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Original Research Paper

Solvability of Infinite Systems of Fractional Equations in the Hahn Sequence Space

M. Hosseini Farahi
Mashhad Branch, Islamic Azad University

R. Allahyari *
Mashhad Branch, Islamic Azad University

M. Hassani
Mashhad Branch, Islamic Azad University

Abstract. We define the Hausdorff measures of noncompactness in the Hahn sequence space. Then, by applying the MNC we consider the solvability of a BVP of fractional type by nonlocal integral boundary conditions in the Hahn sequence space. Eventually, we provide one example to inquire about the performance of the main results.

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1 Introduction

Recently, the implication of MNC has been utilized in sequence spaces for different classes of differential equations ([2, 6, 8, 9, 10, 11, 13, 15,

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*Corresponding Author

16, 17, 18, 19]). Aghajani et al. [4] investigated the solvability of infinite systems of second order differential equations in l_1-spaces. Afterwards, Mohiuddine et al. [14] and Banaś et al. [7] focused in these systems in the l_p spaces.

In this paper, we present the Hausdorff MNC in Hahn sequence space. By applying this MNC, we consider the solvability of infinite systems of a BVP fractional type by nonlocal integral boundary conditions in the Hahn sequence space. Then, we present one example to inquire about the performance of the main results.

Suppose that $(\mho, \|\cdot\|)$ is a real Banach space by zero element 0, $D(\nu,\sigma)$ is the ball centered at ν by radius σ. For $\emptyset \neq \mathfrak{L} \subseteq \mho$, we denote by $\overline{\mathfrak{L}}$ the closure and by Conv$\mathfrak{L}$ the closed convex hull of $\mathfrak{L}$, $\emptyset \neq \mathfrak{N}_\mho \subseteq \mho$ is the family of all relatively compact subsets and $\emptyset \neq \mathfrak{M}_\mho \subseteq \mho$ is the family of nonempty bounded subsets of $\mho$.

Definition 1.1. [1] The function $\widetilde{\mu} : \mathfrak{M}_\mho \to [0,\infty)$ is a measure of noncompactness (MNC) in $\mho$ if it fulfills:

1° $\mathfrak{N}_\mho \supseteq \{\mathfrak{L} \in \mathfrak{M}_\mho : \widetilde{\mu}(\mathfrak{L}) = 0\} = \ker \widetilde{\mu} \neq \emptyset$.

2° $\mathfrak{L} \subset \mathfrak{R} \Rightarrow \widetilde{\mu}(\mathfrak{L}) \leq \widetilde{\mu}(\mathfrak{R})$.

3° $\widetilde{\mu}(\overline{\mathfrak{L}}) = \widetilde{\mu}(\mathfrak{L}) = \widetilde{\mu}(\mathrm{Conv}\mathfrak{L})$.

4° $\widetilde{\mu}(\zeta\mathfrak{L} + (1-\zeta)\mathfrak{R}) \leq \zeta\widetilde{\mu}(\mathfrak{L}) + (1-\zeta)\widetilde{\mu}(\mathfrak{R})$ for $0 \leq \zeta \leq 1$.

5° If $\mathfrak{L}_n \in \mathfrak{M}_\mho$, $\mathfrak{L}_n = \overline{\mathfrak{L}_n}$, and $\mathfrak{L}_{n+1} \subset \mathfrak{L}_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \widetilde{\mu}(\mathfrak{L}_n) = 0$, then $\emptyset \neq \mathfrak{L}_\infty = \bigcap_{n=1}^{\infty} \mathfrak{L}_n$.

Definition 1.2. [5] Let $(\mathfrak{L}, d)$ be a metric space and $\mathfrak{A} \in \mathfrak{M}_\mathfrak{L}$. The Kuratowski MNC of $\mathfrak{A}$, is

$$\beta(\mathfrak{A}) = \inf \Big(0 < \varepsilon : \bigcup_{\imath=1}^{m} K_\imath \supseteq \mathfrak{A}, K_\imath \subset \mathfrak{L}, \varepsilon > \mathrm{diam}(K_\imath)$$

$$(\imath = 1, 2, \ldots, m);\ m \in \mathbb{N}\Big),$$

where $\mathrm{diam}(K_\imath) = \sup\{d(\upsilon,\nu) : \nu, \upsilon \in K_\imath\}$.

The Hausdorff MNC $\chi(\mathfrak{A})$, is

$$\chi(\mathfrak{A}) = \inf \Big(\varepsilon > 0 : \mathfrak{A} \subset \bigcup_{\imath=1}^{m} D(\nu_\imath, \sigma_\imath), \nu_\imath \in \mathfrak{L}, \sigma_\imath < \varepsilon$$

$$(\imath = 1, 2, \ldots, m);\ m \in \mathbb{N} \Big).$$

Definition 1.3. [3] Let $\mho$ be a Banach space, $\emptyset \neq \mathfrak{Q} \subseteq \mho$, also, $\widetilde{\mu}$ is an MNC in $\mho$. The operator $\mathfrak{H} : Q \to Q$ is called a Meir–Keeler condensing operator if $\forall\ 0 < \varepsilon$, $\exists\ 0 < \delta$ so that

$$\varepsilon \leq \widetilde{\mu}(\mathfrak{L}) < \delta + \varepsilon \quad \text{implies} \quad \widetilde{\mu}(\mathfrak{H}(\mathfrak{L})) < \varepsilon,$$

$\forall$ bounded subset $\mathfrak{L} \subseteq Q$.

Theorem 1.4. *[3] Let $\mho$ be a Banach space, $\emptyset \neq \mathfrak{D} = \overline{\mathfrak{D}} \subseteq \mho$ is bounded, and convex and $\widetilde{\mu}$ is an MNC in $\mho$. If $\mathfrak{H} : \mathfrak{D} \to \mathfrak{D}$ is a continuous Meir–Keeler condensing operator and continuous, then $\mathfrak{H}$ has a fixed point.*

Proposition 1.5. *[5] Let Υ be a subset of $C(I, \mho)$, equicontinuous, bounded, and χ be an Hausdorff MNC. Then the function $\chi(\Upsilon(.))$ is continuous and*

$$\sup_{\wp \in I} \chi(\Upsilon(\wp)) = \chi(\Upsilon), \quad \chi\Big(\int_0^{\wp} \Upsilon(\Im) d\Im \Big) \leq \int_0^{\wp} \chi(\Upsilon(\Im)) d\Im.$$

Example 1.6. Let $\mho = C[0,1]$ and $I = [0,1]$. Next, take

$$\Upsilon = \big\{ h(s) := sf + (1-s)g :\ f, g \in \mho, \|f\| \leq 1, \|g\| \leq 1 \big\}.$$

Therefore, clearly, Υ is a subset of $C(I, \mho)$, equicontinuous and bounded. Also,

$$\Upsilon(0) = \big\{ g :\ g \in \mho, \|g\| \leq 1 \big\}$$

and, ...

$$\Upsilon(\wp) = \big\{ \wp f + (1-\wp) g :\ f, g \in \mho, \|f\| \leq 1, \|g\| \leq 1 \big\}.$$

It is easy to prove that $\chi(\Upsilon(\wp)) = 1$ for any $\wp \in I$. Hence, by using Proposition 1.5 we have

$$\chi(\Upsilon) = \sup_{\wp \in I} \chi(\Upsilon(\wp)) = 1,$$

$$\chi\Big(\int_0^{\wp} \Upsilon(\Im) d\Im \Big) \leq \int_0^{\wp} \chi(\Upsilon(\Im)) d\Im = \int_0^{\wp} 1\ d\Im = \wp.$$

Definition 1.7. ([21]) Let $f:\mathbb{R}_+\to\mathbb{R}$, the Caputo fractional derivative of order $0<\alpha$ is

$$^{c}D^{\alpha}f(\wp)=\frac{\int_0^{\wp}\frac{f^{(\mathfrak{m})}(\kappa)}{(\wp-\kappa)^{\alpha-\mathfrak{m}+1}}d\kappa}{\Gamma(\mathfrak{m}-\alpha)},$$

where $\mathfrak{m}-1=[\alpha]$.

Definition 1.8. ([21]) Let $h:(0,\infty)\to\mathbb{R}$, the R-L (Riemann-–Liouville) fractional integral of order α is

$$\frac{1}{\Gamma(\alpha)}\int_0^{\wp}\frac{h(\wp)}{(\wp-\kappa)^{1-\alpha}}d\kappa=I_{0_+}^{\alpha}h(\wp).$$

Lemma 1.9. *[20] Let* $\vartheta,\xi\in\mathbb{R}$, $0<\vartheta<1$, $\xi>0$, $\gamma\neq\frac{\Gamma(\xi+1)}{\vartheta^{\xi}}$ *and* $f_\iota\in C([0,1])$. *Then, the solution of the FDE*

$$\begin{cases}{}^{c}D_{0_+}^{\alpha}\nu(\wp)=f_\iota(\wp,\nu(\wp)),\ 0\leq\wp\leq1,\ \alpha\in(0,1],\\ \nu(0)=\gamma I_{0_+}^{\xi}\nu(\vartheta)=\gamma\int_0^{\vartheta}\frac{(\vartheta-\kappa)^{\xi-1}}{\Gamma(\xi)}\nu(\kappa)d\kappa,\end{cases}\tag{1}$$

is

$$\nu(\wp)=\frac{1}{\Gamma(\alpha)}\int_0^{\wp}(\wp-\kappa)^{\alpha-1}f_\iota(\kappa,\nu(\kappa))d\kappa$$
$$+\frac{\gamma(\Gamma(\xi+1))}{\Gamma(\xi+1)-\gamma\vartheta^{\xi}}\int_0^{\vartheta}\frac{(\vartheta-\kappa)^{\xi+\alpha-1}}{\Gamma(\xi+\alpha)}f_\iota(\kappa,\nu(\kappa))d\kappa.$$

2 Hahn Sequence Space

By $\omega=\mathbb{C}^N$, ($N=\{0,1,2,\ldots\}$ and $\mathbb{C}$ is the complex field) we denote the space of all complex-valued or real sequences.
Each linear subspace of ω is called a sequence space.

In [12] Hahn defined the Banach sequence space with continuous coordinates (BK space) H of all sequences $\nu=(\nu_k)$ so that

$$H=\{\nu:\sum_{k=1}^{\infty}k|\Delta\nu_k|<\infty\text{ and }\lim_{k\to\infty}\nu_k=0\},$$

where $\Delta\nu_k=\nu_k-\nu_{k+1}$, $\forall\ k\in\mathbb{N}$, by norm

$$\|\nu\|_H=\sum_{k=1}^{\infty}k|\Delta\nu_k|+\sup_k|\nu_k|.$$

Hahn showed that H is a Banach sequence space and $H \subset l_1 \bigcap \int c_0$, where

$$\int c_0 = \big\{\nu = (\nu_k) \in \omega : (k\nu_k) \in c_0\big\}.$$

Lemma 2.1. *[16] Let $\mathfrak{A} \subseteq \mathfrak{L}$ be bounded, where $\mathfrak{L}$ is l_p ($p \in [1, \infty)$) or c_0. If $R_n : \mathfrak{L} \to \mathfrak{L}$ is an operator such that $R_n(\nu) = (\nu_0, \nu_1, \dots, \nu_n, 0, 0, \dots)$, so*

$$\chi(\mathfrak{A}) = \lim_{n\to\infty} \big\{ \sup_{\nu\in\mathfrak{A}} \|(I - R_n)\nu\|\big\}.$$

Theorem 2.2. *Let $\mathfrak{A} \subseteq H$ be bounded. So the Hausdorff MNC χ in the Banach space H is defined by:*

$$\chi(\mathfrak{A}) := \lim_{n\to\infty} \Big\{ \sup_{\nu\in\mathfrak{A}} \Big\{ \sum_{k\geq n} (k|\Delta\nu_k|) + \sup_k |\nu_k| \Big\}\Big\}. \tag{2}$$

Proof. Define the operator $R_n : H \to H$ by
$R_n(\nu) = (\nu_1, \nu_2, \dots, \nu_n, 0, 0, \dots)$ for $\nu = (\nu_1, \nu_2, \dots) \in H$. Then

$$\mathfrak{A} \subset R_n\mathfrak{A} + (I - R_n)\mathfrak{A}. \tag{3}$$

From (3), we get

$$\begin{aligned} \chi(\mathfrak{A}) &\leq \chi(R_n\mathfrak{A}) + \chi((I - R_n)\mathfrak{A}) = \chi((I - R_n)\mathfrak{A}) \\ &\leq \operatorname{diam}((I - R_n)\mathfrak{A}) = \sup_{\nu\in\mathfrak{A}} \|(I - R_n)\nu\|, \end{aligned}$$

where

$$\|(I - R_n)\nu\| = \sum_{k=1}^{\infty} (k|\Delta\nu_k|) + \sup_k |\nu_k|,$$

when n is sufficiently large. So

$$\chi(\mathfrak{A}) \leq \lim_{n\to\infty} \sup_{\nu\in\mathfrak{A}} \|(I - R_n)\nu\|. \tag{4}$$

Reciprocally, suppose that $\varepsilon > 0$ and $\{z_1, z_2, \dots, z_j\}$ be a $[\chi(\mathfrak{A}) + \varepsilon]$-net of $\mathfrak{A}$. So

$$\mathfrak{A} \subset \{z_1, z_2, \dots, z_j\} + [\chi(\mathfrak{A}) + \varepsilon]D(H),$$

where $D(H)$ is a unit ball of H. So

$$\sup_{\nu\in\mathfrak{A}} \|(I - R_n)\nu\| \leq \sup_{1\leq\iota\leq j} \|(I - R_n)z_\iota\| + [\chi(\mathfrak{A}) + \varepsilon],$$

then

$$\lim_{n\to\infty} \sup_{\nu\in\mathfrak{A}} \|(I - R_n)\nu\| \leq \chi(\mathfrak{A}) + \varepsilon. \tag{5}$$

Since ε is arbitrary, by (4) and (5), relation (2) holds. □

3 Application

Now, we study the solvability of infinite system (1) in the Hahn sequence space. We give one example to show the performance of main results.

Consider:

(a) Let $f_\iota \in C(I \times \mathbb{R}^\infty, \mathbb{R})$, $(\iota \in \mathbb{N})$ be a function. The function $f : I \times H \to H$ is defined by

$$(z, \nu) \to (f\nu)(\kappa) = (f_1(\kappa, \nu(\kappa)), f_2(\kappa, \nu(\kappa)), f_3(\kappa, \nu(\kappa)), \ldots),$$

so that the family of functions $((f\nu)(\kappa))_{\kappa \in I}$ is equicontinuous, where $I = [0, 1]$.

(b) The following inequalities hold:

$$|f_\iota(\kappa, \nu(\kappa))| \leq |a_\iota(\kappa)||\nu_\iota(\kappa)|,$$

$$|\Delta f_\iota(\kappa, \nu(\kappa))| \leq |a_\iota(\kappa)||\Delta\nu_\iota(\kappa)|,$$

where $a_\iota : I \to \mathbb{R}$ are continuous and $(a_\iota(\kappa))_{\iota \in \mathbb{N}}$ is equibounded.
Put

$$A = \sup_{\iota \in \mathbb{N}} \sup_{\kappa \in I} |a_\iota(\kappa)|.$$

Theorem 3.1. *By having the hypotheses* (a), (b) *and* $\left(\frac{1}{\alpha\Gamma(\alpha)} + \frac{\gamma(\Gamma(\xi+1))\vartheta^{\xi+\alpha}}{(\Gamma(\xi+1)-\gamma\vartheta^{\xi})\xi+\alpha}\right)A < 1$, *the E.q (1) admits at least one solution* $\nu = (\nu_k) \in C(I, H)$ *for each* $\wp \in I$.

Proof. Define the operator $F : C(I, H) \to C(I, H)$ as

$$(F\nu)(\wp) = \frac{1}{\Gamma(\alpha)} \int_0^\wp (\wp - \kappa)^{\alpha-1} f_\iota(\kappa, \nu(\kappa)) d\kappa$$

$$+ \frac{\gamma(\Gamma(\xi+1))}{\Gamma(\xi+1) - \gamma\vartheta^\xi} \int_0^\vartheta \frac{(\vartheta - \kappa)^{\xi+\alpha-1}}{\Gamma(\xi+\alpha)} f_\iota(\kappa, \nu(\kappa)) d\kappa.$$

Also, $C(I, H)$ is equipped by norm

$$\|\nu\|_{C(I,H)} = \sup_{\wp \in I} \|\nu(\wp)\|_H.$$

By, using our assumptions, we get

$$
\begin{aligned}
&\|(F\nu)(\wp)\|_H \\
&= \sum_{k=1}^{\infty} k|\Delta(\frac{1}{\Gamma(\alpha)}\int_0^{\wp}(\wp-\kappa)^{\alpha-1}f_k(\kappa,\nu(\kappa))d\kappa \\
&\quad +\frac{\gamma(\Gamma(\xi+1))}{\Gamma(\xi+1)-\gamma\vartheta^{\xi}}\int_0^{\vartheta}\frac{(\vartheta-\kappa)^{\xi+\alpha-1}}{\Gamma(\xi+\alpha)}f_k(\kappa,\nu(\kappa))d\kappa)| \\
&\quad +\sup_k|\frac{1}{\Gamma(\alpha)}\int_0^{\wp}(\wp-\kappa)^{\alpha-1}f_k(\kappa,\nu(\kappa))d\kappa \\
&\quad +\frac{\gamma(\Gamma(\xi+1))}{\Gamma(\xi+1)-\gamma\vartheta^{\xi}}\int_0^{\vartheta}\frac{(\vartheta-\kappa)^{\xi+\alpha-1}}{\Gamma(\xi+\alpha)}f_k(\kappa,\nu(\kappa))d\kappa| \\
&\leq \sum_{k=1}^{\infty} k|a_k(\kappa)||\Delta\nu_k(\kappa)|\Big(\frac{1}{\Gamma(\alpha)}\int_0^{\wp}(\wp-\kappa)^{\alpha-1}d\kappa \\
&\quad +\frac{\gamma(\Gamma(\xi+1))}{\Gamma(\xi+1)-\gamma\vartheta^{\xi}}\int_0^{\vartheta}\frac{(\vartheta-\kappa)^{\xi+\alpha-1}}{\Gamma(\xi+\alpha)}d\kappa\Big) \\
&\quad +\sup_k|a_k(\kappa)||\nu_k(\kappa)|\Big(\frac{1}{\Gamma(\alpha)}\int_0^{\wp}(\wp-\kappa)^{\alpha-1}d\kappa \\
&\quad +\frac{\gamma(\Gamma(\xi+1))}{\Gamma(\xi+1)-\gamma\vartheta^{\xi}}\int_0^{\vartheta}\frac{(\vartheta-\kappa)^{\xi+\alpha-1}}{\Gamma(\xi+\alpha)}d\kappa\Big) \\
&\leq \Big(\frac{\wp^{\alpha}}{\alpha\Gamma(\alpha)}+\frac{\gamma(\Gamma(\xi+1))\vartheta^{\xi+\alpha}}{(\Gamma(\xi+1)-\gamma\vartheta^{\xi})\xi+\alpha}\Big)\sum_{k=1}^{\infty}k|\Delta\nu_k(\kappa)|+\sup_k|\nu_k(\kappa)|A.
\end{aligned}
$$

Taking the supremum on $\wp$ in $[0,1]$, we get

$$\|F\nu\|_{C(I,H)} \leq \Big(\frac{1}{\alpha\Gamma(\alpha)}+\frac{\gamma(\Gamma(\xi+1))\vartheta^{\xi+\alpha}}{(\Gamma(\xi+1)-\gamma\vartheta^{\xi})\xi+\alpha}\Big)A\|\nu\|_{C(I,H)}.$$

The above inequality can be written as

$$\sigma \leq \Big(\frac{1}{\alpha\Gamma(\alpha)}+\frac{\gamma(\Gamma(\xi+1))\vartheta^{\xi+\alpha}}{(\Gamma(\xi+1)-\gamma\vartheta^{\xi})\xi+\alpha}\Big)A\sigma. \tag{6}$$

Let σ_0 be optimal solution of (6). Take

$$D=D(\nu^0,\sigma_0)=\{\nu=(\nu_\iota)\in C(I,H):\ \|\nu\|_{C(I,H)}\leq\sigma_0\}.$$

Clearly, $\overline{D}=D$ is convex, bounded and F is bounded on D.

Let $y\in D$ and $\varepsilon>0$. By applying (a), $\exists\ 0<\delta$ so that if $\nu\in D$ and $\|\nu-y\|_{C(I,H)}\leq\delta$ then $\|(f\nu)-(fy)\|_{C(I,H)}\leq\frac{\varepsilon}{\Big(\frac{1}{\alpha\Gamma(\alpha)}+\frac{\gamma(\Gamma(\xi+1))\vartheta^{\xi+\alpha}}{(\Gamma(\xi+1)-\gamma\vartheta^{\xi})\xi+\alpha}\Big)A}$. Hence, for each $\wp$ in $[0,1]$, we have

$$
\begin{aligned}
&\|(F\nu)(\wp)-(Fy)(\wp)\|_H\\
&= \sum_{k=1}^{\infty}k|(\frac{1}{\Gamma(\alpha)}\int_0^{\wp}(\wp-\kappa)^{\alpha-1}\Delta(f_k(\kappa,\nu(\kappa))-f_k(\kappa,y(\kappa)))d\kappa\\
&\quad+\frac{\gamma(\Gamma(\xi+1))}{\Gamma(\xi+1)-\gamma\vartheta^{\xi}}\int_0^{\vartheta}\frac{(\vartheta-\kappa)^{\xi+\alpha-1}}{\Gamma(\xi+\alpha)}\Delta(f_k(\kappa,\nu(\kappa))-f_k(\kappa,y(\kappa)))d\kappa)|\\
&\quad+\sup_k|\frac{1}{\Gamma(\alpha)}\int_0^{\wp}(\wp-\kappa)^{\alpha-1}(f_k(\kappa,\nu(\kappa))-f_k(\kappa,y(\kappa)))d\kappa\\
&\quad+\frac{\gamma(\Gamma(\xi+1))}{\Gamma(\xi+1)-\gamma\vartheta^{\xi}}\int_0^{\vartheta}\frac{(\vartheta-\kappa)^{\xi+\alpha-1}}{\Gamma(\xi+\alpha)}(f_k(\kappa,\nu(\kappa))-f_k(\kappa,y(\kappa)))d\kappa|\\
&\le \Big(\frac{1}{\alpha\Gamma(\alpha)}+\frac{\gamma(\Gamma(\xi+1))\vartheta^{\xi+\alpha}}{(\Gamma(\xi+1)-\gamma\vartheta^{\xi})\xi+\alpha}\Big)\sup_{\wp\in I}\Big(\sum_{k=1}^{\infty}k|\Delta(f_k(\wp,\nu(\wp))-f_k(\wp,y(\wp)))|\\
&\quad+\sup_k|(f_k(\wp,\nu(\wp))-f_k(\wp,y(\wp)))|\Big)\\
&= \Big(\frac{1}{\alpha\Gamma(\alpha)}+\frac{\gamma(\Gamma(\xi+1))\vartheta^{\xi+\alpha}}{(\Gamma(\xi+1)-\gamma\vartheta^{\xi})\xi+\alpha}\Big)\|(f\nu)-(fy)\|_{C(I,H)}\le\varepsilon.
\end{aligned}
$$

Then

$$\|(F\nu)-(Fy)\|_{C(I,H)}\le\varepsilon.$$

So, F is continuous.

Now, we prove that $(F\nu)$ is continues in $(0,1)$. Let $\wp_1\in(0,1)$, $\wp>\wp_1$ and $\varepsilon>0$, so that $|\wp-\wp_1|<\varepsilon$, then, we can write

$$
\begin{aligned}
&\|(F\nu)(\wp)-(F\nu)(\wp_1)\|_H\\
&\le \sum_{k=1}^{\infty}k|\frac{1}{\Gamma(\alpha)}\Delta\Big(\int_0^{\wp}(\wp-\kappa)^{\alpha-1}f_k(\kappa,\nu(\kappa))d\kappa-\int_0^{\wp_1}(\wp_1-\kappa)^{\alpha-1}f_k(\kappa,\nu(\kappa))d\kappa\Big)|\\
&\quad+\sup_k|\frac{1}{\Gamma(\alpha)}\Big(\int_0^{\wp}(\wp-\kappa)^{\alpha-1}f_k(\kappa,\nu(\kappa))d\kappa-\int_0^{\wp_1}(\wp_1-\kappa)^{\alpha-1}f_k(\kappa,\nu(\kappa))d\kappa\Big)|\\
&\le \frac{1}{\Gamma(\alpha)}\sum_{k=1}^{\infty}k|a_k(\kappa)||\Delta\nu_k(\kappa)||\Big(\int_0^{\wp}(\wp-\kappa)^{\alpha-1}d\kappa-\int_0^{\wp_1}(\wp_1-\kappa)^{\alpha-1}d\kappa\Big)|\\
&\quad+\frac{1}{\Gamma(\alpha)}\sup_k|a_k(\kappa)||\nu_k(\kappa)||\Big(\int_0^{\wp}(\wp-\kappa)^{\alpha-1}d\kappa-\int_0^{\wp_1}(\wp_1-\kappa)^{\alpha-1}d\kappa\Big)|\\
&\le \frac{A}{\Gamma(\alpha)}\Big(\sum_{k=1}^{\infty}k|\Delta\nu_k(\kappa)|+\sup_k|\nu_k(\kappa)|\Big)\big(\frac{\wp_1^{\alpha}}{\alpha}-\frac{\wp^{\alpha}}{\alpha}\big),
\end{aligned}
$$

since $\wp>\wp_1$ and $0\le\alpha<1$ we have $\frac{\wp_1^{\alpha}}{\alpha}-\frac{\wp^{\alpha}}{\alpha}\le 0$. This proves that $(F\nu)$ is continues on $(0,1)$.

Finally, we show that F satisfies in Theorem 1.4. By Proposition 1.5 and (2), Hausdorff MNC for $D\subset C(I,H)$ is defined by

$$\chi_{C(I,H)}(D)=\sup_{\wp\in I}\chi_H(D(\wp)),$$

where $D(\wp) = \{\nu(\wp) : \nu \in D\}$. Therefore, we get

$$
\begin{aligned}
&\chi_H(FD)(\wp)\\
&= \lim_{n\to\infty}\{\sup_{\nu\in D}\Bigg(\sum_{k\geq n} k|\Delta(\frac{1}{\Gamma(\alpha)}\int_0^{\wp}(\wp-\kappa)^{\alpha-1}f_k(\kappa,\nu(\kappa))d\kappa\\
&\quad +\frac{\gamma(\Gamma(\xi+1))}{\Gamma(\xi+1)-\gamma\vartheta^{\xi}}\int_0^{\vartheta}\frac{(\vartheta-\kappa)^{\xi+\alpha-1}}{\Gamma(\xi+\alpha)}f_k(\kappa,\nu(\kappa))d\kappa)|\\
&\quad +\sup_k\Big|\frac{1}{\Gamma(\alpha)}\int_0^{\wp}(\wp-\kappa)^{\alpha-1}f_k(\kappa,\nu(\kappa))d\kappa\\
&\quad +\frac{\gamma(\Gamma(\xi+1))}{\Gamma(\xi+1)-\gamma\vartheta^{\xi}}\int_0^{\vartheta}\frac{(\vartheta-\kappa)^{\xi+\alpha-1}}{\Gamma(\xi+\alpha)}f_k(\kappa,\nu(\kappa))d\kappa\Big|\Bigg)\\
&\leq \lim_{n\to\infty}\{\sup_{\nu\in D}\Bigg(\sum_{k\geq n} k|a_k(\kappa)||\Delta\nu_k(\kappa)|\Big(\frac{1}{\Gamma(\alpha)}\int_0^{\wp}(\wp-\kappa)^{\alpha-1}d\kappa\\
&\quad +\frac{\gamma(\Gamma(\xi+1))}{\Gamma(\xi+1)-\gamma\vartheta^{\xi}}\int_0^{\vartheta}\frac{(\vartheta-\kappa)^{\xi+\alpha-1}}{\Gamma(\xi+\alpha)}d\kappa\Big)\\
&\quad +\sup_k|a_k(\kappa)||\nu_k(\kappa)|\Big(\frac{1}{\Gamma(\alpha)}\int_0^{\wp}(\wp-\kappa)^{\alpha-1}d\kappa\\
&\quad +\frac{\gamma(\Gamma(\xi+1))}{\Gamma(\xi+1)-\gamma\vartheta^{\xi}}\int_0^{\vartheta}\frac{(\vartheta-\kappa)^{\xi+\alpha-1}}{\Gamma(\xi+\alpha)}d\kappa\Big)\Bigg)\\
&\leq A\Big(\frac{\wp^{\alpha}}{\alpha\Gamma(\alpha)}+\frac{\gamma(\Gamma(\xi+1))\vartheta^{\xi+\alpha}}{(\Gamma(\xi+1)-\gamma\vartheta^{\xi})\xi+\alpha}\Big)\\
&\quad \lim_{n\to\infty}\{\sup_{\nu\in D}\Bigg(\sum_{k\geq n} k|\Delta\nu_k(\kappa)|+\sup_k|\nu_k(\kappa)|\Bigg).
\end{aligned}
$$

Then, we have

$$
\sup_{\wp\in I}\chi_H(FD)(\wp) \leq A\Big(\frac{1}{\alpha\Gamma(\alpha)}+\frac{\gamma(\Gamma(\xi+1))\vartheta^{\xi+\alpha}}{(\Gamma(\xi+1)-\gamma\vartheta^{\xi})\xi+\alpha}\Big)\chi_{C(I,H)}(D).
$$

This implies that

$$
\chi_{C(I,H)}(FD) < A\Big(\frac{1}{\alpha\Gamma(\alpha)}+\frac{\gamma(\Gamma(\xi+1))\vartheta^{\xi+\alpha}}{(\Gamma(\xi+1)-\gamma\vartheta^{\xi})\xi+\alpha}\Big)\chi_{C(I,H)}(D) < \varepsilon. \tag{7}
$$

Then

$$
\chi_{C(I,H)}(D) < \frac{1}{A\Big(\frac{1}{\alpha\Gamma(\alpha)}+\frac{\gamma(\Gamma(\xi+1))\vartheta^{\xi+\alpha}}{(\Gamma(\xi+1)-\gamma\vartheta^{\xi})\xi+\alpha}\Big)}\varepsilon.
$$

Let us choose $\delta = \varepsilon(\frac{1}{A\Big(\frac{1}{\alpha\Gamma(\alpha)}+\frac{\gamma(\Gamma(\xi+1))\vartheta^{\xi+\alpha}}{(\Gamma(\xi+1)-\gamma\vartheta^{\xi})\xi+\alpha}\Big)}-1)$. So, F is a Meir–Keeler condensing operator on $D \subset H$. By using Theorem 1.4, F has a fixed point in D, thus the equations (1) has at least one solution in $C(I,H)$. □

Example 3.2. Consider the equations

$$\begin{cases} {}^C D^{\alpha}_{0_+}\nu(\wp) = \sum\limits_{j=\iota}^{\infty} \dfrac{2j+1}{(j^2+1)(j^2+2j+2)} e^{-3\kappa}\sin(\kappa+e^{\kappa})\nu_j(\kappa), \\ \wp\in[0,1],\ \ \alpha=1,\ \ \vartheta=\frac{1}{3},\ \ \gamma=\frac{1}{4},\xi=\frac{3}{2} \\ \nu(0)=\gamma I^{\xi}_{0_+}\nu(\frac{1}{3})=\gamma\int_0^{\frac{1}{3}}\frac{(\frac{1}{3}-\kappa)^{\frac{3}{2}-1}}{\Gamma(\frac{3}{2})}\nu(\kappa)d\kappa, \end{cases} \tag{8}$$

where $\alpha=1$, $\xi=\frac{3}{2}$, $\gamma=\frac{1}{4}$ and $\vartheta=\frac{1}{3}$.

Take $f_\iota(\wp,\nu(\wp)) = \sum\limits_{j=\iota}^{\infty}\dfrac{2j+1}{(j^2+1)(j^2+2j+2)}e^{-3\kappa}\sin(\kappa+e^{\kappa})\nu_j(\kappa)$. Therefore, (8) is a special case of (1). Clearly, $\sum\limits_{j=\iota}^{\infty}\dfrac{2j+1}{(j^2+1)(j^2+2j+2)}e^{-3\kappa}\sin(\kappa+e^{\kappa})\nu_j(\kappa)$ $(\iota\in\mathbb{N})$ is continuous on $I=[0,1]$. Notice that, for any $\wp\in I$, if $\nu(\wp)=(\nu_\iota(\wp))\in H$, then $(f_\iota(\kappa,\nu(\kappa)))\in H$. Let $\varepsilon>0$ and $\nu(\wp)=(\nu_\iota(\wp))\in H$. So, by taking $y(\wp)=(y_\iota(\wp))\in H$ with $\|\nu(\wp)-y(\wp)\|_H\le\delta(\varepsilon):=2\varepsilon$, we have

$$\|f(\wp,\nu(\wp))-f(\wp,y(\wp))\|_H\le\frac{1}{2}\|\nu(\wp)-y(\wp)\|_H=\varepsilon,$$

which implies that condition (a) holds. Now, for condition (b), we have

$$\begin{aligned} |f_\iota(\kappa,\nu(\kappa))| &\le |\sum_{j=\iota}^{\infty}\frac{2j+1}{(j^2+1)(j^2+2j+2)}e^{-3\kappa}\sin(\kappa+e^{\kappa})\nu_j(\kappa)| \\ &\le |a_\iota(\kappa)||\nu_\iota(\kappa)|. \end{aligned}$$

and

$$\begin{aligned} |\Delta f_\iota(\kappa,\nu(\kappa))| &\le |\Delta\sum_{j=\iota}^{\infty}\frac{2j+1}{(j^2+1)(j^2+2j+2)}e^{-3\kappa}\sin(\kappa+e^{\kappa})\nu_j(\kappa)| \\ &\le |a_\iota(\kappa)||\Delta\nu_\iota(\kappa)|. \end{aligned}$$

$a_\iota(\wp)=\frac{e^{-3\wp}}{2}$ are continuous and $(a_\iota(\wp))_{\iota\in\mathbb{N}}$ is equibounded, by $A\le\frac{1}{2}$ and

$$A\Big(\frac{1}{\alpha\Gamma(\alpha)}+\frac{\gamma(\Gamma(\xi+1))\vartheta^{\xi+\alpha}}{(\Gamma(\xi+1)-\gamma\vartheta^{\xi})\xi+\alpha}\Big)=0.508<1.$$

Then T.h 3.1 grantees that equations (8) has at least one solution in $C([0,1],H)$.

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Masoomeh Hosseini Farahi
PhD of Mathematics
Department of Mathematics
Mashhad Branch, Islamic Azad University, Mashhad, Iran.
E-mail: farahimasoome@gmail.com

Reza Allahyari
Associate Professor of Mathematics
Department of Mathematics
Mashhad Branch, Islamic Azad University, Mashhad, Iran.
E-mail: rezaallahyari@mshdiau.ac.ir

Mahmoud Hassani
Associate Professor of Mathematics
Department of Mathematics
Mashhad Branch, Islamic Azad University, Mashhad, Iran.
E-mail: hassani@mshdiau.ac.ir