$$\Theta = \overline{conv}\mathcal{T}(\Theta) \quad \text{ or } \quad \Theta = \mathcal{T}(\Theta) \cup \{0\} \Longrightarrow \Psi(\Theta) = 0,$$

holds for every subset Θ of $\overline{\mathfrak{A}_{\varrho_0} \cap AAA(\mathbb{R}^+, \mathcal{V})}$. It follows from the Mönch fixed point theorem that $\mathcal{T}$ has a fixed point

$$v \in \overline{\mathfrak{A}_{\varrho_0} \cap AAA(\mathbb{R}^+, \mathcal{V})}.$$

4 Example

In order to illustrate the usefulness of the theoretical results established in the preceding section, we consider the following heat equation with Dirichlet boundary conditions :

$$\begin{cases} \frac{\partial}{\partial t}\omega(t,y) &= \frac{\partial^2}{\partial \xi^2}\omega(t,y) + \int_0^t \aleph(t-\zeta)\frac{\partial^2}{\partial \zeta^2}\omega(t,y)d\zeta \\ & +\frac{1}{52\sqrt{1+t}}\sin\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right) \\ & \times\left(\sin\omega(t,y) + \int_0^t \frac{\sin(t-\zeta)}{1+(t-\zeta)^2}\omega(t,y)d\zeta\right) \\ & +\frac{e^{-t}}{52\sqrt{1+t}}\left(\omega(t,y) + \int_0^t \frac{\sin(t-\zeta)}{1+(t-\zeta)^2}\omega(t,y)d\zeta\right), \\ & \quad t\in\mathbb{R}^+, y\in[0,1], \\ \omega(t,0) = & \omega(t,1) = 0, \quad \omega(0,y) = \omega_0(y), \quad t\in\mathbb{R}^+, \quad y\in[0,1]. \end{cases} \tag{13}$$

Here $\aleph : \mathbb{R} \to \mathbb{R}$ is bounded uniformly continuous, continuously differentiable. Set $\mathcal{V} = L^2(0,1)$ and let $\mathcal{A}$ be be the Laplace operator

$$(\mathcal{A}\omega)(y) = \frac{\partial^2}{\partial \zeta^2}\omega(y),$$

then $\mathcal{A} : D(\mathcal{A}) = H^2(0,1)\cap H_0^1(0,1) \to L^2(0,1)$. Note that, the operator $\mathcal{A}$ has eigenvalues $\{-n^2\pi^2\}_1^{+\infty}$ and generates a C_0-semigroup $(\Phi(t))_{t\geq 0}$ on $\mathcal{V}$ such that

$$\|\Phi(t)\|_{B(\mathcal{V})} \leq \eta e^{-\lambda t},$$

with $\eta = 1$, $\lambda = \pi^2$ for all $t \geq 0$.
We define the operator $\mathcal{B}(t) : \mathcal{B} : \mathcal{V} \to \mathcal{V}$ as follows:

$$\mathcal{B}(t)\omega = \aleph(t)\mathcal{A}\omega \quad \text{for } t\geq 0 \quad \text{and } \omega\in D(\mathcal{A}).$$

Furthermore we set

$$\omega(t)(y) = \omega(t,y) \ \text{ for } t\in\mathbb{R}^+ \text{ and } y\in[0,1].$$
$$\omega(0) = \omega(0,y) \ \text{ for } t\in\mathbb{R}^+ \text{ and } y\in[0,1].$$

Then the system (13) takes the following abstract form

$$\begin{cases} \omega'(t) = \mathcal{A}\omega(t) + \int_0^t \mathcal{B}(t-\zeta)\omega(\zeta)d\zeta \\ \qquad\qquad +f\left(t,\omega(t),\int_0^t g(t,\zeta,\omega(\zeta))d\zeta\right), \ t\geq 0, \\ \omega(0) = \omega_0, \end{cases} \tag{14}$$

where the nonlinear function $f:\mathbb{R}^+\to\mathcal{V}$ given by

$$\begin{aligned}
& f\left(t,\omega(t),\int_0^t g(t,\zeta,\omega(\zeta))d\zeta\right)\\
&= \frac{1}{52\sqrt{1+t}}\sin\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right)\\
&\quad\times\left(\sin\omega+\int_0^t\frac{\sin(t-\zeta)}{1+(t-\zeta)^2}\omega(\zeta)d\zeta\right)\\
&\quad+\frac{e^{-t}}{52\sqrt{1+t}}\left(\omega(t)+\int_0^t\frac{\sin(t-\zeta)}{1+(t-\zeta)^2}\omega(\zeta)d\zeta\right).
\end{aligned}$$

Let

$$\widehat{f}\,(t,\omega(t),\vartheta(t)) \;=\; \frac{1}{52\sqrt{1+t}}\sin\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right)(\sin\omega(t)+\vartheta(t))\,,$$

$$\nabla(t,\omega(t),\vartheta(t)) \;=\; \frac{e^{-t}}{52\sqrt{1+t}}(\omega(t)+\vartheta(t)),$$

and

$$g(t,\zeta,\omega(\zeta))=\frac{2\sin(t-\zeta)}{1+(t-\zeta)^2}\omega(\zeta).$$

Then it is easy to verify that $\widehat{f},\nabla:\mathbb{R}\to\mathcal{V}$ are continuous and $\widehat{f}(t,v(t),\vartheta(t))\in AA(\mathbb{R}\to\mathcal{V})$ and

$$|\nabla(t,\omega(t),\vartheta(t))| \;\leq\; \frac{e^{-t}}{52\sqrt{1+t}}(|\omega|+|\vartheta|),$$

which implies $\nabla(t,\omega(t),\vartheta(t))\in C_0(\mathbb{R}^+\to\mathcal{V})$ and

$$f(t,\omega(t),\vartheta(t))=\widehat{f}(t,\omega(t),\vartheta(t))+\nabla(t,\omega(t),\vartheta(t))\in AAA(\mathbb{R}^+,\mathcal{V}).$$

Observe that

$$|f(t,\omega(t),\vartheta(t))|\leq\frac{1}{52\sqrt{1+t}}(|\omega_2(t)|+|\vartheta(t)|).$$

Moreover, for a bounded subset V_1, V_2 of $\mathcal{V}$, and from properties of measure of noncompactness Ψ, we have

$$\Psi(f(t, V_1, V_2)) \leq \frac{1}{52\sqrt{1+t}} \left(\Psi(V_1) + \Psi(V_2)\right).$$

Moreover, let $p_1 = p_2 = \dfrac{1}{3}$, then, the assumptions $(\mathbb{H}_2)$ hold with

$$\hbar(t) = \rho(t) = \frac{1}{52\sqrt{1+t}}.$$

Similarly, g clearly satisfies. Further, we get

$$|g(t, \jmath, v)| \leq \frac{2|\sin(t-\zeta)|}{1+(t-\zeta)^2} |v| .$$

Now, by the property of measure of noncompactness for bounded subset V of $\mathcal{V}$, we have

$$\Psi(g(t, \jmath, V)) \leq \frac{2|\sin(t-\zeta)|}{1+(t-\zeta)^2} \Psi(V).$$

In addition

$$\sup_{t \in \mathbb{R}^+} \int_0^t \frac{2|\sin(t-\zeta)|}{1+(t-\zeta)^2} d\zeta \leq \pi.$$

Then the assumptions $(\mathbb{H}_1)$ hold with

$$\beta(t, \jmath) = \chi(t, \jmath) = \frac{2\sin(t-\zeta)}{1+(t-\zeta)^2}.$$

Furthermore, from Theorem 3.4, we obtain

$$\begin{aligned} \max\left(2\eta(1+\beta^*)\|\hbar\|_{L^{\frac{1}{p_1}}}, 4\eta(1+2\chi^*)\|\rho\|_{L^{\frac{1}{p_2}}}\right) &= \frac{\sqrt[3]{2}(1+2\pi)}{13} \\ &= 0,80 < 1. \end{aligned}$$

So, all the conditions of Theorem 3.4 are satisfied. Hence by the conclusion of Theorems 3.4, it follows that the problem (1) has at least one a asymptotically almost automorphic mild solution $\omega \in \overline{\mathfrak{A}_{\varrho} \cap AAA(\mathbb{R}^+, \mathcal{V})}$.

Conclusion

In this paper, we have presented an analysis of the existence of asymptotically almost automorphic mild solution for a class of integro-differential equations. Our approach utilizes Mönch's fixed point theorem and measures of non-compactness to obtain the results. Furthermore, we have illustrated the practical applications of our results through a specific example. We hope that our analysis can inspire further research in this area and contribute to the development of more complex systems. In our future work, we aim to expand the study to second-order differential evolution equations, with different types of delay impulsive effects.

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