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LS-Category and Topological Complexity of Product of Several Manifolds

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Abstract. The LS-category and the topological complexity are some homotopy invariants of a topological space, and the topological complexity is a close relative of the LS-category. In this article, we compute the LS-category, the lower and upper bounds for topological complexity of certain manifolds and their products based on known results.

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1 Introduction

The concept of Lusternik-Schnirelman category(LS-category) was introduced by Lusternik and Schnirelmann [13] and topological complexity was introduced by Farber [7]. One can estimate the bound for topological complexity by using LS-category.

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In Section 2, we give basic definitions, results and some well known examples. Section 3 is devoted to discuss about some properties of LS-category and topological complexity. From the definition of LS-category in [2] and [12], $cat(X) = n$ if n is the least positive integer such that X is covered by $n+1$ contractible open subsets of X. But the definition of LS-category in [7] says that $cat(X) = n$ if n is the least positive integer such that X is covered by n contractible open subsets of X. Theorem 2.20 is proved by using the definition of LS-category in [7]. But in [1], authors used the definition of LS-category as in [2] and [12], and hence derived Lemma 3.8, Corollary 3.11, Corollary 4.2 and Corollary 4.5. Motivated by the findings of [1], we extend our study of LS-category and topological complexity on product of manifolds. In Section 4, we give the corrected version of these results as theorems 4.25 to 4.28. Further, we find the LS-category and the topological complexity for product of Dold manifolds, generalized Dold spaces and product spaces along with some applications.

2 Preliminaries

We start this section, by recalling the definition of manifolds and examples. After that we recall the definition and properties of the LS-category and topological complexity.

Consider a Hausdorff space M in which each point has an open neighborhood homeomorphic to $\mathbb{R}^n$. Then M is said to be a manifold of dimension n, or more concisely an n-manifold. The following are some examples of topological manifold.

Example 2.1. Consider $1 \leq m \leq n-1$ and $Gr_m(\mathbb{R}^n)$ denotes the set of all m-dimensional subspaces in $\mathbb{R}^n$. The space $Gr_m(\mathbb{R}^n)$ is known as real Grassmann manifold [14] and it has dimension $m(n-m)$. In particular, $Gr_1(\mathbb{R}^{n+1}) = \mathbb{R}P^n$ is the real projective plane.

Example 2.2. Consider $1 \leq m \leq n-1$ and $Gr_m(\mathbb{C}^n)$ denotes the set of all m-dimensional subspaces in $\mathbb{C}^n$. The space $Gr_m(\mathbb{C}^n)$ is known as complex Grassmann manifold [14] and it has complex dimension $m(n-m)$(real dimension of $Gr_m(\mathbb{C}^n)$ is $2m(n-m)$). In particular, $Gr_1(\mathbb{C}^{n+1}) = \mathbb{C}P^n$ is the complex projective plane. The cell structure

on $Gr_m(\mathbb{C}^n)$ is defined as follows:
An m-tuple $\lambda = (\lambda_1, \ldots, \lambda_m)$ is called a *Schubert symbol* if $1 \leq \lambda_1 < \cdots < \lambda_m \leq n$. Consider $\mathbb{C}^l := \{(z_1, \ldots, z_l, 0, \ldots, 0) \in \mathbb{C}^n\}$. The *Schubert cell* $E(\lambda)$ for the Schubert symbol λ is defined as $E(\lambda) = \{V \in Gr_m(\mathbb{C}^n) | \dim(V \cap Gr_m(\mathbb{C}^{\lambda_j})) = j, \dim(V \cap Gr_m(\mathbb{C}^{\lambda_j - 1})) = j-1$ for $j = 1, 2, \ldots, m\}$. It is clear that $E(\lambda)$ is of even dimension and it gives the cell structure on $Gr_m(\mathbb{C}^n)$ [14]. It is well known that the cup length of $Gr_m(\mathbb{C}^n)$ is $m(n-m)$.

Example 2.3. Consider the increasing sequence of subspaces of $\mathbb{C}^n$ as $\{0\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n$, where $\dim(V_i) = i$. This is called the complete flag on $\mathbb{C}^n$. Denote $Fl(n) = \{V_\bullet = (\{0\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n)\}$ be the set of all complete flags on $\mathbb{C}^n$. Here $Fl(n)$ is known as complete flag manifold with complex dimension $\frac{n(n-1)}{2}$(real dimension of $Fl(n)$ is $n(n-1)$). The cell structure on $Fl(n)$ is defined as follows:
Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{C}^n$. Let $F_\bullet = (\{0\} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^n)$, where $F_i = \{e_1, \ldots e_i\}$ be the standard complete flag of $\mathbb{C}^n$. Consider the symmetric group S_n. For each $\omega \in S_n$, define $E(\omega) = \{V_\bullet \in Fl(n) | \dim(V_p \cap F_q) = \#\{i \leq p | \omega(i) \leq q\}$, for every $1 \leq p, q \leq n\}$. Then $E(\omega)$ is an open cell of real dimension $2l(\omega)$, where $l(\omega) = \#\{i < j | \omega(i) < \omega(j)\}$. For each $\omega \in S_n$, we have $0 \leq l(\omega) \leq \frac{n(n-1)}{2}$. This gives the cell structure on $Fl(n)$ [11]. It is well known that the cup length of $Fl(n)$ is $\frac{n(n-1)}{2}$.

Example 2.4. Consider the space $S^m \times \mathbb{C}P^n$ with free $\mathbb{Z}_2$-action defined by $(x, z) \to (-x, \bar{z})$, it is known as Dold manifold with dimension $m + 2n$ which is defined by Dold in 1956 and it is denoted by $D(m, n)$. The cohomology ring of $D(m, n)$ with $\mathbb{Z}_2$ coefficients is given by $H^*(D(m, n), \mathbb{Z}_2) = \frac{\mathbb{Z}_2[c]}{c^{m+1}} \otimes \frac{\mathbb{Z}_2[d]}{d^{n+1}}$, where $c \in H^1(D(m, n))$ and $d \in H^2(D(m, n))$ are the generators of the ring $H^*(D(m, n))$ with the property that $c^{m+1} = 0$ and $d^{n+1} = 0$ [6].

Throughout this work, we make the following notations fixed for our convenience.

Notation 2.5. If X and X_i are topological spaces, and R is a ring, then
(a) $X^k = \underbrace{X \times X \times \cdots \times X}_{\text{k-times}}$.

(b) $(Gr_m(\mathbb{C}^n))^k = \underbrace{Gr_m(\mathbb{C}^n) \times Gr_m(\mathbb{C}^n) \times \cdots \times Gr_m(\mathbb{C}^n)}_{\text{k-times}}$.

(c) $\prod_{i=1}^{k} X_i = X_1 \times X_2 \times \cdots \times X_k$.

(d) $\prod_{i=1}^{k} \mathbb{R}P^{n_i} = \mathbb{R}P^{n_1} \times \mathbb{R}P^{n_2} \times \cdots \times \mathbb{R}P^{n_k}$.

(e) $1^{\otimes k} = \underbrace{1 \otimes \cdots \otimes 1}_{\text{k-times}}$.

(f) $(1 \otimes 1)^{\otimes(k-1)} = \underbrace{(1 \otimes 1) \otimes \cdots \otimes (1 \otimes 1)}_{\text{(k-1)-times}}$.

(g) $\bigotimes_{i=1}^{k} H^*(X_i, R) = H^*(X_1, R) \otimes H^*(X_2, R) \otimes \cdots \otimes H^*(X_k, R)$.

(h) $\bigotimes_{i=1}^{k} [w_i]^{n_i} = [w_1]^{n_1} \otimes [w_2]^{n_2} \otimes [w_3]^{n_3} \otimes \cdots \otimes [w_k]^{n_k}$.

Definition 2.6. *Let M be an m-dimensional space with a free $\mathbb{Z}_2$-action and N be an n-dimensional $\mathbb{Z}_2$ manifold. Then the diagonal $\mathbb{Z}_2$-action on the product $M \times N$ is free. So the orbit space $(M \times N)/\mathbb{Z}_2$ is an $(m+n)$-dimensional manifold. We call this manifold as a generalized projective product space and denote it by $X(M, N)$ [17]. And also note that Dold manifold [6], projective product space [5], and the generalized Dold manifold [16], are all examples of this class of manifolds. Consider an m-dimensional manifold M and an n-dimensional manifold N with involutions $\tau : M \to M$ and $\sigma : N \to N$ such that σ has nonempty fixed point set. Consider the space*

$$X(M, N) = \frac{M \times N}{(x,y) \sim (\tau(x), \sigma(y))}.$$

Then $X(M, N)$ is the generalized Dold space with dimension $m + n$.

Example 2.7. Let n_i be positive integers, $1 \leq i \leq r$. Define

$P(n_1, n_2, \ldots, n_r) = \frac{\prod_{i=1}^{r} S^{n_i}}{(x_1,x_2,\ldots,x_r) \sim (-x_1,-x_2,\ldots,-x_r)}$, where $x_i \in S^{n_i}$. This is a manifold of dimension $\sum_{i=1}^{r} n_i$, which we call a projective product space [5].

Example 2.8. Let $X(Gr_m(\mathbb{C}^n), n_1, \ldots, n_r) = \frac{Gr_m(\mathbb{C}^n) \times \prod_{i=1}^{r} S^{n_i}}{(y,x_1,\ldots,x_r) \sim (\tau(y),-x_1,\ldots,-x_r)}$,

where τ is the conjugation involution whose fixed point set is the real Grassmann manifold $Gr_m(\mathbb{R}^n)$. This induces a fiber bundle
$Gr_m(\mathbb{C}^n) \hookrightarrow X(Gr_m(\mathbb{C}^n), n_1, \dots, n_r) \xrightarrow{p} P(n_1, \dots, n_r)$, where
$P(n_1, \dots, n_r)$ is the projective product space.

Example 2.9. Let $X(Fl(n), n_1, \dots, n_r) = \frac{Fl(n) \times \prod_{i=1}^{r} S^{n_i}}{(y, x_1, \dots x_r) \sim (\sigma(y), -x_1, \cdots -x_r)}$, where σ is the conjugation involution, whose fixed point set is the real flag manifold. This induces a fiber bundle $Fl(n) \hookrightarrow X(Fl(n), n_1, \dots, n_r)$ $\xrightarrow{p} P(n_1, \dots, n_r)$, where $P(n_1, \dots, n_r)$ is the projective product space.

Definition 2.10. *The Lusternik-Schnirelman category(LS-category) is defined as the smallest integer k such that X may be covered by k open subsets $V_1, V_2, \dots, V_k$ of X with each inclusion $V_i \hookrightarrow X$ is null-homotopic and it is denoted by $cat(X)$. If no such k exists, we will set $cat(X) = \infty$.*

Example 2.11. [3, Example 1.6] X is a contractible space if and only if $cat(X) = 1$.

Definition 2.12. *Consider the space X and a commutative ring R. Then the cup-length [3] of X with coefficients in R denoted by $cup_R(X)$, is the smallest integer k such that all $(k+1)$-fold cup products vanish in the reduced cohomology $\tilde{H}(X; R)$. If such integer does not exist, then $cup_R(X) = \infty$.*

Theorem 2.13. [3, Proposition 1.5] *The cup-length of a space X is less than the LS-category of the space for all coefficients in R. In notation, $cup_R(X) + 1 \leq cat(X)$.*

Theorem 2.14. [3, Proposition 1.37] *Suppose X and Y are path connected spaces such that $X \times Y$ is completely normal. Then $cat(X \times Y) \leq cat(X) + cat(Y) - 1$.*

Theorem 2.15. [3, Proposition 8.23] *Let X be a simply connected symplectic manifold. Then $cat(X) = \frac{dim(X)}{2} + 1$.*

Theorem 2.16. [2, Theorem 3.5] *Let X be a closed, connected n-manifold with $\pi_1(X) = \mathbb{Z}_2$. Then $cat(X) = dim(X) + 1$ if and only if $w^{dim(X)} \neq 0$, where w is the nonzero element of $H^1(X, \mathbb{Z}_2)$.*

Definition 2.17. *Let X be a path connected space and PX denote the space of all continuous paths $\gamma : [0,1] \to X$ in X. We denote by $\pi : PX \to X \times X$, the map associating to any path $\gamma \in PX$, the pair of its initial and end points. That is, $\pi(\gamma) = (\gamma(0), \gamma(1))$. A continuous function $s : X \times X \to PX$ such that the composition $\pi \circ s = id$ is the identity map. Then s is called a* section *of π.*

Definition 2.18. [7] *The topological complexity $TC(X)$ of a path connected space X is the minimal integer k, such that the Cartesian product $X \times X$ may be covered by k open subsets $U_1, U_2, \ldots, U_k$ such that for any $i = 1, 2, \ldots, k$, there exists a continuous motion planning $s_i : U_i \to PX$ with $\pi \circ s_i = id$ over U_i. If no such k exists, then $TC(X) = \infty$.*

Theorem 2.19. [7, Theorem 1] *X is contractible if and only if $TC(X) = 1$.*

Theorem 2.20. [7, Theorem 5] *If X is path connected and paracompact, then $cat(X) \leq TC(X) \leq 2cat(X) - 1$.*

Theorem 2.21. [7, Theorem 11] *For any path connected metric spaces X and Y, $TC(X \times Y) \leq TC(X) + TC(Y) - 1$.*

Consider the homomorphism $\smile : H^*(X;\mathbb{K}) \otimes H^*(X;\mathbb{K}) \to H^*(X;\mathbb{K})$ defined by $(u_1 \otimes v_1) \smile (u_2 \otimes v_2) = (-1)^{|v_1||u_2|} u_1 u_2 \otimes v_1 v_2$ where $\mathbb{K}$ is a field and $|v_1|, |u_2|$ denote the degrees of cohomology classes v_1 and u_2, respectively. The kernel of the homomorphism is called the ideal of the zero-divisors of $H^*(X;\mathbb{K})$. The *zero-divisors-cup-length*(zcl) [7] of $H^*(X;\mathbb{K})$ is the length of the longest nontrivial product in the ideal of the zero-divisors of $H^*(X;\mathbb{K})$.

Theorem 2.22. [7, Theorem 7] *The topological complexity of motion planning is greater than the zero-divisors-cup-length of $H^*(X;\mathbb{K})$.*

Theorem 2.23. [8, Lemma 28.1] *Let X be a simply connected symplectic manifold. Then $TC(X) = dim(X) + 1$.*

Example 2.24. Consider the real projective plane $\mathbb{R}P^n$. The cohomology ring of $\mathbb{R}P^n$ over the coefficient ring $\mathbb{Z}_2$, $H^*(\mathbb{R}P^n, \mathbb{Z}_2)$ is $\frac{\mathbb{Z}_2[\alpha]}{\alpha^{n+1}}$, where $\alpha \in H^1(\mathbb{R}P^n, \mathbb{Z}_2)$. Since $\alpha^n \neq 0$, we have $cup_{\mathbb{Z}_2}(\mathbb{R}P^n) = n$. Therefore, $cat(\mathbb{R}P^n) \geq n + 1$, by Theorem 2.13. Also, by Theorem 1.7 in

[3], we have $cat(\mathbb{R}P^n) \leq n+1$. This implies that the category of the real projective plane $\mathbb{R}P^n$ is $n+1$ [3, Example 1.8]. For $\mathbb{C}P^n$, we have $H^*(\mathbb{C}P^n, \mathbb{Z}) = \frac{\mathbb{Z}[\omega]}{\omega^{n+1}}$, where $\omega \in H^2(\mathbb{C}P^n, \mathbb{Z})$ such that $\omega^n \neq 0$. Since $\mathbb{C}P^n$ is a simply connected symplectic manifold, by Theorem 2.15, we have $cat(\mathbb{C}P^n) = n+1$.

Theorem 2.25. [15, Corollary 2.7] *The LS-category of the Dold manifold $D(m,n)$ is $m+n+1$.*

Theorem 2.26. [15, Theorem 3.8] *If $m = 2^{r-1}$ and $n = 2^{t-1}$, then $2m+2n-1 \leq TC(D(m,n)) \leq 2m+2n+1$.*

Theorem 2.27. [4, Proposition 2.4] *Let $X(M,N)$ be a generalized projective product space as defined in 2.6. Let $\{V_1, \dots, V_q\}$ be an $\langle\tau\rangle$-invariant categorical cover of M. Then $cat(X(M,N)) \leq q + cat(N/\sigma) - 1$.*

Theorem 2.28. [4, Theorem 4.11] *Let M be a compact, simply connected and path connected space with an involution τ such that τ^* is identity. Let N be a simply connected, path connected space with free involution σ. Then $H^*(X(M,N), \mathbb{Z}_2) = H^*(M, \mathbb{Z}_2) \otimes H^*(N/\sigma, \mathbb{Z}_2)$.*

Theorem 2.29. [4, Theorem 5.1] *Let $n_1 \leq \cdots \leq n_r$. Then $cat(X(Gr_m(\mathbb{C}^n), n_1, \dots, n_r)) = m(n-m) + n_1 + r$.*

Theorem 2.30. [4, Proposition 5.2] *Let $n_1 \leq \cdots \leq n_r$. Then $zcl_{\mathbb{Z}_2}(Gr_m(\mathbb{C}^n)) + zcl_{\mathbb{Z}_2}(\mathbb{R}^{n_1}) + r \leq TC(X(Gr_m(\mathbb{C}^n), n_1, \dots, n_r)) \leq 2m(n-m) + 2(n_1 + r) - 1$.*

3 LS-Category and Topological Complexity

This section is dedicated to discuss various characteristics of topological complexity and LS-category. Additionally, we discover the topological complexity and category for flag and Grassmann manifolds.

Theorem 3.1. *Suppose X and Y are path connected spaces such that $X \times Y$ is completely normal. Then $cat(X) \leq cat(X \times Y)$.*

Proof. Consider the inclusion map $i : X \hookrightarrow X \times Y$. Suppose $cat(X \times Y) = n$. Then there exist n open subsets $U_1, U_2, \ldots, U_n$ of $X \times Y$ such that $\bigcup_{k=1}^{n} U_k = X \times Y$ with each inclusion $U_k \hookrightarrow X \times Y$ is null-homotopic. Consider $i^{-1}(U_k)$, for each $k = 1, 2, \ldots, n$, $i^{-1}(U_k)$ is open in X and have the same property of U_k. Also, note that $\bigcup_{k=1}^{n} i^{-1}(U_k) = X$. This implies that $cat(X) \leq cat(X \times Y)$. □

Corollary 3.2. *Suppose X and Y are path connected spaces such that $X \times Y$ is completely normal and Y is contractible. Then $cat(X) = cat(X \times Y)$.*

Proof. The proof follows from Theorem 2.14 and Theorem 3.1. □

Example 3.3. Consider the space $S^n \times \mathbb{R}$. For any n, $cat(S^n) = 2$ [3, Example 1.6]. Also, $\mathbb{R}$ is contractible and so $cat(\mathbb{R}) = 1$. Therefore, $cat(S^n \times \mathbb{R}) = cat(S^n) = 2$.

Theorem 3.4. *Let X and Y be path connected topological spaces. Then $TC(X) \leq TC(X \times Y)$. Further, if Y is contractible, then $TC(X) = TC(X \times Y)$.*

Proof. Consider the inclusion map $i : X \hookrightarrow X \times Y$ and the projection map $p : X \times Y \to X$. Both are continuous and $p \circ i \simeq id_X$. Then by Theorem 3 of [7], $TC(X) \leq TC(X \times Y)$. Equality follows from Theorem 2.19 and Theorem 2.21. □

Example 3.5. Consider the space $\mathbb{R} \times X$, where $X = (S^m)^n$. Since $\mathbb{R}$ is contractible, we have $TC(\mathbb{R} \times X) = n + 1$ if m is odd and $TC(\mathbb{R} \times X) = 2n + 1$ if m is even.

Theorem 3.6. *Let X be a path connected and paracompact space with $dim(X) = 2n$. If $cat(X) = n + 1$ such that $w^n \neq 0$ for some $w \in H^*(X, \mathbb{K})$, where $\mathbb{K}$ is an infinite field, then $TC(X) = 2cat(X) - 1$.*

Proof. Consider $w \otimes 1 - 1 \otimes w \in H^*(X, \mathbb{K}) \otimes H^*(X, \mathbb{K})$. Then
$$(w \otimes 1 - 1 \otimes w)^{2n} = \sum_{i=0}^{2n} (-1)^n \binom{2n}{i} \cdot w^{2n-i} \otimes w^i = (-1)^n \binom{2n}{n} \cdot w^n \otimes w^n \neq 0.$$

This implies that $TC(X) \geq 2n+1 = 2cat(X)-1$. By Theorem 2.20, we have $TC(X) = 2cat(X)-1$. $\square$

The following Example 3.7 shows that the equality $TC(X) = 2cat(X)-1$ is not true in general.

Example 3.7. Let $X = \mathbb{R}P^2$. Then $cat(X) = 3$ and $TC(X) = 4$ [9, Corollary 8.2]. Therefore, $TC(X) = 4 \neq 5 = 2cat(X) - 1$.

Theorem 3.8. *For any positive integers m and n with $m \leq n-1$, $cat(Gr_m(\mathbb{C}^n)) = m(n-m)+1$ and $TC(Gr_m(\mathbb{C}^n)) = 2m(n-m)+1$.*

Proof. Since $Gr_m(\mathbb{C}^n)$ is a simply connected symplectic manifold, the proof follows from Theorem 2.15 and Theorem 2.23. $\square$

Theorem 3.9. *For any positive integer n, $cat(Fl(n)) = \frac{n(n-1)}{2} + 1$ and $TC(Fl(n)) = n(n-1)+1$.*

Proof. $Fl(n)$ is a simply connected symplectic manifold. Therefore, the proof follows from Theorem 2.15 and Theorem 2.23. $\square$

Theorem 3.10. *Let $n_1 \leq \cdots \leq n_r$. Then $cat(X(Fl(n), n_1, \ldots, n_r)) = \frac{n(n-1)}{2} + n_1 + r$.*

Proof. The cohomology of the projective product space [5] and Theorem 2.28 yields that the cup-length of $X(Fl(n), n_1, \ldots, n_r)$ is $\frac{n(n-1)}{2} + n_1 + r - 1$. Let $0 \leq i \leq \frac{n(n-1)}{2}$, and consider $U_i = \bigcup_{l(\omega)\leq i} E(\omega)$. Then for each i, U_i is a subcomplex of $Fl(n)$. Therefore, for each i, there exists a conjugation invariant open neighborhood V_i of U_i such that V_i retracts on U_i. Let $V_{-1} = \emptyset$, then $\{V_i - V_{i-1}\}_{i=0}^{\frac{n(n-1)}{2}}$ is a conjugation invariant categorical cover of $Fl(n)$. It is already known that $cat(P(n_1, \ldots, n_r)) = n_1 + r$ [10, Theorem 1.2]. Therefore, Theorem 2.13 and Theorem 2.27 gives the equality $cat(X(Fl(n), n_1, \ldots, n_r)) = \frac{n(n-1)}{2} + n_1 + r$. $\square$

Theorem 3.11. *Let $1 \leq i \leq k$, $n_1 \leq \cdots \leq n_r$. Then $zcl_{\mathbb{Z}_2}(Fl(n)) + zcl_{\mathbb{Z}_2}(\mathbb{R}^{n_1}) + r \leq TC(X(Fl(n), n_1, \ldots, n_r)) \leq n(n-1) + 2(n_1 + r) - 1$.*

Proof. By Theorem 2.28, we have $zcl_{\mathbb{Z}_2}(X(Fl(n), n_1, n_2, \ldots, n_r)) = zcl_{\mathbb{Z}_2}(Fl(n)) + zcl_{\mathbb{Z}_2}(\mathbb{R}^{n_1}) + r - 1$. By Theorem 2.20, Theorem 2.22 and Theorem 3.10, the proof follows. $\square$

4 LS-Category and Topological Complexity of Product Spaces

In this section, we calculate the LS-category and the topological complexity of product spaces.

Theorem 4.1. *Consider the space* $\prod_{i=1}^{k} X_i$ *with* $cat(X_i) = n_i + 1$ *such that* $[w_i]^{n_i} \neq 0$*, for some* $[w_i] \in H^1(X_i, R)$*. Then* $cat\left(\prod_{i=1}^{k} X_i\right) = \sum_{i=1}^{k} n_i + 1$.

Proof. Consider the cohomology ring of $\prod_{i=1}^{k} X_i$. Then by Künneth formula, we have $H^*\left(\prod_{i=1}^{k} X_i, R\right) = \bigotimes_{i=1}^{k} H^*(X_i, R)$.
Consider $[w_i] \in H^1(X_i, R)$ such that $[w_i]^{n_i} \neq 0$. Now define

$$\begin{aligned} a_1 &= [w_1] \otimes 1^{\otimes(k-1)} \\ a_2 &= 1 \otimes [w_2] \otimes 1^{\otimes(k-2)} \\ &\vdots \\ a_k &= 1^{\otimes(k-1)} \otimes [w_k]. \end{aligned}$$

Then

$$\begin{aligned} a_1^{n_1} &= [w_1]^{n_1} \otimes 1^{\otimes(k-1)} \neq 0 \\ a_2^{n_2} &= 1 \otimes [w_2]^{n_2} \otimes 1^{\otimes(k-2)} \neq 0 \\ &\vdots \\ a_k^{n_k} &= 1^{\otimes(k-1)} \otimes [w_k]^{n_k} \neq 0. \end{aligned}$$

This implies that $a_1^{n_1} a_2^{n_2} \dots a_k^{n_k} = \bigotimes_{i=1}^{k} [w_i]^{n_i} \neq 0$.
Therefore, $cup_R\left(\prod_{i=1}^{k} X_i\right) \geq \sum_{i=1}^{k} n_i + 1$.
By Theorem 2.14, $cat\left(\prod_{i=1}^{k} X_i\right) \leq \sum_{i=1}^{k} cat(X_i) - k + 1 = \sum_{i=1}^{k} (n_i + 1) - k +$

$1 = \sum_{i=1}^{k} n_i + 1$. This implies that $cat\left(\prod_{i=1}^{k} X_i\right) = \sum_{i=1}^{k} n_i + 1$, by Theorem 2.13. □

Corollary 4.2. *Consider the space X with $cat(X) = n+1$ such that $[w]^n \neq 0$, for some $[w] \in H^*(X, R)$. Then $cat(X^k) = kn+1$.*

Corollary 4.3. *For the product of real projective spaces $\prod_{i=1}^{k} \mathbb{R}P^{n_i}$,*

$$cat\left(\prod_{i=1}^{k} \mathbb{R}P^{n_i}\right) = \sum_{i=1}^{k} n_i + 1.$$

Proof. We know that $H^*(\mathbb{R}P^n, \mathbb{Z}_2) = \frac{\mathbb{Z}_2[\alpha]}{\alpha^{n+1}}$. If $\alpha \in H^1(\mathbb{R}P^n, \mathbb{Z}_2)$, then $\alpha_1^n \neq 0$. Therefore, by Theorem 4.1, we get the result. □

Corollary 4.4. *Consider the product $(\mathbb{R}P^n)^k$. Then $cat((\mathbb{R}P^n)^k) = kn+1$.*

Proof. The proof follows from Corollary 4.3. □

Theorem 4.5. *Let $\mathbb{K}$ be an infinite field and X_i be path connected spaces with $TC(X_i) = 2n_i + 1$, for all $1 \leq i \leq n$. Suppose for each i, there exists $[w_i] \in H^*(X_i, \mathbb{K})$ such that $([w_i] \otimes 1 - 1 \otimes [w_i])^{2n_i} \neq 0$. Then* $TC\left(\prod_{i=1}^{k} X_i\right) = \sum_{i=1}^{k} 2n_i + 1$.

Proof. For each i, $([w_i] \otimes 1 - 1 \otimes [w_i])^{2n_i} = (-1)^n \binom{2n_i}{n_i} \cdot [w]^{n_i} \otimes [w]^{n_i} \neq 0$. By Künneth formula, we have $H^*\left(\prod_{i=1}^{k} X_i, \mathbb{K}\right) = \bigotimes_{i=1}^{k} H^*(X_i, \mathbb{K})$. Set

$$\begin{aligned}
a_1 &= ([w_1] \otimes 1^{\otimes(k-1)}) \otimes (1^{\otimes k}) - (1^{\otimes k}) \otimes ([w_1] \otimes 1^{\otimes(k-1)}) \\
a_2 &= (1 \otimes [w_2] \otimes 1^{\otimes(k-2)}) \otimes (1^{\otimes k}) - (1^{\otimes k}) \otimes (1 \otimes [w_2] \otimes 1^{\otimes(k-2)}) \\
&\vdots \\
a_k &= (1^{\otimes(k-1)} \otimes [w_k]) \otimes (1^{\otimes k}) - (1^{\otimes k}) \otimes (1^{\otimes(k-1)} \otimes [w_k]).
\end{aligned}$$

Note that for each $i \in \{1, 2, \ldots k\}$, we have $a_i \in \bigotimes_{i=1}^{k} H^*(X_i, \mathbb{K}) \otimes \bigotimes_{i=1}^{k} H^*(X_i, \mathbb{K})$ such that $a_i^{2n_i} \neq 0$. From this, we have $zcl\left(\prod_{i=1}^{k} X_i\right) \geq \sum_{i=1}^{k} 2n_i$. This implies that $TC\left(\prod_{i=1}^{k} X_i\right) \geq \sum_{i=1}^{k} 2n_i + 1$. Therefore, by Theorem 2.21, we have $TC\left(\prod_{i=1}^{k} X_i\right) = \sum_{i=1}^{k} 2n_i + 1$. □

Corollary 4.6. *Consider the path connected space X with $TC(X) = 2n+1$. Suppose there exists $[w] \in H^*(X, \mathbb{K})$ such that $([w] \otimes 1 - 1 \otimes [w])^{2n} \neq 0$. Then $TC(X^n) = 2kn + 1$.*

Proof. The proof follows from Theorem 4.5. □

Corollary 4.7. *For any positive integers m_i and n_i with $m_i \leq n_i - 1$, $1 \leq i \leq k$, we have $cat\left(\prod_{i=1}^{k} Gr_{m_i}(\mathbb{C}^{n_i})\right) = \sum_{i=1}^{k} m_i(n_i - m_i) + 1$ and*
$TC\left(\prod_{i=1}^{k} Gr_{m_i}(\mathbb{C}^{n_i})\right) = 2\sum_{i=1}^{k} m_i(n_i - m_i) + 1$.

Proof. $Gr_m(\mathbb{C}^n)$ is a simply connected symplectic manifold. Let ω_i be the symplectic 2-form on $Gr_{m_i}(\mathbb{C}^{n_i})$. If $[\omega_i] \in H^2(Gr_{m_i}(\mathbb{C}^{n_i}), \mathbb{R})$ represents the corresponding cohomology class of ω_i, then $[\omega_i]^{m_i(n_i - m_i)} \neq 0$ and $[\omega_i]^{m_i(n_i - m_i)} \otimes [\omega_i]^{m_i(n_i - m_i)} \neq 0$. From Theorem 4.1, Theorem 4.5 and Theorem 3.8, the proof follows. □

Corollary 4.8. *For any positive integers m and n with $m \leq n$, we have*
$cat((Gr_m(\mathbb{C}^n))^k) = km(n-m) + 1$ and
$TC((Gr_m(\mathbb{C}^n))^k) = 2km(n-m) + 1$.

Proof. The proof follows from Theorem 4.7, by replacing $m_i = m$ and $n_i = n$ for all i. □

Corollary 4.9. *For any positive integers $n \geq 2$ and $n_i \geq 2$, $1 \leq i \leq k$, we have the following.*
(a) *$cat\left(\prod_{i=1}^{k} \mathbb{C}P^{n_i}\right) = \sum_{i=1}^{k} n_i + 1$ and $TC\left(\prod_{i=1}^{k} \mathbb{C}P^{n_i}\right) = 2\sum_{i=1}^{k} n_i + 1$.*
(b) *$cat\left((\mathbb{C}P^n)^k\right) = kn + 1$ and $TC\left((\mathbb{C}P^n)^k\right) = 2kn + 1$.*

Proof. (a) As $Gr_1(\mathbb{C}^n) = \mathbb{C}P^{n-1}$, the proof follows from Theorem 4.7 by replacing $m_i = 1$ and $n_i = n_i + 1$ for all i.
(b) The proof follows from (a). □

Corollary 4.10. *For any positive integers n_i, $1 \leq i \leq k$, we have*
$cat\left(\prod_{i=1}^{k} Fl(n_i)\right) = \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} + 1$ *and*
$TC\left(\prod_{i=1}^{k} Fl(n_i)\right) = \sum_{i=1}^{k} n_i(n_i-1) + 1.$

Proof. $Fl(n)$ is a simply connected symplectic manifold. Let ω_i be the symplectic 2-form on $Fl(n)$. If $[\omega_i] \in H^2(Fl(n), \mathbb{R})$ represents the corresponding cohomology class of ω_i, then $[\omega_i]^{\frac{n_i(n_i-1)}{2}} \neq 0$ and $[\omega_i]^{\frac{n_i(n_i-1)}{2}} \otimes [\omega_i]^{\frac{n_i(n_i-1)}{2}} \neq 0$. Thus, the proof follows from Theorem 4.1, Theorem 4.5 and Theorem 3.9. $\square$

Corollary 4.11. *For any positive integer n, we have*
$cat((Fl(n))^k) = k(\frac{n(n-1)}{2}) + 1$ *and* $TC((Fl(n))^k) = kn(n-1) + 1.$

Proof. The proof follows from Theorem 4.10, by replacing $n_i = n$ for all i. $\square$

Theorem 4.12. *For any positive integers $m_1, \ldots, m_k$ and $n_1, \ldots, n_k$, we have* $cat\left(\prod_{i=1}^{k} D(m_i, n_i)\right) = \sum_{i=1}^{k}(m_i + n_i) + 1.$

Proof. By Künneth formula, we have
$H^*\left(\prod_{i=1}^{k} D(m_i, n_i), \mathbb{Z}_2\right) = \bigotimes_{i=1}^{k} H^*(D(m_i, n_i), \mathbb{Z}_2) = \bigotimes_{i=1}^{k} \left(\frac{\mathbb{Z}_2[c_i]}{c_i^{m_i+1}} \otimes \frac{\mathbb{Z}_2[d_i]}{d_i^{n_i+1}}\right),$
where $c_i^{m_i} \otimes d_i^{n_i} \neq 0$. Therefore, the proof follows from Theorem 4.1 and Theorem 2.25. $\square$

Corollary 4.13. *For any positive integers m and n, we have*
$cat((D(m, n))^k) = k(m + n) + 1.$

Proof. Replace $m_i = m$ and $n_i = n$ for all i, in Theorem 4.12. This completes the proof. $\square$

Theorem 4.14. *If $m_i = 2^{r_i - 1}$, $n_i = 2^{t_i - 1}$, $1 \leq i \leq k$, then* $\sum_{i=1}^{k}(2m_i + 2n_i - 2) + 1 \leq TC\left(\prod_{i=1}^{k} D(m_i, n_i)\right) \leq \sum_{i=1}^{k}(2m_i + 2n_i) + 1.$

Proof. By Künneth formula, we have
$H^*\left(\prod_{i=1}^{k} D(m_i,n_i),\mathbb{Z}_2\right) = \bigotimes_{i=1}^{k} H^*(D(m_i,n_i),\mathbb{Z}_2) = \bigotimes_{i=1}^{k}\left(\frac{\mathbb{Z}_2[c_i]}{c_i^{m_i+1}} \otimes \frac{\mathbb{Z}_2[d_i]}{d_i^{n_i+1}}\right)$.
Let $a_i, b_i \in \left[\bigotimes_{i=1}^{k}\left(\frac{\mathbb{Z}_2[c_i]}{c_i^{m_i+1}} \otimes \frac{\mathbb{Z}_2[d_i]}{d_i^{n_i+1}}\right)\right] \otimes \left[\bigotimes_{i=1}^{k}\left(\frac{\mathbb{Z}_2[c_i]}{c_i^{m_i+1}} \otimes \frac{\mathbb{Z}_2[d_i]}{d_i^{n_i+1}}\right)\right]$ such that

$$\begin{aligned}
a_1 &= [(c_1\otimes 1)\otimes(1\otimes 1)^{\otimes(k-1)}]\otimes[(1\otimes 1)^{\otimes k}]\\
&\quad -[(1\otimes 1)^{\otimes k}]\otimes[(c_1\otimes 1)\otimes(1\otimes 1)^{\otimes(k-1)}]\\
a_2 &= [(1\otimes 1)\otimes(c_2\otimes 1)\otimes(1\otimes 1)^{\otimes(k-2)}]\otimes[(1\otimes 1)^{\otimes k}]\\
&\quad -[(1\otimes 1)^{\otimes k}]\otimes[(1\otimes 1)\otimes(c_2\otimes 1)\otimes(1\otimes 1)^{\otimes(k-2)}]\\
&\vdots\\
a_k &= [(1\otimes 1)^{\otimes(k-1)}\otimes(c_k\otimes 1)]\otimes[(1\otimes 1)^{\otimes k}]\\
&\quad -[(1\otimes 1)^{\otimes k}]\otimes[(1\otimes 1)^{\otimes(k-1)}\otimes(c_k\otimes 1)]
\end{aligned}$$

and

$$\begin{aligned}
b_1 &= [(1\otimes d_1)\otimes(1\otimes 1)^{\otimes(k-1)}]\otimes[(1\otimes 1)^{\otimes k}]\\
&\quad -[(1\otimes 1)^{\otimes k}]\otimes[(1\otimes d_1)\otimes(1\otimes 1)^{\otimes(k-1)}]\\
b_2 &= [(1\otimes 1)\otimes(1\otimes d_2)\otimes(1\otimes 1)^{\otimes(k-2)}]\otimes[(1\otimes 1)^{\otimes k}]\\
&\quad -[(1\otimes 1)^{\otimes k}]\otimes[(1\otimes 1)\otimes(1\otimes d_2)\otimes(1\otimes 1)^{\otimes(k-2)}]\\
&\vdots\\
b_k &= [(1\otimes 1)^{\otimes(k-1)}\otimes(1\otimes d_k)]\otimes[(1\otimes 1)^{\otimes k}]\\
&\quad -[(1\otimes 1)^{\otimes k}]\otimes[(1\otimes 1)^{\otimes(k-1)}\otimes(1\otimes d_k)]
\end{aligned}$$

Now let $p_i = 2^{r_i}-1$ and $q_i = 2^{t_i}-1$, then $a_1^{p_1}\dots a_k^{p_k} b_1^{q_1}\dots b_k^{q_k} \neq 0$. So zero-divisors-cup-length is greater than or equal to $\sum_{i=1}^{k}(p_i+q_i) = \sum_{i=1}^{k}(2^{r_i}-1+2^{t_i}-1) = \sum_{i=1}^{k}(2\cdot 2^{r_i-1}+2\cdot 2^{t_i-1}-2) = \sum_{i=1}^{k}(2m_i+2n_i-2)$. Therefore, the proof follows from Theorem 2.20, Theorem 2.22 and Theorem 2.26. □

Corollary 4.15. *If* $m = 2^{r-1}$ *and* $n = 2^{t-1}$*, then* $2k(m+n-1)+1 \leq TC((D(m,n))^k) \leq 2k(m+n)+1$.

Proof. The proof follows from Theorem 4.14, by replacing $m_i = m$ and $n_i = n$ for all i. □

Theorem 4.16. *Let* $1 \leq i \leq k$, $n_{i1} \leq \cdots \leq n_{ir_i}$. *Then* $cat\left(\prod_{i=1}^{k} X(Gr_{m_i}(\mathbb{C}^{n_i}), n_{i1}, \ldots, n_{ir_i})\right) = \sum_{i=1}^{k}(m_i(n_i-m_i)+n_{i1}+r_i-1)+1.$

Proof. By Künneth formula, we have $H^*\left(\prod_{i=1}^{k} X(Gr_{m_i}(\mathbb{C}^{n_i}), n_{i1}, \ldots, n_{ir_i}), \mathbb{Z}_2\right) = \bigotimes_{i=1}^{k} H^*\left(X(Gr_{m_i}(\mathbb{C}^{n_i}), n_{i1}, \ldots, n_{ir_i}), \mathbb{Z}_2\right)$. From the cohomology of the projective product spaces [5] and Theorem 2.28, the cup-length of $\prod_{i=1}^{k} X(Gr_{m_i}(\mathbb{C}^{n_i}), n_{i1}, \ldots, n_{ir_i})$ is $\sum_{i=1}^{k}(m_i(n_i-m_i)+n_{i1}+(r_i-1))$. By Theorem 2.14, we have $cat\left(\prod_{i=1}^{k} X(Gr_{m_i}(\mathbb{C}^{n_i}), n_{i1}, \ldots, n_{ir_i})\right) \leq \sum_{i-1}^{k} cat(X(Gr_{m_i}(\mathbb{C}^{n_i}), n_{i1}, \ldots, n_{ir_i})) - k + 1 = \sum_{i=1}^{k}(m_i(n_i-m_i)+n_{i1}+r_i-1)+1$, by Theorem 2.29. Therefore, the proof follows from Theorem 2.13. □

Theorem 4.17. *Let* $1 \leq i \leq k$, $n_{i1} \leq \cdots \leq n_{ir_i}$. *Then* $\sum_{i=1}^{k}(zcl_{\mathbb{Z}_2}(Gr_{m_i}(\mathbb{C}^{n_i})) + zcl_{\mathbb{Z}_2}(\mathbb{R}^{n_{i1}}) + r_i - 1) + 1 \leq TC\left(\prod_{i=1}^{k} X(Gr_{m_i}(\mathbb{C}^{n_i}), n_{i1}, \ldots, n_{ir_i})\right) \leq 2\sum_{i=1}^{k}(m_i(n_i-m_i)+n_{i1}+r_i-1)+1.$

Proof. By Künneth formula and Theorem 2.28, we have $zcl_{\mathbb{Z}_2}\left(\prod_{i=1}^{k} X(Gr_{m_i}(\mathbb{C}^{n_i}), n_{i1}, \ldots, n_{ir_i})\right) = \sum_{i=1}^{k}(zcl_{\mathbb{Z}_2}(Gr_{m_i}(\mathbb{C}^{n_i})) + zcl_{\mathbb{Z}_2}(\mathbb{R}^{n_i}) + r_i - 1)$. Therefore, the proof follows from Theorem 2.20, Theorem 2.22, Theorem 2.30 and Theorem 4.16. □

Theorem 4.18. *Let* $1 \leq i \leq k$, $n_{i1} \leq \cdots \leq n_{ir_i}$. *Then* $cat\left(\prod_{i=1}^{k} X(Fl(n_i), n_{i1}, \ldots, n_{ir_i})\right)$ *is* $\sum_{i=1}^{k}\left(\frac{n_i(n_i-1)}{2} + n_{i1} + r_i - 1\right) + 1$.

Proof. By Künneth formula, we have $H^*\left(\prod_{i=1}^{k} X(Fl(n_i), n_{i1}, \ldots, n_{ir_i}), \mathbb{Z}_2\right) = \bigotimes_{i=1}^{k} H^*(X(Fl(n_i), n_{i1}, \ldots, n_{ir_i}), \mathbb{Z}_2)$. From the cohomology of the projective product spaces [5] and Theorem 2.28, the cup-length of $\prod_{i=1}^{k} X(Fl(n_i), n_{i1}, \ldots, n_{ir_i})$ is $\sum_{i=1}^{k}\left(\frac{n_i(n_i-1)}{2} + n_{i1} + (r_i - 1)\right)$. By Theorem 2.14, we have $cat\left(\prod_{i=1}^{k} X(Fl(n_i), n_{i1}, \ldots, n_{ir_i})\right) \leq \sum_{i=1}^{k} cat(X(Fl(n_i), n_{i1}, \ldots, n_{ir_i})) - k + 1 = \sum_{i=1}^{k}\left(\frac{n_i(n_i-1)}{2} + n_{i1} + r_i - 1\right) + 1$. Thus, the proof follows from Theorem 2.13. □

Theorem 4.19. *Let* $1 \leq i \leq k$, $n_{i1} \leq \cdots \leq n_{ir_i}$. *Then* $\sum_{i=1}^{k}(zcl_{\mathbb{Z}_2}(Fl(n_i)) + zcl_{\mathbb{Z}_2}(\mathbb{R}^{n_{i1}}) + r_i - 1) + 1 \leq TC\left(\prod_{i=1}^{k} X(Fl(n_i), n_{i1}, \ldots, n_{ir_i})\right) \leq 2\left(\sum_{i=1}^{k}\left(\frac{n_i(n_i-1)}{2} + n_{i1} + r_i - 1\right)\right) + 1$.

Proof. By Künneth formula and Theorem 2.28, we have $zcl_{\mathbb{Z}_2}\left(\prod_{i=1}^{k} X(Fl(n_i), n_{i1}, \ldots, n_{ir_i})\right) = \sum_{i=1}^{k}(zcl_{\mathbb{Z}_2}(Fl(n_i)) + zcl_{\mathbb{Z}_2}(\mathbb{R}^{n_i}) + r_i - 1)$. The proof follows from Theorem 2.20, Theorem 2.22 and Theorem 4.18. □

Theorem 4.20. $TC\left(\prod_{i=1}^{k} S^{n_i}\right) = 2k - n + 1$, *where* n *is the number of odd dimensional sphere in this product.*

Proof. Let $a_i \in H^*(S^{n_i}, \mathbb{Q})$. Consider $S = \prod_{i=1}^{k}(a_i \otimes 1 - 1 \otimes a_i)^p$, where p is one if n_i is odd and p is two if n_i is even. Then $S \neq 0$. Therefore, the zero-divisors-cup-length of $\prod_{i=1}^{k} S^{n_i}$ is at least $2(k-m)+m$. From Theorem 8 of [7], we have $TC(S^m) = 2$ if m is odd and $TC(S^m) = 3$ if m is even and thus the proof follows from Theorem 2.21 and Theorem 2.22. □

Theorem 4.21. *Let $S = \prod_{i=1}^{\infty} S_i^m$, for each i, S_i^m is an m-dimensional sphere. Then $TC(S) = \infty$.*

Proof. Suppose that $TC(S) \lneq \infty$. Then $TC(S) = k$ for some $k \in \mathbb{N}$. Consider $X = \prod_{i=1}^{k} S_i^m$ and $Y = \prod_{i=k+1}^{\infty} S_i^m$. Then $X \times Y = S$ and by [7, Theorem 13], $TC(X) \geq k+1$. This leads a contradiction to Theorem 3.4. Therefore, $TC(S) = \infty$. Also, note that $cat(S) = \infty$. □

Remark 4.22. Since $cup(\mathbb{R}P^\infty) = \infty$ and $cup(\mathbb{C}P^\infty) = \infty$, we have $cat(\mathbb{R}P^\infty) = \infty$ and $cat(\mathbb{C}P^\infty) = \infty$. Similarly, $TC(\mathbb{R}P^\infty) = \infty$ and $TC(\mathbb{C}P^\infty) = \infty$.

The authors Akhtaifar and Asadi Golmankhaneh [1] used unreduced LS-category in their paper. The unreduced LS-category [2, 12] of a space X is the least positive integer n such that X is covered by $n+1$ contractible open subsets of X and we denote it by ${}^{u}cat(X)$. The reduced LS-category of X is the least positive integer n such that X is covered by n contractible open subsets of X and it is denoted by $cat(X)$. Theorem 3.3 in [1] is same as the Theorem 5 in [7]. It is observed that Farber [7] used the definition of reduced LS-category to prove Theorem 5 and it does not hold for unreduced LS-category as shown in the following example.

Example 4.23. Consider the sphere S^n. In Example 1.6 in [3], it is shown that the unreduced LS-category of S^n is 1 for any n. Hence by Theorem 5 in [7], we have $TC(S^n) = 1$ for all n. But by Theorem 8 in [7],

$$TC(S^n) = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 3 & \text{if } n \text{ is even} \end{cases}$$

The authors in [1] assumed Theorem 5 of [7] for unreduced LS-category to calculate the upper bound for topological complexity and proved Lemma 3.8, Corollary 3.11, Corollary 4.2 and Corollary 4.5. From above discussion, the upper bound derived in Lemma 3.8, Corollary 3.11, Corollary 4.2 and Corollary 4.5. in [1] are not true for unreduced LS-category. The Theorem 3.3 in [1] can be modified for unreduced LS-category as follows.

Theorem 4.24. *If X is a path connected and paracompact space, then* ${}^{u}cat(X) + 1 \leq TC(X) \leq 2({}^{u}cat(X)) + 1$.

Proof. By the definition of reduced and unreduced LS-category, ${}^{u}cat(X) + 1 = cat(X)$. Therefore, the result follows from Theorem 2.20. □

Using the Theorem 4.24, the corrected version of Lemma 3.8, Corollary 3.11, Corollary 4.2 and Corollary 4.5. in [1] are given as theorems 4.25 to 4.28, respectively and the proof of which follows directly.

Theorem 4.25. [1, Lemma 3.8] *Let $Gr_m(\mathbb{R}^n)$ denote the real Grassmann of $m-$planes in $\mathbb{R}^n$. Then* $5 \leq TC(Gr_2(\mathbb{R}^4)) \leq 7$.

Theorem 4.26. [1, Corollary 3.11] *For any positive integer $m \geq 1$, we have* $4m + 1 \leq TC((Gr_2(\mathbb{R}^4))^m) \leq 6m + 1$.

Theorem 4.27. [1, Corollary 4.2] *For any positive integer $p \geq 2$, we have* $3(2^p) - 2 \leq TC(Gr_2(\mathbb{R}^{2^p+1})) \leq 2^{p+1} - 3$.

Theorem 4.28. [1, Corollary 4.5] *For any positive integer $p \geq 2$, we have* $3(2^p) \leq TC(Gr_2(\mathbb{R}^{2^p+2})) \leq 2^{p+2} - 1$.

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