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On the Relative Non-Abelian Tensor Product of a Pair of Prime Power Groups

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Abstract. Let (G, N) be a pair of prime power groups. We give a new upper bound for $|N \otimes G|$, where $N \otimes G$ is the non-abelian tensor product of N and G. Among other results, the relative Schur multiplier of free product of groups is determined under some conditions.

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1. Introduction

Let G and H be two groups equipped with an action $(g, h) \mapsto {}^{g}h$ of G on H and an action $(h, g) \mapsto {}^{h}g$ of H on G. The actions should be compatible, see [1]. The non-abelian tensor product $G \otimes H$ is the group generated by symbols $g \otimes h$ for $g \in G$ and $h \in H$, subject to the relations

$$gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h) \quad , \quad g \otimes hh' = (g \otimes h)\ ({}^{h}g \otimes {}^{h}h'),$$

for all $g, g' \in G$ and $h, h' \in H$.
By the non-abelian tensor product of a pair of groups $N \otimes G$, we mean the non-abelian tensor product of groups G and its normal subgroup N when the conjugation action is considered.

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When the concept of non-abelian tensor product of groups introduced by R. Brown and J.-L. Loday [1] in 1987, many people interested to study and apply it to different scopes of group theory. One of these attempts is to find upper bound for the order of this group. G. Ellis [3], has shown that when N is a normal subgroup of a d-generator finite p-group G and $|N| = p^n$, then $|N \otimes G| \leqslant p^{nd}$. In this article we obtain a new upper bound $p^{(n-s)d+m}$, where the group $\dfrac{N}{[N,G]}$ has exponent p^s and $|G^{ab}| = p^m$.

In [4], Ellis introduced the Schur multiplier of the pair of groups to yield sharper results of the usual multiplier $\mathcal{M}(G)$, more study on the pairs of groups and provide non-trivial information on the third integral homology of a group. To see the relation of this subject with the non-abelian tensor product of groups, suppose (G,N) is a pair of groups and denote by $J_2(N,G)$ the kernel of epimorphism $N \otimes G \longrightarrow G$ which maps $n \otimes g$ to $[n,g]$ for all $n \in N$ and $g \in G$. In [4], it is established that the quotient group $\dfrac{J_2(N,G)}{\nabla(N,G)}$ is isomorphic to the Schur multiplier of the pair (G,N) where $\nabla(N,G) = \langle n \otimes n | n \in N \rangle$ is a subgroup of $N \otimes G$. We remind that if $N = G$ then $\bigtriangledown(G,G)$ is denoted by $\bigtriangledown(G)$. Our aim is to compute $\nabla(N,G)$ and give the order of non-abelian tensor product of a pair of groups with respect to the order of its Schur multiplier under some conditions.

One of the suggested problems about the non-abelian tensor product of groups in [2], was the verifying the treatment of tensor product on the free product of groups. For this purpose, N. D. Gilbert [5] computed $J_2(G,G)$ when G is the free product of some groups. To generalize Gilbert's result we will determine $J_2(N_1 * N_2, G_1 * G_2)$ where N_i is a normal subgroup of G_i, $i = 1, 2$.

2. Upper Bound

Let start this section with the following lemma.

Lemma 2.1. *Let N be a normal subgroup of a group G. Let Z be a central subgroup of G contained in N. Then the following sequence is* *exact:*

$$(N \otimes Z) \times (Z \otimes G) \longrightarrow N \otimes G \longrightarrow N/Z \otimes G/Z \longrightarrow 1,$$

if in addition $Z \subseteq [N, G]$, *then the sequence*

$$Z \otimes G \longrightarrow N \otimes G \longrightarrow N/Z \otimes G/Z \longrightarrow 1 \qquad (*)$$

is exact.

Theorem 2.2. *Let* G *be a* d*-generator finite* p*-group with* G^{ab} *of order* p^m. *If* N *is a normal subgroup of* G *of order* p^n *and* $\dfrac{N}{[N,G]}$ *has exponent* p^s, *then*

$$|N \otimes G| \leqslant p^{(n-s)d+m}.$$

Proof. If G is of order p, then the result holds. Suppose that G is a finite p-group and the inequality holds for all p-groups of order less than $|G|$. If $G \cong C_{p^{m_1}} \times C_{p^{m_2}} \times ... \times C_{p^{m_d}}$, where $0 \leqslant m_1 \leqslant m_2 \leqslant ... \leqslant m_d$ and $m_1 + m_2 + ... + m_d = m$ and also $N \cong C_{p^{n_1}} \times C_{p^{n_2}} \times ... \times C_{p^{n_d}}$ where $0 \leqslant n_1 \leqslant n_2 \leqslant ... \leqslant n_d$ and $n_1 + n_2 + ... + n_d = n$, then $|N \otimes G| = p^t$ in which

$$\begin{aligned} t &= \textstyle\sum_{i=1}^{d} n_i + \sum_{i=1}^{d-1} i n_{d-i} + \sum_{j=2}^{d}(\sum_{i=1}^{j-1} min\{n_j, m_i\}) \\ &\leqslant d(n_1 + n_2 + ... + n_{d-1}) + n_d + m_1 + m_2 + ... + m_{d-1} \\ &\leqslant d(n - n_d) + m. \end{aligned}$$

If G is not abelian, choose a subgroup Z in $[N, G] \cap Z(G)$ of order p. By the exact sequence $(*)$ and the isomorphism $Z \otimes G \cong Z \otimes G^{ab} \cong (C_p)^d$ together with induction hypotheses it follows that

$$\begin{aligned} |N \otimes G| &\leqslant |N/Z \otimes G/Z||Z \otimes G| \\ &= p^{(n-1-s)d+m+d} \\ &\leqslant p^{(n-s)d+m}. \quad \square \end{aligned}$$

Note that when $e \times p\ (\frac{N}{[N,G]}) \geqslant e \times p\ (G^{ab})$ this bound is better than that given in [3]. For example, if G is a finite d-generator extra special p-group, then $|N \otimes G| \leqslant p^{2d}$ for all cyclic normal subgroups N of G. Suppose $N \otimes^i G = (((N \otimes G) \otimes G) ... \otimes G)$ is the power tensor of N with $i-1$ copies of G and

$$\gamma_1(N,G) \supseteq \gamma_2(N,G) \supseteq ... \supseteq \gamma_i(N,G) \supseteq ...$$

is the central series defined in [3], where $\gamma_1(N,G) = N$ and $\gamma_i(N,G) = [\gamma_{i-1}(N,G), G]$. Then there is an epimorphism $N \otimes^i G \longrightarrow \gamma_i(N,G)$ with kernel $J_i(N,G)$.

Corollary 2.3. *Let N be a normal subgroup of a d-generator finite p-group G with $|G^{ab}| = p^m$. Suppose that $|\gamma_i(N,G)| = p^{n_i}$ and $e \times p\ (\frac{\gamma_i(N,G)}{\gamma_{i+1}(N,G)}) = p^{s_i}$. Then for any $c \geqslant 1$*

$$|N \otimes^{c+1} G| \leqslant p^t,$$

in which $t = \sum_{i=1}^{c}(n_i - s_i)d^{c-i+1} + m(1 + (c-1)d)$.

Proof. The case $c = 1$ obtains from Theorem 2.2. By exact sequence

$$J_c(N,G) \otimes G \longrightarrow (N \otimes^c G) \otimes G \longrightarrow \gamma_c(N,G) \otimes G \longrightarrow 1$$

and inequality $|J_c(N,G) \otimes G| \leqslant |J_c(N,G)|^d \leqslant |N \otimes^c G|^d$ we have

$$|N \otimes^{c+1} G| \leqslant p^{(n_c - s_c)d + m} |N \otimes^c G|^d = p^t. \quad \square$$

3. The Schur Multiplier of Pair of Groups

Let (G,N) be a pair of groups. The non-abelian exterior product $N \wedge G$ is obtained from the non-abelian tensor product $N \otimes G$ by imposing the additional relations $n \otimes n = 1$ for all $n \in N$. Ellis [4] showed that the Schur multiplier of the pair (G,N), $\mathcal{M}(G,N)$ is isomorphic to $Ker(N \wedge G \longrightarrow G)$. In particular if N is central, then

$$\mathcal{M}(N,G) \cong \frac{N \otimes G^{ab}}{\nabla(N,G)}.$$

Results in [1, 4] give a commutative diagram with exact rows and central extensions as columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
\Gamma(\frac{N}{[N,G]}) \arrow[r, "\psi"] \arrow[d, equal] & J_2(N,G) \arrow[r] \arrow[d] & \mathcal{M}(N,G) \arrow[r] \arrow[d] & 0 \\
\Gamma(\frac{N}{[N,G]}) \arrow[r, "\psi"] & N\otimes G \arrow[r] \arrow[d] & N\wedge G \arrow[r] \arrow[d] & 1 \\
 & {[N,G]} \arrow[r, equal] \arrow[d] & {[N,G]} \arrow[d] & \\
 & 1 & 1 &
\end{tikzcd} \tag{1}$$

where Γ is the Whithead's quadratic functor [9], and the homomorphism $\Gamma(\frac{N}{[N,G]}) \xrightarrow{\psi} N\otimes G$ assigns $\gamma(n[N,G])$ to $n\otimes n$ for all $n\in N$.

Theorem 3.1. *Let N be a normal subgroup of a finite group G and $|\frac{N}{[N,G]}|$ be odd.*
(i) If N has a complement and $[N,G]=[G,G]$, then $\nabla(N,G)$ is isomorphic with $\triangledown(\frac{N}{[N,G]})$ and

$$|N\otimes G| = |N||\mathcal{M}(N,G)||\mathcal{M}(\frac{N}{[N,G]})|.$$

If $\frac{N}{[N,G]}$ has r cyclic component of even order, then the right hand of above formula may be multiplied by 2^i for some $0\leqslant i\leqslant r$.
(ii) If G^{ab} is elementary abelian p-group and $\frac{N}{[N,G]} \cong \prod_{i=1}^{d'}\langle \widehat{n_i}\rangle \cong (C_p)^{d'}$, in which $\widehat{n_i}$ denotes the corresponding element in $\frac{N}{[N,G]}$ and $n_i\in[G,G]$

for $1 \leqslant i \leqslant k,\ 0 \leqslant k \leqslant d'$*, then*

$$\nabla(N,G) \cong \prod_{i=k}^{d'}(n_i{\otimes}n_i)\times\prod_{i=1}^{d'-1}(\prod_{j=k+1}^{d'}(n_i{\otimes}n_j)(n_j{\otimes}n_i)) \cong (C_p)^{\frac{1}{2}(d'-k)(d'+k+1)}.$$

In particular $\nabla(N,G) \cong \triangledown(\frac{N}{[N,G]})$ *if and only if* $k = 0$*.*

Proof. (i) Suppose N has a complement in G. It follows that the exact sequence

$$0 \to \frac{G}{[N,G]} \to \frac{G}{[N,G]} \to G/N \to 1$$

splits. So if $[N,G] = [G,G]$ then $\frac{N}{[N,G]} \otimes \frac{N}{[N,G]} \leqslant \frac{N}{[N,G]} \otimes \frac{G}{[N,G]}$. On the other hand there is a surjective homomorphism

$$\nabla(N,G) \longrightarrow \nabla(\frac{N}{[N,G]}, \frac{G}{[N,G]}). \tag{2}$$

Therefore the result holds by the fact that $\triangledown\frac{N}{[N,G]} \cong \Gamma\frac{N}{[N,G]}$.

(ii) The image of n_i, $k < i \leqslant d'$ in G^{ab}, say $\bar{n}_i$, is of order p and

$$p = o(\widehat{n_i} \otimes \bar{n}_i) \leqslant o(n_i \otimes n_i) \leqslant o(\gamma(\widehat{n_i})) = p.$$

The first inequality holds because of epimorphism (2) and the last inequality satisfies because of ψ. Also

$$p = o((\hat{n_i} \otimes \bar{n}_j)(\hat{n_j} \otimes \bar{n}_i)) \leqslant o((n_i \otimes n_j)(n_j \otimes n_i)) \leqslant o(\hat{n_i} \otimes \hat{n_j}) = p.$$

By the homomorphism $\frac{N}{[N,G]} \otimes \frac{N}{[N,G]} \longrightarrow N \otimes G$ given by $\hat{n_i} \otimes \hat{n_j} \mapsto (n_i \otimes n_j)(n_j \otimes n_i)$, so the last inequality holds.

Note that all elements $n_i \otimes n_i$ and $(n_i \otimes n_j)(n_j \otimes n_i)$ are distinct and if both of i and j are less than or equal k, then $(n_i \otimes n_j)(n_j \otimes n_i) = 0$. $\square$

Invoking Theorem 3.1 for example if G is a finite d-generator non-abelian p-group of nilpotency class 2, p is odd, G^{ab} an elementary abelian group of order p^{n-c} and $Z = Z(G)$ is elementary abelian of order p^r, then $\nabla(Z, G) \cong (C_p)^{\frac{1}{2}(r-c)(r+c+1)}$ and consequently

$$\mathcal{M}(Z, G) \cong (C_p)^{rd-\frac{1}{2}[r^2+r-(c^2+c)]}.$$

Gilbert [5] studied the non-abelian tensor square of free product of groups. We here generalize his result and also determine the schur multiplier of a pair of free product of groups.

Theorem 3.2. *Let N_i be a normal subgroup of group G_i, $i = 1, 2$ and $(G_1 * G_2, N_1 * N_2)$ be a pair of groups, where $*$ denotes the free product of groups. Then*

(i) $J_2(N_1*N_2, G_1*G_2) \cong J_2(N_1, G_1) \oplus J_2(N_2, G_2) \oplus (\frac{N_1}{[N_1, G_1]} \otimes \frac{N_2}{[N_2, G_2]})$,

(ii) $\nabla(N_1 * N_2, G_1 * G_2) \cong \nabla(N_1 \times N_2, G_1 \times G_2)$,

(iii) If N_1 has a complement in G_1 and $[N_1, G_1] = [G_1, G_1]$, then

$$\mathcal{M}(N_1 * N_2, G_1 * G_2) \cong \mathcal{M}(N_1, G_1) \oplus \mathcal{M}(N_2, G_2).$$

Proof. (i) There are homomorphisms

$$i : N_1 \otimes G_1 \to (N_1 * N_2) \otimes (G_1 * G_2), \quad j : N_2 \otimes G_2 \to (N_1 * N_2) \otimes (G_1 * G_2),$$

and the function

$$(N_1 \otimes G_1) \times (N_2 \otimes G_2) \longrightarrow (N_1 * N_2) \otimes (G_1 * G_2),$$

given by $(x, y) \mapsto i(x)j(y)$ which restricts to an injective homomorphism

$$J_2(N_1, G_1) \oplus J_2(N_2, G_2) \longrightarrow J_2(N_1 * N_2, G_1 * G_2).$$

On the other hand there is a homomorphism

$$\frac{N_1}{[N_1, G_1]} \otimes \frac{N_2}{[N_2, G_2]} \longrightarrow J_2(N_1 * N_2, G_1 * G_2), \qquad (3)$$

which maps $\hat{n_1} \otimes \hat{n_2}$ to $(n_1 \otimes n_2)(n_2 \otimes n_1)$ for all $n_i \in N_i$, $i = 1, 2$ and hence the homomorphism

$$\xi : J_2(N_1, G_1) \oplus J_2(N_2, G_2) \oplus (\frac{N_1}{[N_1, G_1]} \otimes \frac{N_2}{[N_2, G_2]}) \longrightarrow J_2(N_1 * N_2, G_1 * G_2),$$

arises so that ξ is injective. By using the surjection

$$\alpha : (N_1 * N_2) \otimes (G_1 * G_2) \longrightarrow (N_1 \otimes G_1) \times (N_1 \otimes G_2) \times (N_2 \otimes G_1) \times (N_2 \otimes G_2),$$

if $\xi(x, y, z) = 1$ then $\alpha\xi(x, y, 1) = \alpha\xi(1, 1, z^{-1})$. Hence $\alpha\xi(1, 1, z^{-1}) = 1$. So $z = 1$ and $\xi(x, y, 1) = 1$ implies that $x = y = 1$.

Now let $t \in (N_1 * N_2) \otimes (G_1 * G_2)$. Write $t = uvw$ where

$$u \in \langle n_1 \otimes g_1 | n_1 \in N_1, g_1 \in G_1 \rangle, \quad w \in \langle n_2 \otimes g_2 | n_2 \in N_2, g_2 \in G_2 \rangle,$$

and $v \in V = \langle n_1 \otimes g_2, n_2 \otimes g_1 | n_i \in N_i, g_i \in G_i \rangle$ (See [5]). Then $\kappa(t) = acb$ where $a \in [N_1, G_1]$, $b \in [N_2, G_2]$ and $c \in [G_1, G_2]$, the Cartesian subgroup of $G_1 * G_2$. Note that κ is the commutator map. If $\kappa(t) = 1$ then $a = b = 1$. Thus $\kappa(t) = \kappa(v) = c = 1$. So $u \in J_2(N_1, G_1)$ and $w \in J_2(N_2, G_2)$. But if v contains no subword $y = (n_1 \otimes n_2)(n_2 \otimes n_1)$ or $y = (n_2 \otimes n_1)(n_1 \otimes n_2)$, then its image $\kappa(v)$ is a freely reduced word in $[G_1, G_2]$ and so $\kappa(v) \neq 1$. Thus we can write $v = x_0 y x_1$ with $x_i \in V$. But $\kappa(v) = \kappa(x_0 x_1) = 1$. So by induction on the number of $n_1 \otimes g_2$, $n_2 \otimes g_1$ needed to express v and by epimorphism (3) we should have ξ is surjective.

(*ii*) The proof is similar to the special case $N_i = G_i$, $i = 1, 2$ given in [7].

(*iii*) The isomorphism

$$(N_1 \times N_2) \otimes (G_1 \times G_2) \cong (N_1 \otimes G_1) \times (N_1 \otimes G_2) \times (N_2 \otimes G_1) \times (N_2 \otimes G_2),$$

implies that

$$\nabla(N_1 \times N_2, G_1 \times G_2) \cong \nabla(N_1, G_1) \oplus \nabla(N_2, G_2) \oplus U,$$

where $U = \langle (n_1 \otimes n_2)(n_2 \otimes n_1) \mid n_1 \in N_1, n_2 \in N_2 \rangle \leqslant (N_1 \otimes G_2)(N_2 \otimes G_1)$. If N_1 has a complement, the homomorphism (3) extends to an isomorphism on U. Since the restriction of the composition map

$$(N_1 \otimes G_2) \times (N_2 \otimes G_1) \xrightarrow{\pi_2} N_2 \otimes G_1 \longrightarrow \frac{N_2}{[N_2, G_2]} \otimes \frac{G_1}{[N_1, G_1]} \longrightarrow \frac{N_2}{[N_2, G_2]} \otimes \frac{N_1}{[N_1, G_1]}$$

to the subgroup U is a left inverse of the homomorphism (3) so the result holds by *(ii)* and diagram (1). $\square$

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