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Finding Target in the Two-Stage Networks based on the CRA Model in DEA-RA

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Abstract.
Considering intermediate data, two-stage networks eliminate the possibility of evaluating the performance of decision-making units in Black box mode. In this article, based on the structure of two-stage networks, Central Resource Allocation (CRA) models with fuzzy data are proposed. Then, two-stage network models are proposed in the form of combining data envelopment analysis and Raito analysis. In general, the models of this article using the CRA structure, Target introduce

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the decision-making units under the assumption of constant return to scale. CRA models in the two-stage network structure in DEA-R provide the possibility of finding suitable targets for decision-making units by solving a linear programming model instead of solving n linear programming problems (for n decision-making units). In conclusion, the models are proposed based on a practical study on 16 Iranian airlines.

AMS Subject Classification: 90-XX; 90-10; 90C15
Keywords and Phrases: Data Envelopment Analysis, Fuzzy Data, Fuzzy Probability Function, DEA-RA

1 Introduction

Data envelopment analysis (DEA) is one of the important and practical tools for evaluating the performance of decision-making units. Since to evaluate the performance of each decision-making unit, a linear programming model can be introduced in data envelopment analysis to calculate the efficiency value and find the appropriate model. Therefore, in the case that the parameters of linear programming models are fuzzy numbers, firstly, more realistic information is considered for the decision-making units, secondly, the calculation of efficiency can be a suitable criterion for separating the decision-making units from each other. In the past few decades, based on Farrell's idea in 1975 [9] and then inventing the data envelopment analysis technique that was developed by Charnes et al. (1978) [3] and then raising the subject of variable technology by Banker et al. (1984) [1] , a suitable platform has been provided to evaluate the performance of decision-making units (DMU). Since DEA based on definitive data can evaluate DMUs, therefore input and output parameters are very important. In this regard, the combination of ratio analysis models and DEA are suitable models for evaluating DMUs based on ratios of input and output. It should be noted that in the DEA-RA models, the parameters are not ratio, but the parameter ratios can be a criterion for evaluating the DMUs. In 2007, Despic et .al proposed the use of DEA-RA models [6]. After that, in 2011, Wei et al. analyzed a ratio analysis of pseudo-inefficiency using DEA-RA models on 21 medical centers in Taiwan [25]. Wei et al. also extended the input-oriented DEA-RA models for efficiency and super-efficiency calculation in 2011 [26]. Later, Mozaffari et al. (2014) made a comparison between cost and revenue efficiencies in DEA and DEA-RA [20]. DEA-R models have been the focus of some researchers in the last decade. In addition to the behavior similar to DEA, it is possible to present hybrid models to calculate efficiency for the case where ratios of inputs to outputs are available. In general, in DEA and DEA-R, at least n linear programming models are solved to evaluate n decision-making units, but in CRA models, it is possible to use a linear model. Moreover, Golany et al. presented a basic input-oriented centralized resource allocation (CRA) model in 1995 [10]. CRA models are important because they can obtain the projection of all DMUs on the efficiency frontier usinga single linear programming problem. This is similar to what Lozano and Villa proposed in 2004 [18]. Later on, Lozano et al. (2009) implemented their resource allocation models in an applied study [19]. After that, Hosseinzadeh Lotfi et al. (2010) combined the enhanced Russel model and CRA models and conducted a study on 30 insurance companies [11]. Later in 2012, Hosseinzadeh Lotfi et al. presented a CRA model with stochastic data [12]. In 2021, Chen et al investigated Operating efficiency in Chinese universities using an extended two-stage network DEA approach [5]. In 2022, Wang et al investigated the non-aeronautical efficiency of Chinese listed airports based on a two-stage network DEA [27]. Then Khoveyni et al presented a two-stage network DEA with shared resources and investigated the drawbacks and measuring the overall efficiency [17]. Also, in 2023, Omrani et al. (2023) proposed a two-stage network DEA with shared input and undesirable output for evaluation of the road transport sector [23]. In 2023, Chen et al. using an aggregated two-stage network DEA approach to analyzed the efficiency of Chinese universities with shared inputs. They proposed aggregated two-stage DEA models to measure the efficiency scores of 52 Chinese universities in 2014 [4].

Since the real world is not consistent with accurate data, that is, many data in the real world are relative, so the issue of inaccurate data is raised in organizations. For a more accurate evaluation of organizations, it is very important to use fuzzy parameters, for example, bank customer satisfaction cannot be an exact number, but a fuzzy number can represent customer satisfaction. Therefore, in the present article, the discussion of two-stage network with fuzzy data is of interest. On the other hand, the fuzzy data envelopment analysis (FDEA) technique was presented for the modelling of imprecise and ambiguous data. The concept of fuzziness was introduced by Max Black in the year 1937.After that, various models involving fuzzy linear programming problems were presented, and numerous methods were proposed for solving them. An example in this area would be the study by Fang and Hu (1999), which presented linear programming problems with fuzzy coefficients in their restrictions [8]. Later, Mahdavi-Amiri and Naseri (2006) presented a duality approach to linear programming with fuzzy numbers using a linear ranking function. Then, in2010, Aydin Keskin et al. presented a categorization method for evaluating and selecting suppliers (fuzzy ART algorithm) [16]. Next, Ebrahimnejad et al. (2014) proposed a three-stage DEA model with two parallel stages [7]. In 2020, Heydari et al. presented a fully fuzzy network DEA-Range adjusted measurement model to evaluate the efficiency of Iranian airlines [14]. Next, in 2022, Tavassoli et al proposed a novel fuzzy network DEA model to evaluate the efficiency of Iran's electricity distribution network with sustainability considerations [24]. In 2023, Hosseinzadeh Lotfi et al. reviewed Ranking, sensitivity and stability analysis in fuzzy DEA [13]. Also, in 2023, Nazari-Shirkouhi et al proposed a hybrid approach using Z-number DEA model and artificial neural network for Resilient supplier Selection [21]. Next in 2023, Huang et al. presented heterogeneous multi-attribute group decision-making based on fuzzy data envelopment analysis cross-efficiency model [15]. Since in an air transportation system, a two-stage network can be proposed and include pre-flight and post-flight operations, so the models proposed in the article can be useful in the evaluation of airlines. In this paper, the domestic flight network of Iranian airlines is evaluated in a two-stage network. Moreover, the input and output parameters are considered to be fuzzy, and after examining a two-stage network with fuzzy data, the feasibility approach is used to solve the linear programming problems. Generally, the main objectives of the current research can be categorized as follows:

A. Evaluating two-stage networks with fuzzy data using centralized resource allocation (CRA) models.

B. In CRA models, using a maximum of two linear programming models, we can obtain the projection of all DMUs on the efficiency frontier,

and this is very noteworthy in comparison with classical DEA models.
C. Using a probability function and determining a confidence interval for the restrictions provides the possibility for the defuzzification of fuzzy programming models.

Research contribution: In this paper, DEA-RA models are presented, which are a combination of CCR, DEA, and FDEA models. In other words, the contribution of this research is a fuzzy technique that is used to solve linear programming models and that finds benchmarks in two-stage networks. In this paper, section two briefly describes input-oriented DEA-RA models and the fuzzy probability theory. Then, in section three, a two-stage network is presented using the fuzzy probability function approach, and in section four, an applied study of Iranian airlines is presented. And finally, the conclusions of the paper are provided in section five.

2 Preliminaries

In this section, a brief introduction is given to the basic concepts relating input-oriented DEA-RA models and the fuzzy probability theory.

2.1 Input-oriented envelopment DEA-RA model

Assuming $\frac{x_{ij}}{y_{rj}}$ ratios are defined for n DMUs, each of which consumes m inputs $X_j = (x_1j, x_2j, \dots, x_mj)$ to produce s outputs $Y_j = (y_1j, x_2j, \dots, y_sj)$ where $X_j > 0$,$Y_j > 0$, the input-oriented envelopment DEA-RA model with fuzzy data can be presented as follows :

$$\begin{aligned}
&\gamma_o^* = Min\gamma_o \\
&s.t. \qquad \sum_{j=1}^{n} \lambda_j(\frac{x_{ij}}{y_{rj}}) \leq \gamma_o(\frac{x_{ij}}{y_{rj}}), \quad i = 1, \dots, m; r = 1, \dots, s; \\
&\qquad\qquad \sum_{j=1}^{n} \lambda_j = 1, \qquad j = 1, \dots, n; \\
&\qquad\qquad \lambda_j \geq 0, \qquad j = 1, \dots, n;
\end{aligned} \tag{1}$$

By solving the linear programming problem above with a minimum objective function, we obtain the efficiency score, which is denoted by γ_o^*.

Definition 2.1. DMU_o is DEA-RA efficient with input orientation when $\gamma_o^* = 1$ and the slack variables are equal to zero in all optimal solutions .

The model 1 is an input-oriented DEA-RA model defined based on the definition of efficiency as a weighted sum of input-to-output ratios. On the other hand, the model 1 can be equivalent to an input-oriented BCC model without explicit outputs.

2.2 Preliminaries of fuzzy logic

In the section, we present the preliminaries relating fuzzy sets.

Definition 2.2. Fuzzy set: Assuming that A is a set, the set $\underset{\sim}{A} = \{(x, \mu_A(x))|x \in A, \mu_A(x) : X \to [0, 1]\}$
$\underset{\sim}{A}$ is called a fuzzy set. Furthermore, $\mu_A(x)$ is called the membership function of the fuzzy set

Definition 2.3. The fuzzy number $\widetilde{A} = (a_1, a_2, a_3, a_4)$ is called a trapezoidal fuzzy number where a_1 is the left element, a_2 and a_3 are middle elements, and a_4 is the right element.

In this research, fuzzy numbers are considered as trapezoidal numbers.

Definition 2.4. A fuzzy subset of A in which the members have membership degrees greater than or equal to α is called the $\alpha - cut$ of A and is denoted by A_α.
$A_\alpha = x/inA|\mu_A(x) \geq \alpha$

Theorem 2.5. *The $\alpha-$cut of the trapezoidal fuzzy number $\widetilde{A} = (a_1, a_2, a_3, a_4)$ is a closed interval defined as:*

$$[\widetilde{A}]_\alpha = \lfloor [\widetilde{A}]^l_\alpha, [\widetilde{A}]^u_\alpha \rfloor = [a_1 + (a_2 - a_1)\alpha, a_4 - (a_4 - a_3)\alpha] \qquad (2)$$

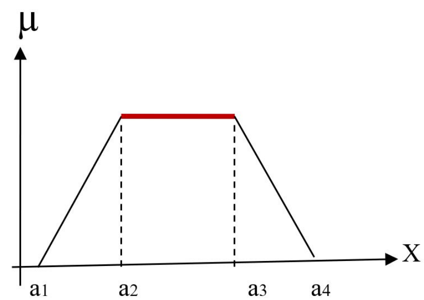

Figure 1: A trapezoidal fuzzy number is illustrated with the assumption of $\widetilde{A} = (a_1, a_2, a_3, a_4)$

Lemma 2.6. *Assume that $\widetilde{a}_1, \widetilde{a}_2, ..., \widetilde{a}_n$ are fuzzy variables with a convex and normal membership function. Let $(0)^L_{\alpha_i}$ and $(0)^U_{\alpha_i}$ be the lower and upper bounds of the $\alpha -$ level set $\widetilde{a}_i, i = 1, ..., n$; then, for any possible level of a_1, a_2, and a_3 where $0 \leq a_1, a_2, a_3 \leq 1$, we can write:*
$(1)\pi(\widetilde{a}_1 + \widetilde{a}_2 + ... + \widetilde{a}_n \leq b) \geq \alpha_1 \Longleftrightarrow (\widetilde{a}_1)^l_{\alpha_1} + (\widetilde{a}_2)^l_{\alpha_1} + ... + (\widetilde{a}_n)^l_{\alpha_1} \leq b$
$(2)\pi(\widetilde{a}_1 + \widetilde{a}_2 + ... + \widetilde{a}_n \geq b) \geq \alpha_2 \Longleftrightarrow (\widetilde{a}_1)^u_{\alpha_2} + (\widetilde{a}_2)^u_{\alpha_2} + ... + (\widetilde{a}_n)^u_{\alpha_2} \geq b$
$(3)\pi(\widetilde{a}_1 + \widetilde{a}_2 + ... + \widetilde{a}_n = b) \geq \alpha_3 \Longleftrightarrow (\widetilde{a}_1)^u_{\alpha_3} + (\widetilde{a}_2)^u_{\alpha_3} + ... + (\widetilde{a}_n)^u_{\alpha_3} \geq b \& (\widetilde{a}_1)^l_{\alpha_3} + (\widetilde{a}_2)^l_{\alpha_3} + ... + (\widetilde{a}_n)^l_{\alpha_3} \leq b$

3 Two-Stage Network Evaluation Using The Fuzzy Probability Function Approach

The purpose of this section is to find suitable benchmarks in DEA-RA in a fuzzy network CRA model using the feasibility approach.

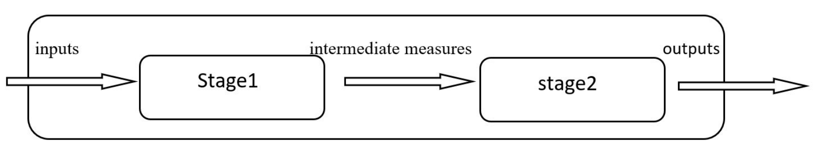

Figure 2: Two-stage network with fuzzy data

Consider n decision-making units as $DMU_j = (\widetilde{x}_j, \widetilde{z}_j, \widetilde{z}_j, \widetilde{y}_j)$ for $j = 1, ..., n$. Assume that the fuzzy data $\widetilde{x}_{ij}$ and $\widetilde{Z}_{lj}$ are available in such a way that makes it possible to produce one input oriented CRA model Therefore, considering $\widetilde{X}_j > \widetilde{0}$ and $\widetilde{z}_j > \widetilde{0}$ as the input and output of the first stage the CRA model in the non-radial DEA-RA form is proposed as follows:

$$
\begin{aligned}
& Min \sum_{l=1}^{d}\sum_{i=1}^{m} w_{il}\theta_{il} \\
& s.t. \sum_{p=1}^{n}\sum_{j=1}^{n} \lambda_{jp}^{1}\left(\frac{\widetilde{x}_{ij}}{\widetilde{z}_{lj}}\right) \leq \theta_{il}\frac{\sum_{j=1}^{n}\widetilde{x}_{ij}}{\sum_{j=1}^{n}\widetilde{z}_{lj}}, \quad && i = 1, \ldots, m; l = 1, \ldots, d; \\
& \sum_{j=1}^{n} \lambda_{jp}^{1} = 1, && p = 1, \ldots, n, \\
& \lambda_{jp}^{1} \geq 0, && p = 1, \ldots, n.
\end{aligned}
\tag{3}
$$

Theorem 3.1. *Model (3) is always feasible.*
Proof: Model (3) is a linear programming problem that by considering $(\lambda_{jp}^1 = 0, p \neq j, \lambda_{jp}^1 = 1, j = p)$ *and* $(\theta_{il} = 1)$, *the restrictions corresponding to the inputs and outputs, as well as the restriction* $(\sum_{j=1}^{n} \lambda_{jp}^1 = 1, \forall p)$ *is established, and a feasible solution for model (3) is always obtained.*

The model (3) is a linear programming problem. Using the probability theory, the probability of fuzzy events is determined. To solve this fuzzy model, we use chance-constrained programming (CCP) as presented by Charnes and Cooper [2]. Therefore, by determining a desirable confidence interval for each restriction, the CCP approach is selected as a solution for defuzzification. Hence, the model (3) is written as follows:

$$
\begin{aligned}
& Min \widetilde{f} \\
& s.t. \\
& \pi(\sum_{l=1}^{f}\sum_{i=1}^{m} w_{il}\theta_{il} \geq \widetilde{f}\ ) \geq \alpha \\
& \pi(\sum_{p=1}^{n}\sum_{j=1}^{n} \lambda_{jp}^{1}\frac{\widetilde{x}_{ij}}{\widetilde{z}_{lj}} - \theta_{il}\frac{\sum_{j=1}^{n}\widetilde{x}_{ij}}{\sum_{j=1}^{n}\widetilde{z}_{lj}} \leq 0) \geq \beta_i, \quad i = 1,\dots,m; l = 1,\dots,d, \\
& \sum_{j=1}^{n} \lambda_{jp}^{1} = 1, \qquad p = 1,\dots,n. \\
& \lambda_{jp}^{1} \geq 0, j, \qquad p = 1,\dots,n.
\end{aligned}
\tag{4}
$$

In the model (4), α_i and β_i are acceptable levels predicted for the first and second restrictions, and both fall within the interval $[0,1]$.In this model, the value of the objective function $\widetilde{f}$ must be the maximum value that the function $\sum_{l=1}^{d}\sum_{i=1}^{m} w_{il}\theta_{il}$ can obtain with a "probability" level of α or higher, under the condition that the probability level of the second restriction is at least equal to β_i. In other words, in the optimal solution, with the probability level of α , the value of $\sum_{l=1}^{d}\sum_{i=1}^{m} w_{il}\theta_{il}$ will be at least equal to $\widetilde{f}$. Meanwhile, all restrictions are estimated at the level of the predicted probabilities. In a CRA model with a probability level of α, the value of $\sum_{l=1}^{d}\sum_{i=1}^{m} w_{il}\theta_{il}$ does not reveal the efficiency of the DMU under evaluation, but rather shows the projection of the DMU on the technical efficiency frontier.

Definition 3.2. A DMU is $\alpha - probability$ efficient if the value of the objective function $\widetilde{f}$ at the probability level α is greater than or equal to one. Otherwise, the DMU is $\alpha - probability$ inefficient.

Nonetheless, for a logical comparison of the efficiency of the DMUs, the same probability level must be considered for the restrictions of α_i and β_i .The fuzzy variables $\widetilde{x}_{ij}$ and $\widetilde{Z}_{lj}$ are convex and normal. Therefore, for any possible level of α_i and β_i where $0 \leq \alpha_i \leq 1$ and $0 \leq \beta_i \leq 1$, Lemma (2.6) holds true. Thus, by applying Lemma (2.6) to the model (4), we can write the model as follows:

$$
\begin{array}{ll}
Min \widetilde{f} & \\
s.t. & \\
(\sum_{l=1}^{f} \sum_{i=1}^{m} w_{il} \theta_{il})_{\alpha}^{u} \geq \widetilde{f} & \\
\sum_{p=1}^{n} \sum_{j=1}^{n} (\lambda_{jp}^{1} \frac{\widetilde{x}_{ij}}{\widetilde{z}_{lj}})_{\alpha_i}^{l} - (\theta_{il} \frac{\sum_{j=1}^{n} \widetilde{x}_{ij}}{\sum_{j=1}^{n} \widetilde{z}_{lj}})_{\alpha_i}^{l} \leq 0, & l = 1, \ldots, d \\
\sum_{j=1}^{n} \lambda_{jp}^{1} = 1, & p = 1, \ldots, n. \\
\lambda_{jp}^{1} \geq 0, & j, p = 1, \ldots, n.
\end{array}
\tag{5}
$$

In the model (5), the variables $\widetilde{x}_{ij}$ and $\widetilde{Z}_{lj}$ are fuzzy. Therefore, based on the rules governing the division of fuzzy numbers, in the second restriction, the expression $(\lambda_{jp} \frac{\widetilde{x}_{ij}}{\widetilde{z}_{lj}})_{\alpha_i}^{l}$ is written as $\frac{(\widetilde{x}_{ij})_{\alpha_i}^{l}}{(\widetilde{z}_{lj})_{\alpha_i}^{u}}$, and the expression $(\theta_{ir} \frac{\sum_{j=1}^{n} \widetilde{x}_{ij}}{\sum_{j=1}^{n} \widetilde{z}_{lj}})_{\alpha_i}^{l}$ is written as $\theta_{ir} \frac{(\sum_{j=1}^{n} \widetilde{x}_{ij})_{\alpha_i}^{l}}{(\sum_{j=1}^{n} \widetilde{z}_{lj})_{\alpha_i}^{u}}$. Thereby, the model (5) can be written as (6).

$$
\begin{aligned}
& Min\widetilde{f} \\
& s.t. \\
& (\sum_{l=1}^{f}\sum_{i=1}^{m} w_{il}\theta_{il})^{u}_{\alpha} \geq \widetilde{f} \\
& \sum_{p=1}^{n}\sum_{j=1}^{n}\lambda^{1}_{jp}\frac{(\widetilde{x}_{ij})^{l}_{\alpha_i}}{(\widetilde{z}_{lj})^{u}_{\alpha_i}} - \theta_{il}\frac{(\sum_{j=1}^{n}\widetilde{x}_{ij})^{l}_{\alpha_i}}{(\sum_{j=1}^{n}\widetilde{z}_{lj})^{u}_{\alpha_i}} \leq 0_i, \quad i=1,\dots,m; l=1,\dots,d, \\
& \sum_{j=1}^{n}\lambda^{1}_{jp} = 1, \quad p=1,\dots,n. \\
& \lambda^{1}_{jp} \geq 0, \quad \forall j,p=1,\dots,n.
\end{aligned} \tag{6}
$$

The model (6) is still a fuzzy programming model. Since $\widetilde{X}_{ij}$ and $\widetilde{Z}_{ij}$ are trapezoidal fuzzy numbers, based on Theorem (2.5), it can be converted into a non-fuzzy programming model as (7).

$$
\begin{aligned}
& Min\widetilde{f} \\
& s.t. \\
& \sum_{l=1}^{d}\sum_{i=1}^{m} w_{il}\theta_{il} \geq \widetilde{f} \\
& \sum_{p=1}^{n}\sum_{j=1}^{n}\lambda^{1}_{jp}\frac{(x_{ij1}+(x_{ij2}-x_{ij1})\alpha_i)}{(z_{lj4}-(z_{lj4}-z_{lj3})\alpha_i)} - \\
& \theta_{il}\frac{\sum_{j=1}^{n}(x_{ij1}+(x_{ij2}-x_{ij1})\alpha_i)}{\sum_{j=1}^{n}(z_{lj4}-(z_{lj4}-z_{lj3})\alpha_i)} \leq 0_i, \quad i=1,\dots,m; l=1,\dots,d, \\
& \sum_{j=1}^{n}\lambda^{1}_{jp} = 1, \quad p=1,\dots,n, \\
& \lambda^{1}_{jp} \geq 0, \quad j,p=1,\dots,n.
\end{aligned} \tag{7}
$$

Therefore, the model (7) is a parametric programming model where:

$$
\begin{aligned}
\hat{x}_{ip} &= \sum_{j=1}^{n} \lambda_{jp}^{*1}(x_{ij1} + (x_{ij2} - x_{ij1})\alpha), \forall i, \forall p, \\
\hat{z}_{lp} &= \sum_{j=1}^{n} \lambda_{jp}^{*1}(z_{lj4} - (z_{lj4} - z_{lj3})\alpha), \forall l, \forall p,
\end{aligned}
\tag{8}
$$

Moreover, in the second stage, n DMUs consume $\widetilde{z}_j$ inputs to produce $\widetilde{y}_j$ outputs. Thereby, the CRA model with fuzzy parameters in DEA-RA for the second network stage is proposed as (9).

$$
\begin{aligned}
& Min \widetilde{f} \\
& s.t. \\
& \sum_{r=1}^{s} \sum_{l=1}^{d} w_{lr}\theta_{lr} \geq \widetilde{f} \\
& \sum_{p=1}^{n} \sum_{j=1}^{n} \lambda_{jp}^{2} \frac{(z_{lj1} + (z_{lj2} - z_{lj1})\alpha_l)}{(y_{rj4} - (y_{rj4} - y_{rj3})\alpha_r)} - \\
& \theta_{lr} \frac{\sum_{j=1}^{n}(z_{lj1} + (z_{lj2} - z_{lj1})\alpha_l)}{\sum_{j=1}^{n}(y_{rj4} - (y_{rj4} - y_{rj3})\alpha_r)} \leq 0, \qquad l = 1, \ldots, d; r = 1, \ldots, s, \\
& \sum_{j=1}^{n} \lambda_{jp}^{2} = 1, \qquad p = 1, \ldots, n. \\
& \lambda_{jp}^{2} \geq 0, \qquad j, p = 1, \ldots, n.
\end{aligned}
\tag{9}
$$

Therefore, for the second network stage, the projection of DMU_s $p = 1, \ldots, n$ is proposed as follows.

$$
\begin{aligned}
\hat{z}_{lp} &= \sum_{j=1}^{n} \lambda_{jp}^{*2}(z_{lj1} + (z_{lj2} - z_{lj1})\alpha_l), \forall l, \forall p, \\
\hat{y}_{rp} &= \sum_{j=1}^{n} \lambda_{jp}^{*2}(y_{rj4} - (y_{rj4} - y_{rj3})\alpha_r), \forall r, \forall p,
\end{aligned}
\tag{10}
$$

Finally, based on the fuzzy borders $\widetilde{X}_j$, $\widetilde{Z}_j$, and $\widetilde{Y}_j$,the CRA model in

DEA-RA for the overall network is proposed as follows while considering a fuzzy probability function and using the method of determining a confidence interval

$$
\begin{aligned}
& Min(0.1)\widetilde{f}_1 + (0.9)\widetilde{f}_2 \\
& s.t. \\
& \sum_{l=1}^{d}\sum_{i=1}^{m} w_{il}\theta_{il} \geq \widetilde{f}_1 \\
& \sum_{r=1}^{s}\sum_{l=1}^{d} w_{lr}\theta_{lr} \geq \widetilde{f}_2 \\
& \sum_{p=1}^{n}\sum_{j=1}^{n} \lambda^1_{jp}\frac{(x_{ij1}+(x_{ij2}-x_{ij1})\alpha_i)}{(z_{lj4}-(z_{lj4}-z_{lj3})\alpha_i)} - \\
& \theta_{il}\frac{\sum_{j=1}^{n}(x_{ij1}+(x_{ij2}-x_{ij1})\alpha_i)}{\sum_{j=1}^{n}(z_{lj4}-(z_{lj4}-z_{lj3})\alpha_i)} \leq 0, \qquad i=1,\ldots,m; l=1,\ldots,d \\
& \sum_{p=1}^{n}\sum_{j=1}^{n} \lambda^2_{jp}\frac{(z_{lj1}+(z_{lj2}-z_{lj1})\alpha_l)}{(y_{rj4}-(y_{rj4}-y_{rj3})\alpha_r)} - \\
& \theta_{il}\frac{\sum_{j=1}^{n}(z_{lj1}+(z_{lj2}-z_{lj1})\alpha_l)}{\sum_{j=1}^{n}(y_{rj4}-(y_{rj4}-y_{rj3})\alpha_r)} \leq 0, \qquad l=1,\ldots,d; r=1,\ldots,s \\
& \sum_{j=1}^{n}\lambda^1_{jp} = 1, \qquad p=1,\ldots,n. \\
& \sum_{j=1}^{n}\lambda^2_{jp} = 1, \qquad p=1,\ldots,n. \\
& \lambda^1_{jp} \geq 0, \lambda^2_{jp} \geq 0, \qquad j,p=1,\ldots,n.
\end{aligned}
\tag{11}
$$

In which the projection of the DMUs is as (12):

$$\hat{x}_{ip} = \sum_{j=1}^{n} \lambda_{jp}^{*2}(x_{ij1} + (x_{ij2} - x_{ij1})\alpha_i), \forall l, \forall p,$$
$$\hat{z}_{lp} = \sum_{j=1}^{n} \lambda_{jp}^{*2}(z_{lj1} + (z_{lj2} - z_{lj1})\alpha_l), \forall l, \forall p, \tag{12}$$
$$\hat{y}_{rp} = \sum_{j=1}^{n} \lambda_{jp}^{*2}(y_{rj4} - (y_{rj4} - y_{rj3})\alpha_r), \forall r, \forall p,$$

Based on the α-cutting method, model (11) has been transformed from a fuzzy linear programming problem into a parametric linear programming problem depending on α, where α is the manager's risk level in the model.The corresponding model (11) in DEA reduces the total inputs of the network in the first and second stages, but the model (11) reduces the ratio of the total inputs to the outputs of the first and second stages of the network separately.On the other hand, the proposed model is based on the non-radial model. The general purpose of model (11) is to find the target in the two-stage network DEA-R.

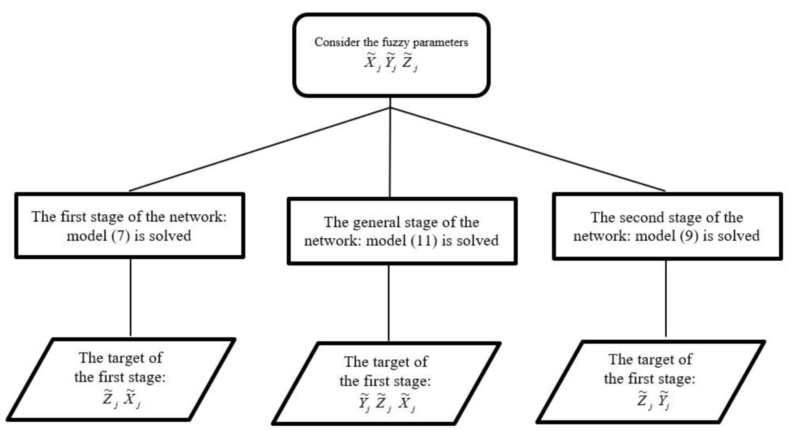

Figure 3: Flowchart of Find the target in DEA-RA.

Therefore, Flowchart (figure(3)) shows the calculation of target detection in the proposed models.

4 Applied Study

With the increasing expansion of air transportation, the airline industry is considered one of the most important pillars of the growth and development of any country. In this article, a two-stage network for domestic flights of 16 Iranian airlines is considered based on the annual report of these companies in 2017. In the first stage, pre-flight services and in the second stage, post-flight services are included. The data of this research is adapted from Mozafari et al.'s 2020 article [22]. Figure (4) shows the network of Iran's domestic airlines.

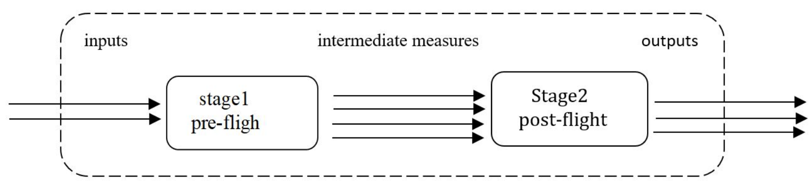

Figure 4: Network of domestic flights in Iranian airline

In the first stage, we observe different α values ranging between 0.1 to 1 with 0.1 distances. By setting $\alpha = 1$ and accepting a 100% risk, the number of personnel (X^1_{1j}) is increased to 32 and the number of empty seats (X^1_{2j}) to 176, benchmark vector is obtained for the DMUs, meaning z^1_{1j} (number of flights), z^1_{2j} (mean delay time), z^1_{3j} (number of passengers), and z^1_{4j} (number of flights) equaling 21, 2548.000, 1917329, and 54, respectively (see ,Table (1-2). In the second stage, by setting $\alpha = 1$ and considering a 100% risk, the number of airplanes (z^2_{1j}) is increased to 7, the mean delay time (z^2_{2j}) to 1306.000 minutes, the number of passengers (z^2_{3j}) to 556006, and the number of flights (z^2_{4j}) to 50. In addition, the target vector of the outputs of the second stage, meaning y^2_{1j}, y^2_{2j}, y^2_{3j}, and y^2_{4j}, is obtained equal to 9, 89700, 117, and 4870, respectively.

Table 1: Benchmarks of DMUs based on different α values in the overall two-stage network using the model (9)

Stage all	α	0.1	0.2	0.3	0.4	0.5
6*stage 1	x_{1j}^1	28	307	308	308	40
	x_{2j}^1	164	163	165	168	165
	z_{1j}^1	24	68	67	67	16
	z_{2j}^1	2732.500	13584.400	13526.600	13468.800	3371.154
	z_{3j}^1	1980925	3763109	3683946	3604782	1906060
	z_{4j}^1	57	39	38	38	53
6*satge 2	z_{1j}^2	3	4	4	5	5
	z_{2j}^2	117.100	131.200	145.300	159.400	173.500
	z_{3j}^2	1072763	1091602	1110441	1129280	1148119
	z_{4j}^2	61	61	61	61	62
	y_{1j}^2	24	24	23	23	23
	y_{2j}^2	88.920	88.840	88.760	88.680	88.600
	y_{3j}^2	292	283	275	267	259
	y_{4j}^2	3.514	3.328	3.142	2.956	2.770

In Table (10, when setting $\alpha = 0.5$, i.e. accepting a 50% risk, the number of personnel is increased to 40, the number of empty seats to 165, the number of airplanes to 16, and the number of flights to 53, and a benchmark vector is obtained by solving the model (9), as the outputs of the overall network, which consists of y_{1j}^2,y_{2j}^2, y_{3j}^2, and y_{4j}^2 equaling 23, 88600, 259, and 2770, respectively.

Table 2: Benchmarks of DMUs based on different α values in the overall two-stage network using the model (9)

Stage all	α	0.6	0.7	0.8	0.9	1
6*stage 1	x_{1j}^1	308	308	309	309	231
	x_{2j}^1	174	177	801	182	171
	z_{1j}^1	65	65	64	64	49
	z_{2j}^1	13353.200	13295.400	13237.600	13179.800	9813.384
	z_{3j}^1	3446454	3367290	3288127	3208963	2821525
	z_{4j}^1	38	38	37	37	39
6*satge 2	z_{1j}^2	5	6	7	7	7
	z_{2j}^2	187.600	201.700	1144.800	1225.400	1306.000
	z_{3j}^2	1166958	1185797	5448045	550405	556006
	z_{4j}^2	62	62	50	50	50
	y_{1j}^2	23	23	9	9	9
	y_{2j}^2	88.520	88.440	89.760	89.730	89.700
	y_{3j}^2	250	242	120	119	117
	y_{4j}^2	2.584	2.398	4.986	4.928	4.870

5 Conclusion

In many organizations, finding efficiency and then finding a suitable target for managers is very important, but there are two forms. Firstly, the calculation of efficiency and target requires solving mathematical models, and secondly, sometimes the evaluation of decision-making units of organizations does not indicate the exact efficiency score for that unit. Therefore, the use of multi-stage network structure in DEA and DEA-R can more accurately evaluate the units of an organization. On the other hand, CRA models propose a suitable target for decision-making units by solving a linear programming model without calculating the efficiency score. In this article, two-stage network models are proposed in CRA-R based on fuzzy data. For future researches, it is suggested to calculate returns to scale and find target in fuzzy multi-stage DEA or DEA-R networks.

References

[1] R.D. Banker, A. Charnes & W.W. Cooper, Some models for estimating technical and scale inefficiencies in data envelopment analysis, *Management science* , 30(9), (1984), 1078-1092.

[2] A. Charnes & W.W. Cooper. Chance-constrained programming, *Management science*, 6(1), (1959), 73-79.

[3] A. Charnes, W.W. Cooper & E. Rhodes, Measuring the efficiency of decision-making units, *European journal of operational research*, 3(4), (1979), 339.

[4] Y. Chen, Y. Pan, H. Liu, H. Wu & G. Deng, Efficiency analysis of Chinese universities with shared inputs: An aggregated two-stage network DEA approach, *Socio-Economic Planning Sciences*, 90, (2023), 101728.

[5] Y. Chen, X. Ma, P. Yan & M. Wang, Operating efficiency in Chinese universities: An extended two-stage network DEA approach, *Journal of management science and engineering*, 6(4), (2021), 482-498.

[6] O. Despić, M. Despić, & J. C. Paradi, DEA-R: Ratio-based comparative efficiency model, its mathematical relation to DEA and its use in applications, *Journal of productivity analysis*, 28(1-2) ,(2007), 33-44.

[7] A. Ebrahimnejad, M. Tavana, F.H. Lotfi, R. Shahverdi & M. Yousefpour, A three-stage data envelopment analysis model with application to banking industry, *Measurement*, 49, (2014), 308-319.

[8] S.C. Fang, C.F. Hu, H.F. Wang, & S.Y. Wu, Linear programming with fuzzy coefficients in constraints, *Computers & Mathematics with Applications*, 37(10), (1999), 63-76.

[9] M.J. Farrell,, The measurement of productive efficiency, *Journal of the Royal Statistical Society Series A: Statistics in Society*, 120(3), (1957), 253-281.

[10] B. Golany, & E. Tamir, Evaluating efficiency-effectiveness-equality trade-offs: a data envelopment analysis approach, *Management science*, 41(7), (1995), 1172-1184.

[11] F.H. Lotfi, A.A. Noora, G.R. Jahanshahloo, J. Gerami & M.R. Mozaffari,, Centralized resource allocation for enhanced Russell models, *Journal of computational and applied mathematics*, 235(1), (2010), 1-10.

[12] F.H. Lotfi, N. Nematollahi, M. H.Behzadi, M. Mirbolouki & Z. Moghaddas, Centralized resource allocation with stochastic data, *Journal of Computational and Applied Mathematics*, 236(7), (2012), 1783-1788.

[13] F. H. Lotfi, M. Sanei, A.A. Hosseinzadeh, S. Niroomand & A. Mahmoodirad, Uncertainty in Data Envelopment Analysis: Fuzzy and Belief Degree-Based Uncertainties, *Elsevier Science* & *Technology*, 2023.

[14] C. Heydari, H. Omrani & R. Taghizadeh, A fully fuzzy network DEA-Range Adjusted Measure model for evaluating airlines efficiency: A case of Iran, *Journal of Air Transport Management*, 89, (2020), 101923.

[15] Y. Huang & M. Wang, Heterogeneous multi-attribute group decision making based on a fuzzy data envelopment analysis cross-efficiency model, *Expert Systems with Applications*, 238, (2024), 121914.

[16] G.A. Keskin, S. İlhan & C. Özkan, The Fuzzy ART algorithm: A categorization method for supplier evaluation and selection. *Expert Systems with Applications* 37(2) (2010), 1235-1240.

[17] M. Khoveyni & R. Eslami, Two-stage network DEA with shared resources: Illustrating the drawbacks and measuring the overall efficiency, *Knowledge-Based Systems*, 250, (2022), 108725.

[18] S. Lozano & G. Villa, Centralized resource allocation using data envelopment analysis, *Journal of productivity analysis*, 22, (2004), 143-161.

[19] S. Lozano, G. Villa & R. Brännlund, Centralised reallocation of emission permits using DEA, *European journal of operational research*, 193(3), (2009), 752-760.

[20] M.R. Mozaffari, P. Kamyab, J. Jablonsky & J.Gerami, Cost and revenue efficiency in DEA-R models, *Computers & industrial engineering*, 78, (2014), 188-194.

[21] S. Nazari-Shirkouhi, M.Tavakoli, K.Govindan & S.Mousakhani, A hybrid approach using Z-number DEA model and Artificial Neural Network for Resilient supplier Selection , *Expert Systems with Applications*, 222, (2023), 119746.

[22] S. Ostovan, M.R. Mozaffari, A. Jamshidi & J. Gerami,Evaluation of two-stage networks based on average efficiency using DEA and DEA-R with Fuzzy data, *International Journal of Fuzzy Systems*, 22, (2020), 1665-1678.

[23] H. Omrani, Z. Yang, A. Karbasian & T.Teplova, Combination of top-down and bottom-up DEA models using PCA: A two-stage network DEA with shared input and undesirable output for evaluation of the road transport sector, *Socio-Economic Planning Sciences*, 89, (2023), 101706.

[24] M. Tavassoli, S. Ketabi & M. Ghandehari, A novel fuzzy network DEA model to evaluate efficiency of Iran's electricity distribution network with sustainability considerations, *Sustainable Energy Technologies and Assessments*, 52, (2022), 102269.

[25] C.K. Wei, L.C. Chen, R.K. Li & C.H. Tsai, Using the DEA-R model in the hospital industry to study the pseudo-inefficiency problem, *Expert systems with applications*, 38.3, (2011), 2172-2176.

[26] C.K. Wei, L.C. Chen, R.K. Li & C.H. Tsai, Exploration of efficiency underestimation of CCR model: Based on medical sectors with DEA-R model, *Expert systems with applications*, 38.4, (2011), 3155-3160.

[27] Z. Wang, X. Zhu, H. Zhang & A. Boocock, Study on non-aeronautical efficiency of Chinese listed airports based on two-stage

network DEA, *Research in Transportation Business & Management*, 46, 100867.

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